Partial Differential Equations and Functional Analysis

Winter 2017/18
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Problem Sheet 3.

Due in class, Friday, November 3, 2017.

Problem 1. (2+3 points)

Let $\Omega \subset \mathbb{R}^n$ be bounded, open and connected, and let α, $\beta \in (0,1]$.

(i) Determine the maximal γ such that $f \in C^{0,\alpha}(\Omega)$ and $g \in C^{0,\beta}(\Omega)$ implies $fg \in C^{0,\gamma}(\Omega)$. Prove your statement.

(ii) Determine the maximal δ such that $f \in C^{0,\alpha}(\mathbb{R})$ and $g \in C^{0,\beta}(\Omega)$ implies $f \circ g \in C^{0,\delta}(\Omega)$. Prove your statement.

Problem 2. (3+2+5∗ points)

Let $A \subset \mathbb{R}^n$ and $\alpha \in (0,1]$.

(i) Show that $(C^{0,\alpha}(A), \|\cdot\|_\alpha)$ is a Banach space.

(ii) Find $u \in C^0([0,1])$ with $[u]_\alpha = \infty$ for all $\alpha > 0$.

(iii∗) Show that $C^{0,\alpha}([0,1])$ is not separable.
Hint: It suffices to construct an uncountable subset $A \subset C^{0,\alpha}([0,1])$ such that for $f, g \in A$ with $f \neq g$: $[f-g]_\alpha \geq 1$. Consider the function

$$\phi(x) = \begin{cases} 1 - 4\left|x - \frac{1}{2}\right|, & x \in [\frac{1}{4}, \frac{3}{4}]; \\ 0, & \text{else.} \end{cases}$$

Construct $\phi_k(x) = 4^{-k\alpha}\phi(4^k x)$ and consider the set $A = \left\{\sum_{k\in\mathbb{N}} a_k \phi_k | a : \mathbb{N} \to \{0,1\}\right\}$.

Problem 3. (2+3 points)

Suppose $E \subset \mathbb{R}^n$ is Lebesgue measurable with $\mathcal{L}^n(E) < \infty$. Define $d : L^1(E) \times L^1(E) \to \mathbb{R}$,

$$d(f,g) := \int_E \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} d\mathcal{L}^n \ .$$

(i) Prove that d is a metric on $L^1(E)$.

(ii) We say that a sequence of measurable functions $f_k : E \to \mathbb{R}$ converges in measure to a measurable function f (and write $f_k \to f$ in measure) if for all $\varepsilon > 0$

$$\lim_{k\to\infty} \mathcal{L}^n\left(\{x \in E \ : \ |f_k(x) - f(x)| > \varepsilon\}\right) = 0.$$

Prove that $\lim_{k\to\infty} d(f_k, f) = 0$ if and only if $f_k \to f$ in measure.

Problem 4. (2+3 points)

Suppose $\Omega \subset \mathbb{R}^n$ is such that $0 < |\Omega| = \int_\Omega 1 \mathrm{d}x < \infty$. Let $u : \Omega \to \mathbb{R}$ be measurable. Define $\Phi_u : [1, \infty) \to [0, \infty]$ by

$$\Phi_u(p) = \left(\frac{1}{|\Omega|} \right)^{\frac{1}{p}} \|u\|_{L^p(\Omega)} .$$

(i) Prove that Φ_u is nondecreasing (in particular for $p < q$ if $u \in L^q(\Omega)$ then $u \in L^p(\Omega)$).

(ii) Prove that $u \in L^\infty(\Omega)$ if and only if the limit $\lim_{p\to\infty} \Phi_u(p)$ is finite and that in this case $\lim_{p\to\infty} \Phi_u(p) = \|u\|_{L^\infty(\Omega)}$.