# Partial Differential Equations and Functional Analysis

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## Problem sheet 14

**These problems are for your preparation for the exam, they need not be handed in.**

**Problem 1.** Let $X$ and $Y$ be Banach spaces and $T \in \mathcal{L}(X, Y)$. We say that $T$ is *completely continuous* if it has the following property:

$$\text{If } x_k \rightharpoonup x_* \text{ then } Tx_k \to Tx_*.$$

(i) Show that $x_k \rightharpoonup x_*$ in $X$ implies $Tx_k \rightharpoonup Tx_*$ in $Y$.

(ii) Show that if $T$ is a compact operator then it is completely continuous.

(iii) Assume additionally that $X$ is reflexive and show that then $T$ is completely continuous if and only if $T$ is a compact operator.

**Problem 2.** Define $f_k \in L^\infty(\mathbb{R})$ by

$$f_k(x) = \sin(kx).$$

Show that $f_k \overset{*}{\rightharpoonup} 0$ but $f_k^2 \overset{*}{\rightharpoonup} \frac{1}{2}$.

**Problem 3.** Let $U \subset \mathbb{R}^3$ be open and bounded and let $f \in L^{\frac{6}{5}}(U)$. Show that there is one and only one $u \in W_0^{1,2}(U)$ solving

$$\int_U \nabla u \cdot \nabla \varphi - \int_U f\varphi = 0 \quad \forall \varphi \in W_0^{1,2}(U).$$

**Problem 4.** Let $U \subset \mathbb{R}^n$ be open and bounded and let $f \in L^2(U)$. For $u \in W_0^{1,2}(U)$ define

$$I(u) = \int_U |\nabla u|^2 + \sin(u) - fu.$$

(i) Show that $\inf_{u \in W_0^{1,2}(U)} I(u) > -\infty$.

(ii) Take a sequence $(u_k)_k \subset W_0^{1,2}(U)$ such that

$$\lim_{k\to\infty} I(u_k) = \inf_{u \in W_0^{1,2}(U)} I(u).$$

Prove that there is a subsequence $(u_{k_l})_l$ and $u_* \in W_0^{1,2}(U)$ such that $u_{k_l} \rightharpoonup u_*$ in $W_0^{1,2}(U)$ and $u_{k_l} \to u_*$ a.e. in $U$.

(iii) Prove that $I(u_*) = \inf_{u \in W_0^{1,2}(U)} I(u)$.

**Problem 5.** Let $H$ be a Hilbert space and $T : H \to H$ linear such that

$$(Tx, y) = (x, Ty) \text{ for all } x, y \in H.$$

Prove that $T$ is bounded.