Γ-convergence

(1)

We say functionals $E_\varepsilon : (X,d) \to [0,\infty]$ Γ-converge to $E : X \to [0,\infty]$

If $\forall u_\varepsilon \to u_0$ we have $\liminf_{\varepsilon\to 0} E_\varepsilon[u_\varepsilon] \geq E[u_0]$ (liminf)

and $\forall u_0 \; \exists u_\varepsilon \to u_0$ s.t. $\limsup_{\varepsilon \to 0} E_\varepsilon[u_\varepsilon] \leq E[u_0]$ (limsup)

Clearly this result is topology dependent. To get an idea. Properties...

Properties: 1: Convergence of minima problems.

Suppose $u_\varepsilon = \operatorname{argmin} E_\varepsilon$ and $u_\varepsilon \to u_0$, then $u_0 \in \operatorname{argmin} E$, and $\lim E_\varepsilon[u_\varepsilon] = \min E$. $= E[u_0]$

$$\operatorname{argmin} E \leq E[u_0] \leq \liminf E_\varepsilon[u_\varepsilon] \quad \text{(liminf)}$$

Further $\limsup E_\varepsilon[u_\varepsilon] \leq \limsup E_\varepsilon[\tilde{u}_\varepsilon] \leq E[u_0]$ (limsup)

To see that $E[u_0] = \min E$, suppose $\exists \tilde{u}_0$ s.t. $E[\tilde{u}_0] < E[u_0]$

By (limsup) $\limsup E_\varepsilon[u_\varepsilon] \leq E[\tilde{u}_0] < E[u_0]$ $\lightning$

Property 2: Continuity under perturbation.

Suppose $E_\varepsilon \xrightarrow{\Gamma} E$, then If $G : X \to \mathbb{R}$ is continuous,

$$E_\varepsilon + G \xrightarrow{\Gamma} E + G.$$

Property 3: Lower semi-continuity of the limit E.

Let $u_i \to u_0$. WTS $\liminf_{i \to \infty} E[u_i] \geq E[u_0]$

By ($\overline{\lim}$) $\lim_{i\to\infty} E[u_i] = \lim_{i\to\infty} \lim_{\varepsilon\to 0} E[u_{i,\varepsilon}] = d$, $\lim_{i\to\infty} \lim_{\varepsilon\to 0} d(u_{i,\varepsilon}, u_0) = 0$.

We may diagonalize on ε to find $u_{i_\varepsilon,\varepsilon} \to u_0$ and

$\lim_{\varepsilon\to 0} E_\varepsilon[u_{i_\varepsilon,\varepsilon}] = A \Rightarrow A \geq E[u_0]$ by (liminf).

Aside on diagonalization. Given sequences $a_{i,n}, b_{i,n} \in \mathbb{B}$ s.t

$$\overline{\lim_{i\to\infty}}\ \overline{\lim_{n\to\infty}}\ a_{i,n} \leq A, \quad \lim_{i\to\infty} \lim_{n\to\infty} b_{i,n} = \mathbb{B},$$

We may find subsequence i_n s.t $\overline{\lim_{n\to\infty}}\, a_{i_n,n} \leq A$, $\lim_{n\to\infty} b_{i_n,n} = \mathbb{B}$

Example: We want to prove Γ-convergence of

$$E[u] = \int_\Omega F(u)\,dx. \quad \text{for } f(s) = \sqrt{s^2(s-1)^2}$$

$$E \xrightarrow[\Gamma]{} ? \text{ to } E.$$

If $(X,d) = (L^2, \|\cdot\|_{L^2})$, then

$$u_\varepsilon \to u_0 \text{ in } L^2 \Rightarrow \lim_{\varepsilon\to 0} E[u_\varepsilon] = E[u_0].$$

So $E \xrightarrow[\Gamma]{} E$ in the L^2-strong topology

But in applications typically, what you want to do is consider a sequence $\varepsilon\to 0$ s.t $\sup E[u_\varepsilon] < \infty$. You want to understand where these functions go, for instance if u_ε is a minimizer, can we say where $E[u_\varepsilon]$ converges... Do u_ε converge. From the energy we have

$\sup_{\varepsilon>0} \|u_\varepsilon\|_{L^2} < \infty \Rightarrow u_\varepsilon \xrightarrow[L^2]{} u_0$, which does not give strong convergence.

Does $E_\varepsilon \xrightarrow{\Gamma} E$ in L^2-weak? (3)

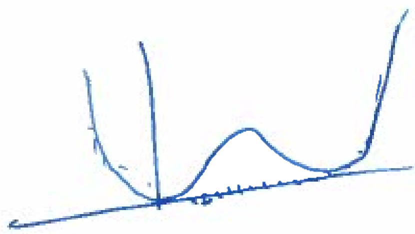
Let $f(x) = \chi_{(0,1/2)}$ on $(0,1)$ be extended periodically

Define $u_\varepsilon(x) = f(x/\varepsilon) \rightharpoonup 1/2$.

But $E[u_\varepsilon] = \int_{\Omega=(0,1)} f(u_\varepsilon) = \int_{(0,1)} 0 \to 0.$

But $E[1/2] > 0$, so $\liminf_{\varepsilon \to 0} E[u_\varepsilon] \geq E[u_0]$ does not hold so $E_\varepsilon \not\xrightarrow{\Gamma} E$. It turns out

$$E_\varepsilon \xrightarrow{\Gamma} \int_\Omega f^{**}(u)$$ where f^{**} is the convex envelope of f



Won't prove this today, but we'll use a method which also applies in this case

Thm. Let $\Omega \subseteq \mathbb{R}^N$ be an open, bdd Lipschitz domain. Define the energy

$$E[u] := \int_\Omega f(\nabla u)\,dx \quad \text{for } u \in \mathring{H}^1(\Omega;\mathbb{R}^d)$$

with f ~~continuous~~ LSC and

$$\frac{1}{c}|s|^2 - C \leq f(s) \leq C(|s|^2+1), \quad f : \mathbb{R}^{d\times N} \to [0,\infty)$$

$$E \xrightarrow{\Gamma} E_Q[u] := \int_\Omega Qf(\nabla u)\,dx \quad \text{in } (\mathring{H}^1, \text{weak}) \ (L^2, \text{strong})$$

Where QF is the quasiconvex envelope of f. (4)

Definition: A function $g : \mathbb{R}^{d \times N} \to \mathbb{R}$ is quasi convex if $\forall\ \phi \in C_c^1(Q(0,1); \mathbb{R}^d)$ the inequality

$$\int_Q g(\xi + \nabla\phi)\,dx \geq g(\xi) \quad \text{holds.}$$

Note if g is convex, it is quasi convex ...

$$g(\xi) \overset{FTC}{=} g\left(\int_Q \xi + \nabla\phi\right) \overset{\text{Jensen's}}{\leq} \int_Q g(\xi + \nabla\phi)$$

The Quasi convex envelope of f:

$$Qf(x_0) = \sup\{g(x_0) : g \leq f \text{ and } g \text{ is quasiconvex}\}$$

$\Rightarrow$ Qf is quasiconvex.

We now prove the Γ-convergence. To do we focus on the ($\underline{\lim}$)

Suppose that $u_\varepsilon \to u_0$ in L^2, $\fint_\Omega u_\varepsilon \equiv 0$.

WTS $\underline{\lim}_{\varepsilon \to 0} E_\varepsilon[u_\varepsilon] \geq E_0[u_0]$.

Define the measure $\lambda_\varepsilon := f(\nabla u_\varepsilon) \mathcal{L}^N \llcorner_\Omega$. WLOG

$\underline{\lim}_{\varepsilon \to 0} = \lim_{\varepsilon \to 0} < \infty$. So $\lambda_\varepsilon(\Omega) \leq C < \infty \quad \forall\ \varepsilon \to 0$.

By weak-$\star$ convergence for measures,

$$\lambda_\varepsilon \overset{\star}{\rightharpoonup} \lambda \in \mathcal{M}(\Omega).$$

$$\varliminf_{\varepsilon\to 0} E[u_\varepsilon] = \varliminf_{\varepsilon\to 0} \lambda_\varepsilon(\Omega) \geq \lambda(\Omega)$$
$$= \lambda_{abs}(\Omega) + \lambda_{sing}(\Omega)$$
$$\geq \int_\Omega \frac{d\lambda}{d\mathcal{L}^N}\, dx$$

If $\left[\frac{d\lambda}{d\mathcal{L}^N}(x_0) \geq Qf(\nabla u(x_0))\right]_{(\star)}$ a.e, then the above shows $\varliminf E[u_\varepsilon] \geq E_Q[u_0]$. (Blow-up Method of Fonseca + Müller)

We prove $(\star)$. Let x_0 be a Lebesgue point for λ.

$$\frac{d\lambda}{d\mathcal{L}^N}(x_0) = \lim_{r\to 0} \frac{\lambda(Q[x_0,r])}{\mathcal{L}^N(Q[x_0,r])} = \lim_{r\to 0}\lim_{\varepsilon\to 0} \frac{\lambda_\varepsilon(Q[x_0,r])}{r^N}$$

$$= \lim_{r\to 0}\lim_{\varepsilon\to 0} \fint_{Q[x_0,r]} f(\nabla u_\varepsilon)\, dx.$$

We rescale $Q[x_0,r]$ to $Q[0,1]$.

Define the function

$$V_{\varepsilon,r}(y) = \frac{u_\varepsilon(x_0+ry) - u_0(x_0)}{r}$$

Then we have

$$\frac{d\lambda}{d\mathcal{L}^N}(x_0) = \lim_{r\to 0} \lim_{\varepsilon\to 0} \fint_{Q[0,1]} f(\nabla v_{\varepsilon,r}).$$

We need some slightly more information. We choose x_0 to be a Lebesgue point of the gradient of u_0:

$$\lim_{r\to 0} \frac{1}{r^2} \fint_{Q[x_0,r]} |u(x) - u(x_0) - \nabla u(x_0)(x - x_0)|^2 \, dx = 0.$$

Up to rescaling, we have that

$$\lim_{r\to 0} \lim_{\varepsilon\to 0} \int_{Q[0,1]} |v_{\varepsilon,r}(y) - v_0(y)|^2 \, dy = 0.$$

$$v_0(y) := \nabla u(x_0)\, y.$$

We now diagonalize to find a sequence r_ε s.t. $v_\varepsilon := v_{\varepsilon, r_\varepsilon}$

$$\frac{d\lambda}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon\to 0} \int_{Q[0,1]} f(\nabla v_{\varepsilon,r_\varepsilon})\, dy$$

$$v_\varepsilon \to v_0 = \nabla u(x_0) y \text{ in } L^2$$

Recall the definition of quasi convex

$$\int g(\xi + \nabla\phi) \geq g(\xi) \quad \forall\, \phi \in C_c^1(Q;\mathbb{R}^d)$$

$$\approx \quad \int g(\nabla\Phi) \geq g(\xi) \quad \forall\, \Phi \in H^1(Q;\mathbb{R}^d),$$
$$\nabla\Phi \equiv \xi \text{ on } \partial Q.$$

We show we can modify v_ε to be affine on the boundary of the cube, while preserving energy. (7)

Define $$v_{\varepsilon,n} = \phi_n v_\varepsilon + (1-\phi_n) v_0 \quad \text{for}$$

$$\phi_n \equiv 1 \text{ in } Q[0, 1-1/n]$$

$$\phi_n \equiv 0 \text{ in } Q \setminus Q[0, 1-1/_{n+1}]$$

Technical $$\psi_n \equiv 1 \text{ in } Q \setminus Q[0, 1-1/n]$$

$$\psi_n \equiv 0 \text{ in } Q[0, 1-1/_{n-1}]$$

$$\int_Q f(\nabla v_{\varepsilon,n}) \geq \int_Q Qf(\nabla v_{\varepsilon,n}) \geq Qf(\nabla u(x_0))$$

Need to connect to $d\lambda / d\mathcal{L}^N (x_0)$

$$\int_Q f(v_{\varepsilon,n}) = \int_{Q[1-1/n]} f(\nabla v_\varepsilon) + \int_{Q\setminus Q[1-1/n]} C(1 + \|\nabla v_{\varepsilon,n}\|^2)$$

$$\Rightarrow \overline{\lim_{\varepsilon\to 0}} \int_Q f(\nabla v_{\varepsilon,n}) \leq \overline{\lim_{\varepsilon\to 0}} \int_Q f(\nabla v_\varepsilon) + \overline{\lim_{\varepsilon\to 0}} \int_{Q\setminus Q[1-1/n]} C(1+\|\nabla v_{\varepsilon,n}\|^2)$$

$$\leq \frac{d\lambda}{d\mathcal{L}^N}(x_0) + \dots$$

So we need to estimate

$$\overline{\lim_{\varepsilon\to 0}}\int_{Q\setminus Q[1-1/n]} C\left(1+\|\nabla v_{\varepsilon,n}\|^2\right)$$

$$\nabla v_{\varepsilon,n} = \nabla\phi_n (v_\varepsilon - v_0) + \phi_n \nabla v_\varepsilon + (1-\phi_n)\nabla v_0$$

$$\overline{\lim_{\varepsilon\to 0}} \cdots \le \overline{\lim_{\varepsilon\to 0}}\int_{Q\setminus Q[1-1/n]} C\left(1+\|\nabla\phi_n\|^2|v_\varepsilon - v_0|^2 + (1-\phi_n)^2|\nabla v_0|^2 + \phi_n^2|\nabla v_\varepsilon|^2\right)$$

$$\le C\mathcal{L}^N(Q\setminus Q[1-1/n]) + \overline{\lim_{\varepsilon\to 0}}\int_{Q} \Psi_n \phi_n^2 |\nabla v_\varepsilon|^2$$

But $|\nabla v_\varepsilon|^2 \mathcal{L}^N \lfloor_\Omega \rightharpoonup \mu \in \mathcal{M}(\Omega)$. so we keep estimating ...

$$\le C\mathcal{L}^N(Q\setminus Q[1-1/n]) + \int \Psi_n \phi_n^2 \, d\mu$$

$$\le C\mathcal{L}^N(Q\setminus Q[1-1/n]) + \mu(Q\setminus Q[1-1/n]) \to 0$$

as $n\to\infty$.

$$Qf(\nabla u(x_0)) \le \lim_{n\to\infty}\lim_{\varepsilon\to 0}\int_Q f(\nabla v_{\varepsilon,n}) \le \frac{d\lambda}{d\mathcal{L}^N}(x_0) + $$

Concluding the $\underline{\lim}$ bdd.