Partial Differential Equations and Functional Analysis

Winter 2017/18
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Problem Sheet 9.

Due in class, Friday, December 15, 2017.

Problem 1. (1+2+1+1 points)

Let l_∞ be the space of bounded sequences $x : \mathbb{N} \to \mathbb{R}$. Let $\tau : l_\infty(\mathbb{R}) \to l_\infty(\mathbb{R})$ defined by $(\tau x)_n = x_{n+1}$ be the left translation operator. Let $\mathbf{1} = (1, 1, \ldots) \in l_\infty(\mathbb{R})$ be the constant sequence. Consider the subspace $A = \{x - \tau x + c\mathbf{1} \,|\, x \in l_\infty(\mathbb{R}), c \in \mathbb{R}\}$.

(i) Show that for $z = x - \tau x + c\mathbf{1} \in A$, $\|z\|_{l_\infty} \geq |c|$ and that the map $C : A \to \mathbb{R}$ mapping z to c is well-defined and linear.

Hint: Use the sum $\frac{1}{N}\sum_{i=1}^N z_i$ *for* $N \in \mathbb{N}$ *large.*

(ii) Show that there is $L \in (l_\infty(\mathbb{R}))'$ with $L = C$ on A, $\|L\| = 1$, $L\mathbf{1} = 1$, and $L \circ \tau = L$. Show that then if $x \in l_\infty(\mathbb{R})$ with $x_n \geq 0$ for every $n \in \mathbb{N}$, then $Lx \geq 0$ and that $\liminf_{n\to\infty} x_n \leq Lx \leq \limsup_{n\to\infty} x_n$ for every $x \in l_\infty(\mathbb{R})$.

Hint: For positivity write $x = c\mathbf{1} - y$ *for some* $c \in \mathbb{R}$, $y \in l_\infty(\mathbb{R})$ *with* $\|y\| \leq |c|$, *remember that for any* $\varepsilon > 0$, *there exists* $N \in \mathbb{N}$ *such that* $\liminf x - \varepsilon \leq x_n \leq \limsup x + \varepsilon$ *for all* $n \geq N$.

(iii) Show that there is no sequence $c \in l_1$ such that $L(x) = \sum_{k=0}^\infty c_k x_k$ for all $x \in l_\infty$.

Hint: Assume for the sake of contradiction that there is such a sequence c, *and compute* $\sum_{k=0}^\infty c_k e_k^{(i)}$ *and* $L(e^{(i)})$ *for standard basis vectors, i.e.,* $e_k^{(i)} = \delta_{ki}$.

(iv) Compute $L(1, 0, 1, 0, 1, 0, \ldots)$.

Problem 2. (5 points)

Let X be a real normed space. Prove that if X' is separable then X is separable.

Hint: Let $\{T_n : n \in \mathbb{N}\}$ *be dense in* X'. *Show that there exist* $x_n \in X$ *with* $\|x_n\| = 1$ *and* $T_n(x_n) \geq \frac{1}{2}\|T_n\|$, *and set* $Y := \operatorname{span}\{x_n : n \in \mathbb{N}\}$. *Assume for the sake of contradiction that* $\overline{Y} \neq X$. *Prove that then there is* $T \in X'$ *with* $\|T\| = 1$ *and* $T|_Y = 0$.

Problem 3. (2+3 points)

(i) Suppose $U \subset \mathbb{R}^n$ open, $T : C_c^\infty(U) \to \mathbb{R}$ such that $|T(f)| \leq K\|f\|_{L^2}$ for all $f \in C_c^\infty(U)$. Prove that there is one and only one $\overline{T} \in \mathcal{L}(L^2(U); \mathbb{R})$ such that $\overline{T}(f) = T(f)$ for all $f \in C_c^\infty(U)$.

(ii) Suppose a sequence $k \mapsto f_k \in W^{1,2}(U)$ and $f \in L^2(U)$ satisfy $f_k \to f$ in $L^2(U)$, and $\|\partial_i f_k\|_{L^2(U)} \leq K$, $i = 1, \ldots, n$. Prove that $f \in W^{1,2}(U)$ and $\|\partial_i f\|_{L^2(U)} \leq K$, $i = 1, \ldots, n$.

Hint: Consider $T : \phi \mapsto \int_U f \partial_i \phi \, d\mathcal{L}^n$.

Problem 4. (2+1+2 points)

Suppose $(X, \|\cdot\|)$ is a real normed space, and $M \subset X$ is closed, convex and $0 \in M^o$, i.e., there is $r > 0$ with $B(0,r) \subset M$. For $x \in M$ define

$$p(x) := \inf\{\lambda > 0 : \frac{x}{\lambda} \in M\}.$$

(i) Show that $p(x) < \infty$ for all $x \in X$, and prove that p is sublinear.

(ii) Prove that $M = p^{-1}([0,1])$.

(iii) Suppose additionally that M is bounded (i.e., there is $R > 0$ such that $M \subset B(0,R)$), and that M is symmetric with respect to 0, i.e., $x \in M \Rightarrow -x \in M$. Prove that p is a norm on X.