Welcome to MCF!

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Course Comments: 2 instructors. I leave end of June. Exams via Zoom.

Course Notes will be put on my webpage.

Good Resources: Bellettini's "Lecture Notes on Mean Curvature Flow ..."

Papers by Evans and Spruck

"Distributional solns to Mean curvature Flow" - Notes by Tim Laux.

Course Goals: - Define MCF (non-parametrically)

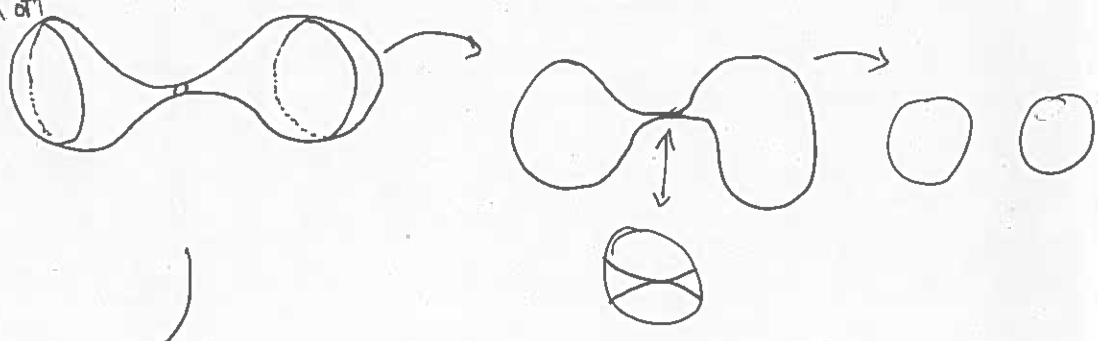
- Construct short time smooth solutions using Level-set approach. (so-called viscosity solutions for MCF)
- Time permitting show convergence of some numerical schemes to MCF: (ATW & MBO)
- Introduce Distributional solutions to Mean curvature flow
- Convergence of Allen-Cahn to MCF via the Relative Energy method.

Why is MCF so confusing?

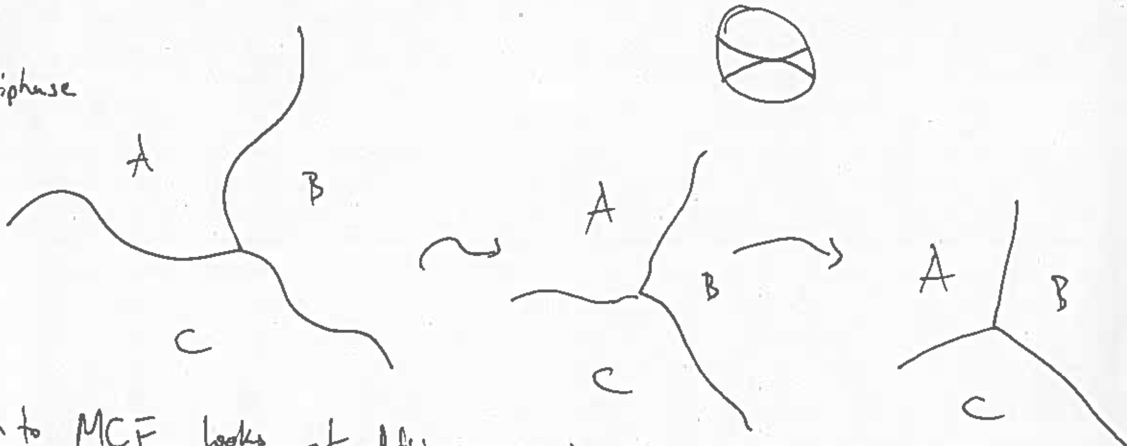
- A goal of this class is to introduce methods that work or can be extended to the "weak" setting. This means these methods have a Hope of defining mean curvature flow past a topological singularity.

Vanishing bubbles

Pinch off



Multiphase



The parametric approach to MCF looks at defining an evolving surface in terms of Embeddings. Let $M \subset\subset \mathbb{R}^N$ be a smooth compact manifold of $\dim(M) = N-1$. Let $f: M \to \mathbb{R}^N$ be the naturally associated smooth Immersion. Then we say F is a parametric (or strong) soln of MCF if $F: [0,T] \times M \to \mathbb{R}^N$ and

$F(0,x) = f(x)$

$\partial_t F(t,x) =$ Mean curvature vector

$F(t,\cdot)$ is a smooth Immersion for all $t \in [0,T)$.

Since $F(t,M)$ is always topologically equivalent to M, we see that this "strong" solution concept cannot account for topological changes. (2)

- What can be relaxed?
  - our notion of smooth surface could weaken
  - we could change our notion of the mean curvature vector (a second-order object)
  - We could weaken our enforcement of the time derivative.

It turns out the first of these ideas is the most useful ... but also complicated.

In analogy to PDE theory a second-order PDE can often be recast as a first-order condition on derivatives.

$$-\Delta u = 0 \text{ in } \Omega \iff \int \nabla u \cdot \nabla \varphi = 0 \quad \forall \varphi \in C^1(\Omega).$$

$$C^2(\Omega) \xrightarrow[\text{weaken}]{} u \in W^{1,2}(\Omega).$$

Alternatively, there is the notion of viscosity solution, which is ~~primary~~ primarily meant for second-order equations. Its utility is often driven by the fact it gives one access to a maximum principle

$$-\Delta u = 0 \text{ in } \Omega \xrightarrow[\text{weaken}]{} \forall \varphi \in C^2(\Omega) \text{ with } \varphi(x_0) = u(x_0) \text{ and } \phi \geq u, \ -\Delta\phi \leq 0$$

$$\text{Likewise for } \varphi \leq u \text{ as } -\Delta\phi \geq 0$$

It turns out that both of these ideas can be generalized to the setting of surfaces, but lead to different results (sometimes).

Weak solution concepts:

In all of these a helpful idea is to represent an evolving surface as the boundary of a set.

Given a set A, we write $\chi = \chi_A = \begin{cases} 1 & \text{on } A, \\ 0 & \text{outside of } A. \end{cases}$

For time dependent sets we use $t \mapsto A(t) \subseteq \mathbb{R}^n$

(Level-set/viscosity solutions to MCF)

The idea here is to represent the evolving boundary $\partial A(t)$ as the 0-set of a function. For instance the signed distance $d(x,t) := \operatorname{dist}(x, A(t)) - \operatorname{dist}(x, \mathbb{R}^n \setminus A(t))$ then $\{d(x,t) = 0\} = \partial A(t)$. So if we can find an evolution for d, we can recover a solution of MCF, by looking at the zero-level set.

$$\partial_t d = \operatorname{tr}\left(\nabla^2 d\,(Id - d\nabla^2 d)^{-1}\right) \quad \text{in a nbhd of } \partial A(t).$$

(Flat flows / Minimizing Movements-type scheme.)

Suppose we want to solve the ODE $\partial_t u = -f(u)$, where f is the gradient of a convex function F.

The second easiest way to solve this is Implicit-Euler $\partial_t u \approx \underbrace{\frac{u_i - u_{i-1}}{(\Delta t)} = -f(u_i)}_{\text{Implicit Euler}} \approx -f(u)$

But as f is the gradient of a function, there is a simple "Energetic" Reformulation

$$u_i = \operatorname{argmin}\left\{ \frac{\|u - u_{i-1}\|^2}{2(\Delta t)} + F(u) \right\} \longrightarrow \text{Minimizing Movements scheme for Gradient flow } \partial_t u = -\nabla F(u).$$

It turns out that formally MCF is the "gradient flow" of the perimeter functional with respect to the appropriate metric (L^2-surface). Almgren - Taylor - and Wang Introduced the minimizing movements type scheme for $\{A_i\}$

$$A_i = \underset{A}{\operatorname{argmin}}\left\{ \frac{1}{(\Delta t)} \int_{A \Delta A_{i-1}} |\operatorname{dist}(x, \partial A_{i-1})| + \operatorname{Per}(A) \right\}.$$

These sequences can be time parametrized as

$$A^{(\Delta t)}(t) = \left\{ A_i \quad \text{if} \quad t \in [i(\Delta t), (i+1)(\Delta t)) \right.$$

As $(\Delta t) \to 0$, $t \longmapsto A^{(\Delta t)}(t)$ - Converges to $t \longmapsto A(t)$, the Flat flow.

(BV/Varifold solution Concepts.)

For me, these solution concepts carry out the analogy of Sobolev functions the closest.

Given a collection of sets with smooth boundary, and bounded perimeter:

$$\operatorname{Per}(A_i) \le C < \infty \quad \forall\, i \in \mathbb{N},$$

one can show that up to a subsequence $\chi_{A_i} \longrightarrow \chi_A$ for some set A

which is of "finite perimeter". This means that $D\chi_A$ is a measure and

$|D\chi_A|(\mathbb{R}^N) < \infty$. Geometric measure theory shows that these surfaces have C^1-like boundaries. For smooth surfaces

$$\underbrace{\int_{\partial A} \vec{H} \cdot \varphi \, d\mathcal{H}^{N-1}}_{\text{Second order in } \partial A} = \underbrace{-\int_{\partial A} (Id - n \otimes n) : \nabla\varphi \, d\mathcal{H}^{N-1}}_{\text{first order in } \partial A} \qquad \forall\, \varphi \in C^1(\mathbb{R}^N; \mathbb{R}^N).$$

Mean Curvature vector

normal vector

As sets with "finite Perimeter" have a normal n, the RHS can be used to define MC in non-smooth settings.

In the next couple of lectures ~~we~~ we will prove existence of strong solutions to MCF using the Level-set approach. (4)

For this, it will be helpful if we understand the signed distance better and this is what we try to do today.

Some conventions: $(a\otimes b)_{ij} = a_i b_j$, $\nabla u = (\partial_1 u, \dots, \partial_N u)$ a row vector.

In this section ~~we~~ will will typically use the outer unit normal vector to be consistent with Bellettini (I apologize).

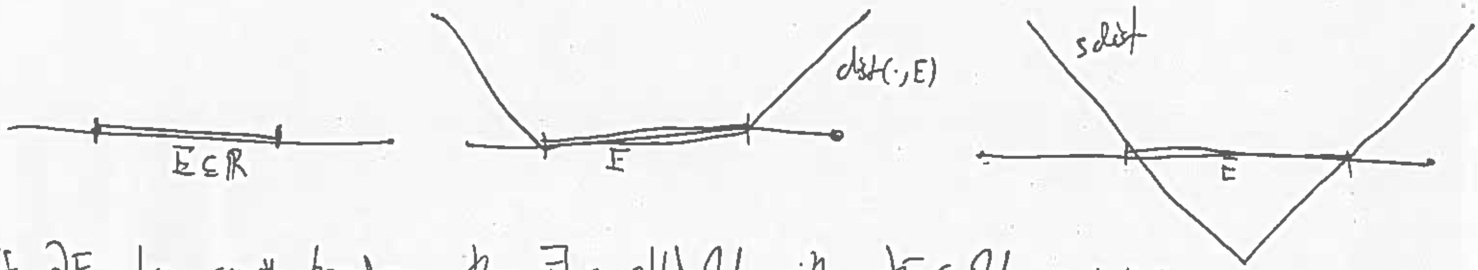
We define the distance function to a set E as

$$\mathrm{dist}(x,E) = \inf_{y\in E} |x-y|.$$

By Rademacher's theorem $\mathrm{dist}(\cdot,E)$ is differentiable a.e and $|\nabla \mathrm{dist}(\cdot,E)| = 1$ a.e. outside of E

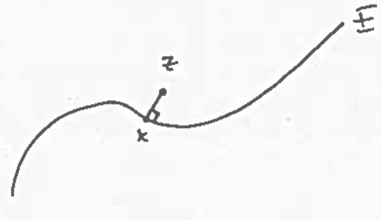
The signed distance function is a ~~bit~~ bit better:

$$\mathrm{sdist}(x,E) = \mathrm{dist}(x,E) - \mathrm{dist}(x,\mathbb{R}^N\setminus E)$$



If ∂E has smooth boundary, then $\exists$ a nbhd $\mathcal{U}$ with $\partial E \subset \mathcal{U}$ such that

$d(\cdot,E) = \mathrm{sdist}(\cdot,E) \in C^\infty(\mathcal{U})$. Further one can show that $\exists!$ closest point $x\in\partial E$ $\forall z\in\mathcal{U}$. We write $x = pr(z) = pr(z,\partial E)$.



In fact, it must be the normal projection onto ∂E.

For $z\notin E$, we have that $\nabla d(z) = \frac{z-x}{|z-x|}$ $\left(\text{For } z\in E,\ \nabla d(z) = -\left(\frac{z-x}{|z-x|}\right)\right)$

where $x = pr(z)$. This can be reshuffled to say that

- $pr(z) = z - d(z)\nabla d(z)$, since $|z-x| = d(z)$.
- $|\nabla d(z)| = 1$ in $\mathcal{U}$
- $\nabla d(z) = \nabla d(pr(z))$
- $\nabla d(x) =$ outer normal to ∂E at x.

As $\mathrm{pr} = \mathrm{Id}$ on ∂E, it is unsurprising that $\nabla \mathrm{pr}(x) = \Pi_{T_x(\partial E)}$ is the projection onto the tangent space at $x \in \partial E$: $\nabla \mathrm{pr} = \mathrm{Id} - \nabla d \otimes \nabla d - d\, \nabla^2 d = \mathrm{Id} - \nabla d \otimes \nabla d$ (on ∂E)

$$= \mathrm{Id} - n \otimes n = \Pi_{T_x(\partial E)}$$

Since $|\nabla d| = 1$ in $\mathcal{U}$, we differentiate to find

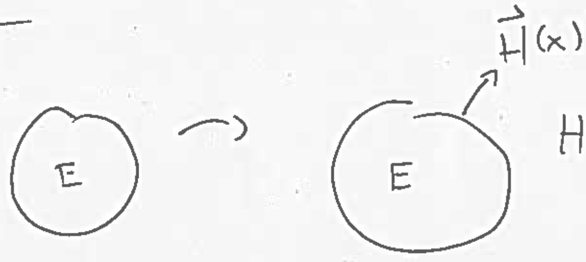
$$\nabla^2 d\, \nabla d = 0 \iff \nabla d \in \ker(\nabla^2 d)$$

We define the mean curvature vector to be

$$\vec{H}(x) = \Delta d(x) \nabla d(x) \qquad (\text{for Now}).$$

and the mean curvature to be

$$H = \Delta d(x) \qquad (\text{Remark: No average in our definition})$$

If  $H > 0$

With this convention mean curvature flow is written $V = -H$ (A questionable convention).

The second fundamental form is $\langle \nabla^2 d(x) v, w \rangle \nabla d(x) = S_x(v, w)$

As $\nabla d \in \ker(\nabla^2 d)$, $S_x(v,w) = S_x(v^T, w^T)$, where $v^T = \Pi_{T_x(\partial E)} v$ is the tangential component.

If we diagonalize $\nabla^2 d(x)$ with respect to eigenvectors $v_1, \dots, v_{N-1}, \nabla d(x)$, we have

$$\nabla^2 d(x) = \sum_{i=1}^{N-1} \mathcal{H}_i\, v_i \otimes v_i$$

and

$$H = \Delta d = \sum_{i=1}^{N-1} \mathcal{H}_i \quad \text{is the sum of the principal curvatures}$$

Example of sphere with radius r: $B(r)$.

$$d(z) = |z| - r, \quad \nabla d = \frac{z}{|z|}, \quad \nabla^2 d = \frac{1}{|z|} - \frac{z \otimes z}{|z|^3}, \quad \Delta d(z) = \frac{N-1}{|z|} \rightsquigarrow \Delta d(x) = \frac{N-1}{r}$$

We will later show that for a set A evolving with surface velocity V (outer normal), the signed distance function satisfies the transport equation

$$\partial_t d + V \cdot \nabla d = 0 \xleftrightarrow{\text{MOF}} \partial_t d + V(\mathrm{pr}(z)) \cdot \nabla d = 0$$