Partial Differential Equations and Functional Analysis

Winter 2017/18
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Problem Sheet 8.

Due in class, Friday, December 8, 2017.

Problem 1. (2+3 points)

Suppose $U \subset \mathbb{R}^n$ is open and bounded, with smooth boundary ∂U.

(i) Show that for $u \in C_c^\infty(U)$

$$\int_U |\Delta u|^2 = \sum_{i,j=1}^n \int_U |\partial_i \partial_j u|^2.$$

(ii) For $f \in L_2(U)$ consider for $u \in C^4(U) \cap C^1(\bar{U})$ the biharmonic equation

$$\begin{aligned} -\Delta^2 u &= f \text{ in } U, \\ u &= 0 \text{ on } \partial U, \\ \nu \cdot \nabla u &= 0 \text{ on } \partial U, \end{aligned}$$

where ν denotes the normal to ∂U. Determine a weak formulation of this boundary value problem in a suitable subspace of $W^{2,2}(U)$, and prove that is has a unique solution.

Problem 2. (5 points)

Let $U \subset \mathbb{R}^n$ be open, bounded with smooth boundary ∂U, and let $f \in L^2(U)$. Denote by $\bar{u} \in W_0^{1,2}(U)$ the unique solution of

$$\int_U \nabla \bar{u} \cdot \nabla \phi \, d\mathcal{L}^n - \int_U f \phi \, d\mathcal{L}^n = 0 \text{ for all } \phi \in W_0^{1,2}(U).$$

Show that for every finite dimensional subspace $V \subset W_0^{1,2}(U)$ there exists a unique $\bar{v} \in V$ such that

$$\int_U \nabla \bar{v} \cdot \nabla \psi \, d\mathcal{L}^n - \int_U f \psi \, d\mathcal{L}^n = 0 \text{ for all } \psi \in V,$$

and that

$$\|\nabla(\bar{u} - \bar{v})\|_{L^2} = \inf_{v \in V} \|\nabla(\bar{u} - v)\|_{L^2}.$$

Conclude that there is constant C that depends only on U such that for every finite dimensional $V \subset W_0^{1,2}(U)$,

$$\|\bar{u} - \bar{v}\|_{W^{1,2}(U)} \le C \inf_{v \in V} \|\bar{u} - v\|_{W^{1,2}(U)}.$$

Hint: Observe that $\int_U (\nabla \bar{u} - \nabla \bar{v}) \nabla \psi \, d\mathcal{L}^n = 0$ *for* $\psi \in V$.

Problem 3. (2+3 points)

Let $(X, \mathcal{S}, \mu)$ be a σ-finite measure space, and let $\nu : \mathcal{S} \to \mathbb{R}$ be a measure on $(X, \mathcal{S})$ with $\nu(X) < \infty$ that is absolutely continuous with respect to μ (i.e., $\mu(E) = 0 \Rightarrow \nu(E) = 0$ for all $E \in \mathcal{S}$).

(i) Prove that there is $h \in L^2(\mu + \nu)$ such that for all $g \in L^2(\mu + \nu)$

$$\int_X g \; d\nu = \int_X gh \; d(\mu + \nu).$$

Hint: Show that $|\int_X g \; d\nu| \leq \sqrt{\nu(X)} \|g\|_{L^2(\mu+\nu)}$, *and apply Riesz' representation theorem.*

(ii) Show that there is $f \in L^1(\mu)$ such that for all $E \in \mathcal{S}$

$$\nu(E) = \int_E f d\mu.$$

Hint: Use that μ *is* σ*-finite to show that* $0 \leq h < 1$ $(\mu + \nu)$*-a.e. Then, for* $E \in \mathcal{S}$ *with* $\mu(E) < \infty$ *consider* $g_k = \frac{1-h^k}{1-h} \chi_E$.

Problem 4. (2+3 points)

Consider $F(u) := \sin(u)$.

(i) Prove that $F : C([0, 1]) \to C([0, 1])$ is Frechet differentiable und compute $DF(0)$.
Hint: Consider $|\sin(u(x) + th(x)) - \sin(u(x)) - \cos(u(x))th(x)|$ *and use Taylor expansion.*

(ii) Prove that $F : L^2(0, 1) \to L^2(0, 1)$ is not Frechet differentiable.
Hint: Assume for the sake of contradiction that there is $T := DF(0)$ *(see (i) for the candidate). Construct a sequence* $k \mapsto f_k$ *of functions with* $f_k \to 0$ *and* $\lim_{k\to\infty} \frac{\|F(f_k)-T(f_k)\|_{L^2}}{\|f_k\|_{L^2}} \neq 0$.