Partial Differential Equations and Functional Analysis

Winter 2017/18
Prof. Dr. Stefan Müller
Richard Schubert

Problem Sheet 11.

Due in class, Friday, January 12, 2018.

Problem 1. (3+2 points)

Let X be an infinite dimensional normed $\mathbb{R}$ vector space. One can construct a linear map $S : X \to \mathbb{R}$ which is not bounded. To do this, one uses the fact, that every vector space has a basis $\mathcal{B}$, and extracts a countable collection of basis vectors $\mathcal{C} = \{e_i : i \in \mathbb{N}\} \subset \mathcal{B}$. Then one defines $S(e_i) = i\,\|e_i\|$ for $e_i \in \mathcal{C}$ and $S(b) = 0$ for $b \in \mathcal{B} \setminus \mathcal{C}$. By the the property of the basis this defines a linear map uniquely on the whole space X (recall the construction after Lemma 4.1 in the lecture).

(i) Let X be a Banach space and let $S : X \to \mathbb{R}$ be linear and not bounded. Let $Y = \operatorname{graph} S \subset X \times \mathbb{R}$. Show that the map $T : Y \to X$ given by $T((x, Sx)) = x$ is bijective and continuous, but not open.

(ii) Let X be a Banach space and let $S : X \to \mathbb{R}$ be linear and not bounded. Let $Y = \operatorname{graph} S$ and define $F : X \to Y$ by $Fx = (x, Sx)$. Show that F is closed, but not continuous.

Problem 2. (1+1+1+2 points)

We say that a sequence x_n in a Banach space X converges *weakly* to x (notation: $x_n \rightharpoonup x$) if $x'(x_n) \to x'(x)$ for all x' in the dual space X'.

(i) Let X and Y be Banach spaces, and consider the Banach space $Z = X \times Y$ with norm $\|(x, y)\|_Z = \|x\|_X + \|y\|_Y$. Prove that for every $z' \in Z'$ there is one and only one $x' \in X'$ and one and only one $y' \in Y'$ such that $z'((x, y)) = x'(x) + y'(y)$ for all $x \in X$ and all $y \in Y$. Deduce that

$$(x_n, y_n) \rightharpoonup (x, y) \text{ in } Z \iff x_n \rightharpoonup x \text{ in } X, \text{ and } y_n \rightharpoonup y \text{ in } Y.$$

(ii) Let X be a Banach space, $Y \subset X$ a closed subspace, and $k \mapsto y_k \in Y$ a sequence. Prove

$$y_k \rightharpoonup y \text{ in } Y \iff y_k \rightharpoonup y \text{ in } X.$$

(iii) Let X and Y be Banach spaces, and let $T \in \mathcal{L}(X, Y)$ be invertible. Prove that

$$x_k \rightharpoonup x \text{ in } X \iff Tx_k \rightharpoonup Tx \text{ in } Y.$$

(iv) Let $1 < p < \infty$, and $U \subset \mathbb{R}^n$ open. Prove that

$$f_k \rightharpoonup f \text{ in } W^{1,p}(U) \Leftrightarrow f_k \rightharpoonup f \text{ in } L^p(U) \text{ and } \partial_i f_k \rightharpoonup \partial_i f \text{ in } L^p(U),\ i \in \{1, \dots, n\}.$$

Hint: Consider a suitable embedding of $W^{1,p}(U)$ into a subspace of $L^p(U) \times \cdots \times L^p(U)$ ($n+1$ copies), and use (i)-(iii).

Problem 3. (2+2+1 points)

(i) Give an example of a sequence $k \mapsto f_k \in L^2([0,1])$ such that $f_k \to 0$ almost everywhere, $f_k \rightharpoonup 0$ weakly in $L^2([0,1])$, but $f_k \nrightarrow 0$ strongly in $L^2([0,1])$.

(ii) Consider a sequence $k \mapsto f_k \in L^\infty([0,1])$ such that there exists $f \in L^\infty([0,1])$ with $f_k \to f$ almost everywhere and $\|f_k\|_{L^\infty} \leq M$ for all $k \in \mathbb{N}$. Let $h \in L^2([0,1])$. Show that $f_k h \to fh$ in $L^2([0,1])$.

(iii) Consider sequences $k \mapsto f_k \in L^\infty([0,1])$ and $k \mapsto g_k \in L^2([0,1])$ such that there are f, g with $f_k \to f$ almost everywhere, $\|f_k\|_{L^\infty} \leq M$ for all $k \in \mathbb{N}$, and $g_k \rightharpoonup g$ weakly in $L^2([0,1])$. Prove that $f_k g_k \rightharpoonup fg$ in $L^2([0,1])$.

Problem 4. (5 points)

Let X be a set and let $\mathcal{S} \subset 2^X$ and suppose that $\bigcup_{W \in \mathcal{S}} W = X$. Let $\mathcal{B}$ denote the collection of sets obtained by taking finite intersections of sets in $\mathcal{S}$ and let $\mathcal{T}$ denote the collection of sets formed by (arbitrary) union of sets in $\mathcal{B}$. More formally:

$$\mathcal{B} := \left\{ \bigcap_{i=1}^{k} W_i : k \in \mathbb{N} \setminus \{0\}, W_i \in \mathcal{S} \forall i \in \{1, \dots, k\} \right\},$$

$$\mathcal{T} := \left\{ \bigcup_{\alpha \in A} V_\alpha : V_\alpha \in \mathcal{B} \forall \alpha \in A \right\}.$$

Show that $\mathcal{T}$ is the coarsest topology containing $\mathcal{S}$.