Partial Differential Equations and Functional Analysis

Winter 2017/18
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Problem Sheet 12.

Due in class, Friday, January 19, 2018.

Problem 1. (3+2 points)

(i) Let $1 < p < \infty$, $U \subset \mathbb{R}^n$ open, and let $\mathcal{W}$ be the set of all cubes that are contained in U. Prove that

$$f_k \rightharpoonup f \text{ in } L^p(U) \quad \Longleftrightarrow \quad \sup_k \|f_k\|_{L^p(U)} < \infty \text{ and } \int_W f_k \, d\mathcal{L}^n \to \int_W f \, d\mathcal{L}^n \quad \forall W \in \mathcal{W}.$$

(ii) Let $1 < p < \infty$ and let $h \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ be $\mathbb{Z}^n$-periodic, i.e. $h(x+z) = h(x)$ for all $x \in \mathbb{R}^n$ and all $z \in \mathbb{Z}^n$. Define $f_k : (0,1)^n \to \mathbb{R}$ by $f_k(x) := h(kx)$. Prove that $f_k \rightharpoonup C$ in $L^p((0,1)^n)$ where the constant C is given by $C = \int_{(0,1)^n} h \, d\mathcal{L}^n$.

Problem 2. (1+2+2 points)

Suppose $1 < p < \infty$, $U \subset \mathbb{R}^n$ open and bounded, and let $f : \mathbb{R}^n \times U \to [0, \infty)$ be continuous.

(i) Prove that

$$u_k \to u \text{ in } W^{1,p}(U) \implies \int_U f(\nabla u(x), x) dx \leq \liminf_k \int_U f(\nabla u_k(x), x) dx.$$

Hint: Fatou.

(ii) Suppose additionally that $F \mapsto f(F, x)$ is convex for all $x \in U$. Prove that

$$u_k \rightharpoonup u \text{ (weakly) in } W^{1,p}(U) \implies \int_U f(\nabla u(x), x) dx \leq \liminf_k \int_U f(\nabla u_k(x), x) dx.$$

Hint: Mazur's lemma and corollary. You may assume without loss of generality that $\ell := \lim_k \int_U f(\nabla u_k(x), x) dx$ *exists. (Otherwise first choose a subsequence.)*

(iii) Under the assumptions of (ii) suppose additionally that $f(F, x) \geq C|F|^p$ for some $C > 0$. Let $g \in L^{p'}(U)$. Show that there exists $u \in W^{1,p}_0(U)$ such that

$$\int_U (f(\nabla u(x), x) - g(x)u(x)) dx \leq \int_U (f(\nabla v(x), x) - g(x)v(x)) dx \quad \forall v \in W^{1,p}_0(U).$$

Problem 3. (1+3+1+2∗+3∗ points)

Let X be a separable real Banach space, and let $b : \mathbb{N} \to X$ be such that $\|b_k\|_X = 1$ for all $k \in \mathbb{N}$, and $\mathrm{span}(b(\mathbb{N}))$ dense in X. Define $\|\cdot\|_b : X' \to \mathbb{R}$ by

$$\|x'\|_b := \sum_{k=0}^{\infty} \frac{1}{2^k} |x'(b_k)|.$$

(i) Prove that $\|\cdot\|_b$ defines a norm on X'.

(ii) Let $f_* \in X'$ and let $f : \mathbb{N} \to X'$ be a bounded sequence, i.e. $\sup_j \|f_j\|_{X'} \le M < \infty$. Prove that $f_j \overset{*}{\rightharpoonup} f_*$ if and only if $\|f_j - f_*\|_b \to 0$.

(iii) Show that the closed unit ball $B := \{x' \in X' \ : \ \|x'\|_{X'} \le 1\}$ is compact in $(X', \|\cdot\|_b)$.

Consider the special case $X := l_1$. Note that we may choose $b_k := e_k$, where e_k denotes the k-th unit vector.

(iv∗) Let $x : \mathbb{N} \to l_1$ be a sequence with $x^{(k)} \rightharpoonup 0$ in l_1. For $\varepsilon > 0$ and $K \ge 1$ set

$$A_K := \{x' \in l_\infty \ : \ \|x'\|_{l_\infty} \le 1 \text{ and } |x'(x^{(k)})| \le \varepsilon \ \forall k \ge K\}.$$

Prove that A_K is weakly-∗ closed in l_∞.

(v∗) Prove that a sequence in l_1 converges weakly if and only if it converges strongly.

Hint: Apply Baire to $(B, \|\cdot\|_b)$ in order to see that one of the A_K has nonempty interior. For $B_\delta(x_0') \subset A_K$ find the right direction $x' - x_0'$ to estimate $\sum_{j=N}^{\infty} \left| x_j^{(k)} \right|$.

Note, however, that the weak topology and the strong topology are different. Indeed, by the remark after Proposition 8.6. every non-empty, weakly open set contains a line. Thus the ball $B := \{x \in l_1 : \|x\| < 1\}$ is strongly open, but not weakly open.

Problem 4. (3+2+5∗ points)

Let $E \subset L^2(-\pi, \pi)$ be the set of all functions

$$f_{m,n}(t) = e^{imt} + m e^{int},$$

where m, n are integers and $0 \le m < n$. Let E_1 be the set of all $g \in L^2(-\pi, \pi)$ such that some sequence in E converges weakly to g. (E_1 is called the *weak sequential closure* of E.)

(i) Find explicit representations for all $g \in E_1$.

Hint: Prove first, that $(e^{imt}, e^{int}) = 2\pi\delta_{nm}$. Then consider $\|f_{m,n}\|$ and note that weakly convergent sequences are bounded. What is the weak limit of $f_n(t) = e^{int}$ for $n \to \infty$? (see problem 1)

(ii) Show that $E_1 \cup \{0\}$ is weakly sequentially closed (i.e. it is its own weak sequential closure) but E_1 is not.

(iii∗) Show that $E_1 \cup \{0\}$ is weakly closed.