Partial Differential Equations and Functional Analysis

Winter 2017/18
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Problem Sheet 5.

Due in class, Friday, November 17, 2017.

Problem 1. (2+1+2 points)

Suppose $U \subset \mathbb{R}^n$ is open.

(i) Prove that if $u \in W^{1,1}(U)$ then $u^+ \in W^{1,1}(U)$ where $u^+(x) := \max\{u(x),\ 0\}$, and show that for all $i = 1, \dots, n$ and a.e. $x \in U$

$$\partial_i u^+(x) = \begin{cases} \partial_i u(x) & \text{if } u(x) > 0 \\ 0 & \text{else.} \end{cases}$$

Hint: Lecture notes.

(ii) Suppose $u,\ v \in W^{1,1}(U)$. Prove that also the pointwise maximum $\max(u, v) \in W^{1,1}(U)$ and the pointwise minimum $\min(u, v) \in W^{1.1}(U)$, where $\max(u, v)(x) := \max\{u(x), v(x)\}$ and $\min(u, v)(x) := \min\{u(x), v(x)\}$. Compute the weak derivatives.

(iii) Suppose $E \subset U$ is measurable, $u \in W^{1,1}(U)$ and $u = 0$ a.e. on E. Prove that $\partial_i u = 0$ a.e. on E for $i \in \{1, \dots, n\}$.

Problem 2. (2+1+2 points)

(i) Let $p \in [1, \infty)$ and $k \in \mathbb{N}$. Prove that $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$, i.e. $W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$.

Hint: For $f \in W^{k,p}(\mathbb{R}^n)$ consider first $f_j(x) = \varphi(\frac{x}{j})f(x)$ with $\varphi \in C_c^\infty(B(0,1))$ and $\varphi = 1$ on $B(0, \frac{1}{2})$.

Define the Fourier transform of a function $f \in L^1(\mathbb{R}^n, \mathbb{C})$ as the function $\mathcal{F}(f) : \mathbb{R}^n \to \mathbb{C}$ given by

$$\mathcal{F}(f)(k) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ik\cdot x} \,\mathrm{d}x.$$

For functions $f, g \in L^1(\mathbb{R}^n, \mathbb{C}) \cap L^2(\mathbb{R}^n, \mathbb{C})$ we have $(f, g)_{L^2} = (\mathcal{F}(f), \mathcal{F}(g))_{L^2}$ (Parseval's identity). By density $\mathcal{F}$ can be extended to an isometry $\mathcal{F} : L^2(\mathbb{R}^n, \mathbb{C}) \to L^2(\mathbb{R}^n, \mathbb{C})$ and Parseval's identity still holds.

(ii) Suppose $f \in W^{1,2}(\mathbb{R}^n, \mathbb{C})$. Prove that for $j = 1, \dots, n$: $(\mathcal{F}(\frac{\partial}{\partial x_j} f))(k) = ik_j(\mathcal{F}(f))(k)$ for a.e. $k \in \mathbb{R}^n$.

(iii) Suppose $f \in L^2(\mathbb{R}^n, \mathbb{C})$. Prove that $f \in W^{1,2}(\mathbb{R}^n, \mathbb{C})$ if and only if the function $k \mapsto |k|(\mathcal{F}(f))(k)$ is in $L^2(\mathbb{R}^n, \mathbb{C})$. Show that for all $f \in W^{1,2}(\mathbb{R}^n, \mathbb{C})$

$$\|f\|^2_{W^{1,2}(\mathbb{R}^n,\mathbb{C})} = \int_{\mathbb{R}^n} \left(1 + |y|^2\right) |\mathcal{F}(f)(y)|^2 \,dy.$$

Definition: A normed space $(X, \|\cdot\|)$ is called uniformly convex if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|x\| = \|y\| = 1$ and $\left\|\frac{1}{2}(x+y)\right\| > 1 - \delta$ implies $\|x - y\| < \varepsilon$.

Problem 3. (3+2 points)

Let $2 \leq p < \infty$.

(i) Prove that for $x,\, y \in \mathbb{R}$

$$|x+y|^p + |x-y|^p \leq 2^{p-1}\left(|x|^p + |y|^p\right).$$

Hint: Show that $(|a|^p + |b|^p)^{1/p} \leq (|a|^2 + |b|^2)^{1/2}$ *for* $a,\, b \in \mathbb{R}$, *and apply this with* $a := x + y$ *and* $b := x - y$. *To show the inequality for* a, b *you can assume* $|a| = 1$ *(why?) and consider the pth power of the inequality for* $c = |b|^2$.

(ii) Suppose $U \subset \mathbb{R}^n$ open. Prove that $L^p(U)$ is uniformly convex.

Problem 4. (1+1+2+1 points)

Let $1 < p < 2$.

(i) Let $h : \mathbb{R} \to \mathbb{R}, h(x) = |x|^p$. Consider $g(t) = h(1+t) + h(1-t) - 2h(1)$ and show that

$$g(t) = \int_0^t (t-s)\left(h''(1+s) + h''(1-s)\right)\mathrm{d}s.$$

Hint: Consider first $h_\epsilon(x) := (|x|^2 + \epsilon^2)^{p/2}$.

(ii) Show that there exist constants $c_1, c_2 > 0$ such that

$$g(t) \geq c_1 t^2 \text{ for } 0 \leq t \leq 2$$
$$g(t) \geq c_2 t^p \text{ for } t \geq 2$$

(iii) Show that there exists $\alpha > 0$ s.t. for all $x, y \in \mathbb{R}$

$$(|x|^p + |y|^p)^{1-\frac{p}{2}}\left(h(x) + h(y) - 2h\left(\frac{x+y}{2}\right)\right)^{p/2} \geq \alpha|x-y|^p.$$

Hint: It suffices to consider the case $\left|\frac{x+y}{2}\right| = 1$ *(why?).*

(iv) Suppose $U \subset \mathbb{R}^n$ open. Prove that $L^p(U)$ is uniformly convex.