Partial Differential Equations and Functional Analysis

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Problem Sheet 7.

Due in class, Friday, December 1, 2017.

Problem 1. (5 points)

Suppose $A \subset L^2(\mathbb{R}^n)$. For $f \in A$ denote by $\mathcal{F}(f)$ its Fourier transform. Prove that

$$A \text{ is precompact} \Leftrightarrow \begin{cases} \sup_{f\in A} \|f\|_{L^2} < \infty, \text{ and} \\ \limsup_{R\to\infty} \sup_{f\in A} \int_{\mathbb{R}^n\setminus B_R(0)} |f(x)|^2 \mathrm{d}x = 0, \text{ and} \\ \limsup_{R\to\infty} \sup_{f\in A} \int_{\mathbb{R}^n\setminus B_R(0)} |\mathcal{F}(f)(k)|^2 \mathrm{d}k = 0. \end{cases}$$

Hint: For the backward implication it is useful to show that for $\varepsilon > 0$ there is a decomposition $f = f_1 + f_2$ with $\operatorname{supp} \mathcal{F}(f_1) \subset B_R(0)$ for some $R > 0$ and $\|\mathcal{F}(f_2)\|_{L^2} < \epsilon$.

Problem 2. (1+1+3 points)

Let $\Omega \neq \emptyset$ be a set. Suppose $H \subset \{f : \Omega \to \mathbb{R}\}$ is a real Hilbert space of functions $\Omega \to \mathbb{R}$ with inner product $(\cdot,\cdot)$. We call a function $K : \Omega \times \Omega \to \mathbb{R}$ a reproducing kernel for H if

(A) $K(x,\cdot) \in H$ for all $x \in \Omega$, and

(B) $f(x) = (f, K(x,\cdot))$ for all $f \in H$ and all $x \in \Omega$.

(i) Prove that if a reproducing kernel for H exists, then it is unique.

(ii) Prove that if a reproducing kernel for H exists, then the functionals $\delta_t : H \to \mathbb{R}$, $f \mapsto f(t)$ are bounded linear functionals for all $t \in \Omega$.

Hint: Cauchy-Schwarz.

Remark: We will see soon that the converse is also true.

(iii) Consider $\Omega = \mathbb{R}$ and set $K(x,y) := \frac{1}{2}e^{-|x-y|}$.

(a) Show that $K(x,\cdot) \in W^{1,2}(\mathbb{R})$ for all $x \in \mathbb{R}$.

(b) Prove that K is reproducing kernel for $W^{1,2}(\mathbb{R})$. Precisely, for $f \in W^{1,2}(\mathbb{R})$ denote by $\bar{f}$ its continuous representative, and show that for all $x \in \mathbb{R}$

$$\bar{f}(x) = (\bar{f}, K(x,\cdot))_{W^{1,2}(\mathbb{R})} = \int_{\mathbb{R}} \bar{f}(t)K(x,t)dt + \int_{\mathbb{R}} \bar{f}'(t)\partial_2 K(x,t)dt.$$

Hint: Split the second integral into $\int_{-\infty}^{x} + \int_{x}^{\infty}$, and integrate by parts. Recall that $W^{1,2}(\mathbb{R}) = W_0^{1,2}(\mathbb{R})$ and note carefully that $\partial_2 K(x,t)$ jumps at $t = x$.

Problem 3. (5 points)

Let two functions $f : \{z \in \mathbb{C} : |z| < R\} \to \mathbb{C}$ and $g : \mathbb{C} \to \mathbb{C}$ be given by the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with convergence radius R, and the power series $g(z) = \sum_{n=0}^{\infty} b_n z^n$ with convergence radius ∞, and suppose $z = g(f(z))$ for all $z \in \mathbb{C}$ with $|z| < R$. Suppose further that X is a Banach space, and that $T \in \mathcal{L}(X)$ with $\limsup_{m\to\infty} \|T^m\|^{\frac{1}{m}} < R$. Prove that $g(f(T)) = T$.

Hint: Let P_k und Q_k be polynomials and set $R_k := P_k \circ Q_k$. Show that if $R_k(z) \to z$ uniformly on $\overline{B}(0, \rho)$ and $R_k(z) = \sum_{n=0}^{N_k} c_{k,n} z^n$, then $c_{k,1} \to 1, k \to \infty$ and $\sup_{n\geq 2} |c_{k,n}|\rho^n \leq c_k \to 0, k \to \infty$ (Cauchy's integral formula).

Remark: Using this result one can show that there is a map $\sqrt{\cdot} : \mathcal{L}(X) \supset B_1(\mathrm{Id}) \to \mathcal{L}(X)$ with the property that $\sqrt{S}^2 = S$

Problem 4. (3+2 points)

(i) Give an example of $T \in \mathcal{L}(l_2, l_2)$ with $\mathcal{R}(T) \neq l_2$ but $\overline{\mathcal{R}}(T) = l_2$.

Hint: For the latter property it suffices to show that $\mathcal{R}(T)$ contains all sequences for which only finitely many elements are not zero.

(ii) Let X be a Banach space, H a Hilbert space, and $K : X \to H$ a compact operator. Show that there is a sequence $k \mapsto K_k$ of operators $K_k : X \to H$ with finite dimensional range such that $K_k \to K$ in $\mathcal{L}(X, H)$.
Hint: For $k \in \mathbb{N}$ use that $K(B(0, 1)) \subset \bigcup_{i=1}^{N(k)} B(y_i, \frac{1}{k})$ and set $H_k := \mathrm{span}\{y_1, \ldots, y_{N(k)}\}$. Use the orthogonal projection $P_k : H \to H_k$.