Summary of the course

Functional analysis and partial differential equations

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This is only a summary of the main results and arguments discussed in class and *not* a complete set of lecture notes. These notes can thus not replace the careful study of the literature. As discussed in class, among other the following two books are recommended:

Al H.W. Alt, Linear functional analysis, Springer.

Br H. Brezis, Functional analysis, Sobolev spaces and partial differential equations, Springer.

These notes are based on the books mentioned above and further sources which are not always mentioned specifically (see that notes at the end for further discussion and recommendation for further reading). These notes are only for the use of the students in the class 'Functional analysis and partial differential equations' at Bonn University, Fall term 2016–2017.

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Contents

1 Structures

We consider the following increasingly richer structures on a set or a vector space.

- Neighbourhoods and convergence (topological spaces)
- Distance (metric spaces)
- Length in a vector space (normed spaces, Banach spaces)
- Length and angle/ scalar product in a vector space (Pre-Hilbert spaces, Hilbert spaces)

In these notes **all vector spaces will be vector spaces of the fields**

$$\mathbb{K} = \mathbb{R} \quad \text{or } \mathbb{K} = \mathbb{C}. \tag{1.1}$$

1.1 Topological spaces

Let X be a set. Then 2^X denotes the set of all subsets of X (including the empty set).

Definition 1.1. *A topological space $(X, \mathcal{T})$ is pair consisting of a set X and a subset $\mathcal{T}$ of 2^X with the following properties.*

(i) $\emptyset \in \mathcal{T}$, $X \in \mathcal{T}$.

(ii) If $U \in \mathcal{T}$, $V \in \mathcal{T}$ *then* $U \cap V \in \mathcal{T}$.

(iii) If Λ *is an arbitrary set and* $U_\lambda \in \mathcal{T}$ *for all* $\lambda \in \Lambda$ *then* $\bigcup_{\lambda \in \Lambda} U_\lambda \in \mathcal{T}$.

The set $\mathcal{T}$ is called a topology on X. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X then $\mathcal{T}_1$ is called *finer* (or *stronger*) than $\mathcal{T}_2$ if $\mathcal{T}_1 \supset \mathcal{T}_2$. In this case $\mathcal{T}_2$ is called *coarser* (or *weaker*) than $\mathcal{T}_1$.

A set $A \subset X$ is called *open* if $A \in \mathcal{T}$. It is called *closed* if the complement $A^c := X \setminus A$ is open. By definition of a topological space a finite intersection and an arbitrary union of open sets is open. It follows from the formula $(\bigcap_{\lambda \in \Lambda} A_\lambda)^c = \bigcup_{\lambda \in \Lambda} A_\lambda^c$ that a finite union and an arbitrary intersection of closed sets is closed.

Example 1.2. *(i) (finest topology)* $\mathcal{T} = 2^X$ *is a topology on* X.

(ii) (coarsest topology) $\mathcal{T} = \{\emptyset, X\}$ *is a topology on* X.

(iii) *(standard topology on $\mathbb{R}$) Let $X = \mathbb{R}$. The standard topology on $\mathbb{R}$ is defined by $U \in \mathcal{T}_{st}$ if and only if for each $x \in U$ there exists an $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset U$.* **If nothing else is said we consider in the following the standard topology on $\mathbb{R}$.**

(iv) *Let $X = \mathbb{R}$. Then*

$$\mathcal{T} = \{U \subset \mathbb{R} : U = \emptyset \text{ or } \mathbb{R} \setminus U \text{ countable }\} \tag{1.2}$$

is a topology on X (exercise).
Here we say that a set A is 'countable' if A is empty or finite or if there exists a bijective map $j : \mathbb{N} \to A$.

(v) *(relative topology) If $A \subset X$, if $\mathcal{T}$ is a topology on X and if $\mathcal{T}_A := \{U \cap A : U \in \mathcal{T}\}$ then $\mathcal{T}_A$ is a topology on A.*

(vi) *(intersection of topologies) If B is a set and $\mathcal{T}_\beta$ is a topology on X for all $\beta \in B$ then $\bigcap_{\beta \in B} \mathcal{T}_\beta$ is a topology on X (exercise).*
In particular given any set $\mathcal{S} \subset 2^X$ then

$$\mathcal{T} := \bigcap_{\mathcal{U} \text{ topology on } X,\, \mathcal{U} \supset \mathcal{S}} \mathcal{U} \tag{1.3}$$

is a topology on X, namely the coarsest topology which contains $\mathcal{S}$.

(vii) *(product topology) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, let*

$$\mathcal{S} := \{U \times Y : U \in \mathcal{T}_X\} \cup \{X \times V : V \in \mathcal{T}_Y\} \tag{1.4}$$

and let $\mathcal{T}$ be the coarsest topology on $X \times Y$ which contains $\mathcal{S}$. In particular $\mathcal{T}$ contains all sets of the form $U \times V$, with $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$.

Definition 1.3. *Let $(X, \mathcal{T})$ be a topological space and let $A \subset X$. The interior A^o, the closure $\overline{A}$ and the boundary ∂A are defined by*

$$A^o := \bigcup_{U \subset A, U \in \mathcal{T}} U, \qquad \overline{A} := \bigcap_{K \supset A, K^c \in \mathcal{T}} K, \qquad \partial A := \overline{A} \setminus A^o. \tag{1.5}$$

By the definition of a topology A^o is open, in fact A^o is the largest open set contained in A. Similarly $\overline{A}$ is closed and is the smallest closed set which contains A. The boundary is also closed since it is the intersection of $\overline{A}$ and $X \setminus A^o$.

Definition 1.4. *Let $(X, \mathcal{T})$ be a topological space. A set $A \subset X$ is called dense, if $\overline{A} = X$. The space X is separable if there exists a countable dense subset.*

Definition 1.5. *Let $(X, \mathcal{T})$ be a topological space and let $x \in X$. A set $U \subset X$ is an open neighbourhood of x if $U \in \mathcal{T}$ and $x \in U$.*

Definition 1.6 (Continuous maps)**.** *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let f be a map from X to Y. Then f is continuous if*

$$V \in \mathcal{T}_Y \quad \Longrightarrow \quad f^{-1}(V) \in \mathcal{T}_X, \tag{1.6}$$

i.e. if the preimage of every open set is open.
The map f is continuous at a point $x \in X$ if for every open neighbourhood V of $f(x) \in Y$ there exists an open neighbourhood U of $x \in X$ such that $f(U) \subset V$.
The map f is called a homeomorphism if f is bijective and f and f^{-1} are continuous.

Notation $f^{-1}(V) := \{x \in X : f(x) \in V\}$. Note the properties

$$\begin{aligned} &f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B), \quad f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B), \\ &f^{-1}(Y \setminus A) = X \setminus f^{-1}(A). \end{aligned} \tag{1.7}$$

Remark. It follows directly from the definition that the composition of continuous maps is continuous: if f_1 is continuous from $(X_1, \mathcal{T}_1)$ to $(X_2, \mathcal{T}_2)$ and f_2 is continuous from $(X_2, \mathcal{T}_2)$ to $(X_3, \mathcal{T}_3)$ then $f_2 \circ f_1$ is continuous from $(X_1, \mathcal{T}_1)$ to $(X_3, \mathcal{T}_3)$.

Remark. The map f is continuous if and only if it is continuous at every $x \in X$ (exercise).

If $\mathcal{T}_X = 2^X$ then every map $f : X \to Y$ is continuous.
If $\mathcal{T}_X = \{\emptyset, X\}$ then the constant map is continuous. If, in addition, $(Y, \mathcal{T}_Y)$ is a Hausdorff space (see below) then the constant map is the only continuous map.
If f is a continuous map from $(X, \mathcal{T}_X)$ to $(Y, \mathcal{T}_Y)$, if $A \subset X$ and if $\mathcal{T}_A$ is the relative topology on A (see Example 1.2 (v)), the $f_{|A}$ is a continuous map from $(A, \mathcal{T}_A)$ to $(Y, \mathcal{T}_Y)$.
The finer the topology on X the more continuous maps exist.

Definition 1.7 (convergence)**.** *Let $(X, \mathcal{T})$ be a topological space. We say that a sequence $x : \mathbb{N} \to X$ converges to x^* (notation: $x_k \to x^*$) if and only if for every open neighbourhood U of x^* the set $\{k \in \mathbb{N} : x_k \in X \setminus U\}$ is finite. The point x^* is called a limit point of the sequence.*

Examples. If $\mathcal{T}$ is the standard topology on $\mathbb{R}$ this agrees with the definition of convergence in Analysis 1.
If $\mathcal{T} = 2^X$ then only sequences which are constant, up to finitely many

terms, are convergent.
If $\mathcal{T} = \{\emptyset, X\}$ then every sequence is convergent and every x^* is a limit point.

Definition 1.8 (Hausdorff space)**.** *A topological space $(X, \mathcal{T})$ is called Hausdorff if sets consisting of a single point are closed and for different points there exist disjoint open neighbourhoods, i.e.,*

$$x \neq y \quad \Longrightarrow \exists U_x, U_y \in \mathcal{T} \text{ such that } x \in U_x, y \in U_y \text{ and } U_x \cap U_y = \emptyset. \tag{1.8}$$

An immediate consequence is

Proposition 1.9. *If $(X, \mathcal{T})$ is a Hausdorff space then every sequence has at most one limit point.*

Notation If $(X, \mathcal{T})$ is a Hausdorff space and the sequence $x : \mathbb{N} \to X$ converges to x^* we write $x^* = \lim_{k\to\infty} x_k$.

[11.10. 2017, Lecture 1]

[13.10. 2017, Lecture 2]

Definition 1.10 (Compactness)**.** *A subset K of a toplogical space is compact if every cover of K by open sets contains a finite subcover. The space $(X, \mathcal{T})$ is called a compact topological space if X is compact.*

More explicitely this definition reads as follows: if Λ is any index set, if $U_\lambda \in \mathcal{T}$ for each $\lambda \in \Lambda$ and $K \subset \cup_{\lambda\in\Lambda} U_\lambda$ then there exists a finite subset $\Lambda' \subset \Lambda$ such that $K \subset \cup_{\lambda\in\Lambda'} U_\lambda$.

Lemma 1.11. *Assume that $f : (X_1, \mathcal{T}_1) \to (X_2, \mathcal{T}_2)$ is continuous and $K \subset X_1$ is compact. Then $f(K) \subset X_2$ is compact.*

Proof. This follows directly from the definitions of continuity and compactness (exercise). □

Theorem 1.12. *Let $f : (X, \mathcal{T}) \to \mathbb{R}$ be continuous and $K \subset X$ be compact. Then f attains its maximum and minimum on K, i.e. there exist $a, b \in K$ such*

$$f(a) \leq f(x) \leq f(b) \quad \forall x \in K.$$

Proof. Exercise. One possibility is to show that every compact set $K' \subset \mathbb{R}$ is bounded and the infimum $m = \inf\{x : x \in K'\}$ and the supremum $M = \sup\{x : x \in K'\}$ belong to K'. □

Definition 1.13 (Connectedness)**.** *Let $(X, \mathcal{T})$ be a topological space. Then X is connected if X cannot be written as a non-trivial disjoint union of two open sets, i.e., if*

$$U \in \mathcal{T} \text{ and } X \setminus U \in \mathcal{T} \quad \Longrightarrow U = \emptyset \text{ or } U = X. \tag{1.9}$$

A subset $A \subset X$ is connected if the topological space $(A, \mathcal{T}_A)$ is connected.

Theorem 1.14 (Intermediate value theorem). *Let $(X, \mathcal{T})$ be a connected topological space and let $f : X \to \mathbb{R}$ be continuous (where $\mathbb{R}$ is equipped with the standard topology). Suppose that there exist $x, y \in X$ with $f(x) < 0 < f(y)$. Then there exists $z \in X$ with $f(z) = 0$.*

Proof. Assume that $0 \notin f(X)$ and set $U = f^{-1}(0, \infty)$. By the continuity of f the set U is open and by assumption U is not empty. Moreover

$$X \setminus U = f^{-1}(\mathbb{R} \setminus (0, \infty)) = f^{-1}((-\infty, 0)) \tag{1.10}$$

is also open and not empty. This contradicts the assumption that X is connected. □

1.2 Metric spaces

1.2.1 Definition and examples

Definition 1.15. *A pair (X, d) is called a metric space if $d : X \times X \to [0, \infty)$ has the following properties.*

(i) (definiteness) $d(x, y) = 0 \Leftrightarrow x = y$

(ii) (symmetry) $d(y, x) = d(x, y)\ \forall x, y \in X$

(iii) (triangle inequality) $d(x, y) \le d(x, z) + d(z, y)\ \forall x, y, z \in X$

A map $d : X \times X \to [0, \infty)$ with the above properties is called a metric on X.

If $A \subset X$ and if d_A denotes the restriction of d to $A \times A$ then (A, d_A) is again a metric space.

If $d : X \times X \to [0, \infty)$ satisfies only (ii), (iii) and $d(x, x) = 0$ for all $x \in X$, then d is called a semimetric (or pseudometric). If d is a semimetric one can define an equivalence relation by $x \sim y$ if and only if $d(x, y) = 0$. The quotient space $\tilde{X} := X/\sim$ consists of equivalence classes $[x] := \{z \in X : x \sim z\} = \{z \in X : d(z, x) = 0\}$. It follows from the triangle inequality that the expression $\tilde{d}([x], [y]) := d(x, y)$ is well-defined. Moreover one easily sees that $\tilde{d}$ is a metric (not just a semimetric) on $\tilde{X}$. In this way one can improve every semimetric to a metric by passing to equivalence classes. This idea was e.g. used in the definition of the Lebesgue spaces $L^p(X, \mu)$ which consist of equivalence classes of functions that only differ on sets of measure zero.

The notion of metric is very flexible.

Lemma 1.16. *Assume that $\psi : [0, \infty) \to [0, \infty)$ is continuous, nondecreasing, concave and satisfies $\psi(0) = 0$, $\psi(x) > 0$ for $x > 0$. If d is a metric, then $\psi \circ d$ is also a metric. If d is a semimetric, then $\psi \circ d$ is a semimetric.*

Proof. We only need to verify the triangle inequality. This follows from the estimate $\psi(a+b) \le \psi(a) + \psi(b)$ for all $a, b > 0$ whose proof is left as an exercise. □

A function $\psi \in C([0,\infty)) \cap C^2((0,\infty)$ is nondecreasing and concave if and only if $\psi' \ge 0$ and $\psi'' \le 0$. In particular $\psi(t) = \frac{t}{1+t} = 1 - \frac{1}{1+t}$ satisfies the assumptions of Lemma 1.16.

Examples. (i) Let $X = \mathbb{R}^n$, $d_\infty(x,y) = \max_{i=1,\dots n} |x_i - y_i|$.
(ii) $X = \mathbb{R}^n$, $1 \le p < \infty$, $d_p(x,y) = \left(\sum_{i=1}^p |x_i - y_i|^p\right)^{1/p}$. For $p = 2$ we obtain the standard Euclidean metric on $\mathbb{R}^n$.
(iii) Let $1 \le p < \infty$, $l_p := \{a : \mathbb{N} \to \mathbb{R} : \sum_{k=0}^\infty |a_k|^p < \infty\}$, $d_p(x,y) := \left(\sum_{k=0}^\infty |x_k - y_k|^p\right)^{1/p}$. Then (l_p, d_p) is a metric space.
(iv) Let l_∞ be the space of bounded sequences $a : \mathbb{N} \to \mathbb{R}$ and $d_\infty(x,y) = \sup_{k\in\mathbb{N}} |x_k - y_k|$. Then (l_∞, d_∞) is a metric space.
(v) $X = \mathbb{R}$, $d(x,y) = \frac{|x-y|}{1+|x-y|}$ (see Lemma 1.16).
(vi) Denote by $\mathbb{R}^\mathbb{N}$ the space of all sequences $a : \mathbb{N} \to \mathbb{R}$. Then $d(x,y) = \sum_{k=0}^\infty 2^{-k} \frac{|x_k - y_k|}{1+|x_k-y_k|}$ is a metric on $\mathbb{R}^\mathbb{N}$.
(vii) (Pull-back metric) Let X be set, let (Y, d_Y) be a metric space and let $f : X \to Y$ be injective. Then $d_X(x_1, x_2) := d_Y(f(x_1), f(x_2))$ is a metric on X.
(viii) Let $X = \mathbb{R} \cup \{-\infty\} \cup \{\infty\} = [-\infty, \infty]$, define $f : X \to [-1, 1]$ by

$$f(x) = \begin{cases} -1 & \text{if} \quad x = -\infty \\ \frac{x}{1+|x|} & \text{if} \quad x \in \mathbb{R} \\ 1 & \text{if} \quad x = \infty \end{cases} \tag{1.11}$$

and set $d_X(x_1, x_2) = |f(x_1) - f(x_2)|$. By (vii) the pair (X, d_X) is a metric space.
(ix) Let V be a finite set, let $E \subset V \times V$ be a symmetric set and consider the graph $\Gamma = (V, E)$. A curve in Γ is a map $\gamma : \{0, 1, \dots k\} \to V$ with $(\gamma(j), \gamma(j+1)) \in E$ and k is called the length of γ. The graph is called connected if for every two points in $x, y \in V$ there exists a curve with $\gamma(0) = x$ and $\gamma(k) = y$. We define $d(x,x) = 0$ and for $x \neq y$ we define $d(x,y)$ be the length of the shortest curve from x to y. Then d is a metric on V. By Lemma 1.16 $\tilde{d} = d/(1+d)$ is another metric on V. If (V, E) is not connected we can define a metric on V by $\hat{d}(x,y) = \tilde{d}(x,y)$ if a curve from x to y exists and $\hat{d}(x,y) = 1$ otherwise.

Let (X, d) be a metric space, let $A_1, A_2, A \subset X$, let $x \in X$ and let $r > 0$. We define the diameter of a set

$$\operatorname{diam}(A) := \sup\{d(x,y) : x, y \in A\}, \tag{1.12}$$

the distance of two sets

$$\operatorname{dist}(A_1, A_2) := \inf\{d(x,y) : x \in A_1, y \in A_2\}, \tag{1.13}$$

the distance of a point from a set

$$\operatorname{dist}(x, A) := \inf\{d(x,y) : y \in A\}, \tag{1.14}$$

the r-neigbhourhood of a set

$$B_r(A) := \{y \in X : \operatorname{dist}(y, A) < r\} \tag{1.15}$$

and the ball of radius r around x

$$B(x,r) := B_r(\{x\}) = \{y \in X : d(x,y) < r\}. \tag{1.16}$$

A set A is called bounded if $\operatorname{diam}(A) < \infty$ and the space (X, d) is called bounded if $\operatorname{diam}(X) < \infty$.

We also define the Hausdorff distance of two sets A and B by

$$d_H(A,B) := \inf\{r > 0 : A \subset B_r(B) \text{ and } B \subset B_r(A)\}. \tag{1.17}$$

It is easy to show (exercise) that

$$d_H(A,B) = \max\big(\sup_{x\in A} \operatorname{dist}(x,B), \sup_{y \in B} \operatorname{dist}(y,A)\big). \tag{1.18}$$

Proposition 1.17 (Topology induced by a metric)**.** *Let (X, d) be a metric space and let $\mathcal{T}_d$ consist of all the sets with the following property:*

$$\forall x \in U \ \exists \varepsilon > 0 \quad B(x,\varepsilon) \subset U. \tag{1.19}$$

Then $\mathcal{T}_d$ is a topology on X and $(X, \mathcal{T}_d)$ is a Hausdorff space.

Proof. This is an easy exercise. The main point is that by the triangle inequality for every $z \in B(x,r)$ we have $B(z,s) \subset B(x,r)$ with $s = r - d(x,z) > 0$. □

In the following we will always consider the topology $\mathcal{T}_d$ on (X, d) (unless stated otherwise) and we will often write only $\mathcal{T}$ instead of $\mathcal{T}_d$. One easily sees that $B(x,r)$ is open and we call this set the open ball of radius r around x.

Lemma 1.18. *Let (X, d) be a metric space. Let*

$$\mathcal{C} := \{A \subset X : A \textit{ closed, bounded, non-empty}\}.$$

Then the Hausdorff distance d_H defined in (1.17) *is a metric on $\mathcal{C}$.*

Proof. Exercise. □

1.2.2 Convergence and continuity in metric spaces

Proposition 1.19 (Convergence in a metric space)**.** *Let (X, d) be a metric space, let $x : \mathbb{N} \to X$ be a sequence. Then the following two statements are equivalent.*

(i) The sequence x converges to x^* in $(X, \mathcal{T}_d)$.

(ii) $\forall \varepsilon > 0 \; \exists k_0 \; \forall k \geq k_0 \quad d(x_k, x^*) < \varepsilon$.

Proof. For the implication $(i) \implies (ii)$ apply Definition 1.7 with $U = B(x^*, \varepsilon)$. For the converse implication one uses the fact the if $U \in \mathcal{T}_d$ und $x^* \in U$ then by definition of $\mathcal{T}_d$ there exists an $\varepsilon > 0$ such that $B(x^*, \varepsilon) \subset U$. □

A key feature in a metric space is that open sets (or equivalently closed sets) are completely characterized in terms of convergence of *sequences*. More precisely we have

Proposition 1.20. *Let (X, d) be a metric space and $A \subset X$. Then the following two statements are equivalent*

(i) A is closed, i.e. $X \setminus A \in \mathcal{T}_d$.

(ii) A is sequentially closed, i.e. for every sequence $x : \mathbb{N} \to A$ which converges to a point $x^* \in X$ we have $x^* \in A$.

Moreover for an arbitrary set $A \subset X$ the closure $\overline{A}$ (see Definition 1.3) agrees with the sequential closure, i.e.,

$$\overline{A} = \{x^* \in X : \exists \, \textit{sequence} \, x : \mathbb{N} \to A, \lim_{k \to \infty} x_k = x^*\}. \tag{1.20}$$

Proof. Exercise. □

In metric spaces continuity can also be characterized in terms of sequences.

Proposition 1.21 (Continuity in metric spaces)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$. Then the following three statements are equivalent.*

(i) f is continuous as a map from $(X, \mathcal{T}_{d_X})$ to $(Y, \mathcal{T}_{d_Y})$.

(ii) ($\varepsilon - \delta$ definition of continuity)

$$\forall x \in X \; \forall \varepsilon > 0 \; \exists \delta > 0 \quad d_X(z, x) < \delta \implies d_Y(f(z), f(x)) < \varepsilon. \tag{1.21}$$

(iii) (sequential continuity) For every $x^* \in X$ and every sequence $x : \mathbb{N} \to X$ which converges to x^* the sequence $k \mapsto f(x_k)$ converges to $f(x^*)$.

Proof. Exercise, see Analysis 1. □

[13.10. 2017, Lecture 2]

[18.10. 2017, Lecture 3]

Metrizability. The foregoing results show that spaces where the topology is induced by a metric are more easy to handle. Such topological spaces are called metrizable. One might wonder whether every topological space which is Hausdorff is metrizable. This is not true in general.

A necessary condition for metrizability is that for each point $x \in X$ there exists a countable family of open sets such that every open neighbourhood of x contains one of these sets (for a metric space one may take the family of balls $B(x,r)$ with $r \in \mathbb{Q}$). We say that each point has a countable neighbourhood basis (or that the space is 'first countable').

One sufficient condition is given by Uryson's theorem[1]: assume there exist a countable family of open sets such that every element of $\mathcal{T}$ contains a set in the family ('$(X,\mathcal{T})$ is second countable') and that the following stronger version of the Hausdorff property holds: for every closed set $A \subset X$ and every $x \in X \setminus A$ there exist $U, V \in \mathcal{T}$ such that $A \subset U$, $x \in V$ and $U \cap V = \emptyset$ ('$(X,\mathcal{T})$ is regular'). Then there exist a metric d on X such that $\mathcal{T} = \mathcal{T}_d$.

It is not difficult to see that the nonstandard topology on $\mathbb{R}$ given in Example 1.2 (iv) does not have a countable neighbourhood basis and is thus not metrizable. One can also verify that for this topology there exist sets which are sequentially closed but not closed (exercise).

Definition 1.22. *Let d_1 and d_2 be metrics on X. We say that d_1 is stronger than d_2 if the topology induced by d_1 is stronger than the one induced by d_2. We say that d_1 and d_2 are equivalent if they induce the same topology (equivalently, if d_1 is stronger than d_2 and d_2 is stronger than d_1).*

Proposition 1.23. *Let d_1 and d_2 be metrics on X. Then then the following statements are equivalent.*

(i) The metric d_1 is stronger than d_2.

(ii) The identity map $x \mapsto x$ is continuous as a map from $(X, \mathcal{T}_{d_1})$ to $(X, \mathcal{T}_{d_2})$.

(iii) Every sequence which converges in d_1 converges also in d_2.

(iv) $\forall x \in X\, \forall \varepsilon > 0\, \exists \delta > 0 \quad d_1(x,y) < \delta \Longrightarrow d_2(x,y) < \varepsilon$

Proof. *(i)* $\Longleftrightarrow$ *(ii)*: this follows directly from the definition of continuity and Definition 1.22.
(ii) $\Longleftrightarrow$ *(iii)*: this follows since continuity and sequential continuity are equivalent.
(ii) $\Longleftrightarrow$ *(iv)*: this follows from the $\varepsilon - \delta$ characterization of continuity. □

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[1]See, e.g., N. Dunford, J.T. Schwartz, Linear operators, Part I, Interscience Publishers, 1966, Theorem I.19,page 24

Examples. (i) Let $X = \mathbb{R}$. The standard metric $d(x,y) = |x-y|$ and the metric $\tilde{d}(x,y) = |\arctan x - \arctan y|$ are equivalent.
(ii) Let l_1 denote the spaces of summable sequences $a : \mathbb{N} \to \mathbb{R}$. The metric $d_1(x,y) := \sum_{k=0}^{\infty} |x_k - y_k|$ is stronger than the metric $d_\infty(x,y) = \sup_{k\in\mathbb{N}} |x_k - y_k|$ (since $d_\infty \leq d_1$), but the two metrics are not equivalent. To see this consider the sequence $j \mapsto x^{(j)}$ where $x_k^{(j)} = 1/j$ for $k \leq j$ and $x_k^{(j)} = 0$ for $k > j$. Then $d_\infty(x^{(j)}, 0) \to 0$ as $j \to \infty$, but $d_1(x^{(j)}, 0) = 1$.

Definition 1.24. *A metric space (X,d) is called separable if $(X, \mathcal{T}_d)$ is separable.*

1.2.3 Completeness

A fundamental concept in metric spaces is completeness.

Definition 1.25 (Cauchy sequence and completeness)**.** *Let (X,d) be a metric space.*
(i) A sequence $x : \mathbb{N} \to X$ is called a Cauchy sequence if

$$\forall \delta > 0 \ \exists k_0 \ \forall k, j \geq k_0 \quad d(x_j, x_k) < \delta. \tag{1.22}$$

(ii) The space (X,d) is called complete if every Cauchy sequence converges.

Examples. (i) Let $X = \mathbb{Q}$, equipped with the standard metric on $\mathbb{R}$. Then X is not complete.
(ii) Let $1 \leq p \leq \infty$. Then the spaces (l_p, d_p) of sequences introduced after Definition 1.15 are complete.
(iii) The space (l_1, d_∞) is not complete (consider the sequence $x^{(j)}$ defined by $x_k^{(j)} = 1/(k+1)$ for $k \leq j$ and $x_k^{(j)} = 0$ for $k > j$). This sequence converges in $(l_\infty, d_\infty$ to x^* with $x_k^* = 1/(k+1)$ for all $k \in \mathbb{N}$ and hence is a Cauchy sequence in (l_1, d_∞). It has, however, no limit in l_1, since $x^* \neq l_1$.
(iv) If (X,d) is a complete metric space and A is a dense subsets of X with $A \neq X$ then the metric space (A, d_A), where d_A denotes the restriction of d to $A \times A$ is not complete.

Remark. Completeness is really a metric property and not a topological property. There exists a metric $\tilde{d}$ on $\mathbb{R}$ which are equivalent to the standard metric such that $(\mathbb{R}, \tilde{d})$ is not complete (see homework problems). The point is that the notion of 'Cauchy sequence' depends on the metric and not just on the topology induced by the metric.

Proposition 1.26. *Let (X,d) be a complete metric space and let $A \subset X$ be closed and let d_A be the restriction of d to $A \times A$. Then the space (A, d_A) is complete.*

Proof. Let $x : \mathbb{N} \to A$ be a Cauchy sequence in A. Then x is also a Cauchy sequence in X and hence has a limit x_*. Since A is closed $x_* \in A$. Thus (A, d_A) is complete. □

An important result states that every non-complete space can be seen as a dense subset in a complete metric space (up to isometry).

Definition 1.27 (Isometry). *Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is called an isometric immersion if*

$$d(f(x), f(y)) = d(x, y). \tag{1.23}$$

The map f is called an isometry if it is in addition bijective.

Note that an isometric immersion is automatically injective. Thus if f is an isometric immersion than f is an isometry from X to $f(X)$. The spaces X and $f(X)$ (equipped with the restriction of the metric d_Y) are indistinguishable as metric spaces. In particular a sequence $x : \mathbb{N} \to \mathbb{R}$ is convergent in X if and only if $f \circ x$ is convergent in $f(X)$ and x is a Cauchy sequence in X if and only if $f \circ x$ is a Cauchy sequence in $f(X)$.

Theorem 1.28 (Completion). *Let (X, d) be a metric space. Then there exists a complete metric space $(\tilde{X}, \tilde{d})$ and an isometric immersion $j : X \to \tilde{X}$ such that $j(X)$ is dense in $\tilde{X}$.*

The space $\tilde{X}$ is called the completion of X (and is unique up to isometries).

Proof. We will see a short proof later. Here we just sketch the standard proof which is based on considering the space of Cauchy sequences in X modulo converging sequences. This proof is modelled on the construction of the real numbers $\mathbb{R}$ from the rational numbers $\mathbb{Q}$ (see e.g. Alt's book for details).

Let

$$\hat{X} = \{x : \mathbb{N} \to X : x \text{ Cauchy sequence}\}$$

Define

$$x \sim y \quad :\Longleftrightarrow \quad \lim_{k \to \infty} d(x_k, y_k) = 0.$$

It is easy to see that $\sim$ is an equivalence relation. The equivalence class of $x \in \hat{X}$ is defined by

$$[x] := \{z \in \hat{X} : z \sim x\}.$$

and the space of equivalences is defined by $\tilde{X} = X/\sim$. On $\tilde{X}$ we define

$$\tilde{d}([x], [y]) := \lim_{j \to \infty} d(x_j, y_j). \tag{1.24}$$

It follows from the definition of the equivalence classes and the triangle inequality that $\tilde{d}$ is well-defined, i.e., the right hand side only depends on the equivalence classes of the Cauchy sequences x and y. Since x and y are Cauchy sequences it is also easy to see $j \mapsto d(x_j, y_j)$ is a Cauchy sequence in $\mathbb{R}$ and that therefore $\lim_{j\to\infty} d(x_j, y_j)$ exists. It is easy to see that $\tilde{d}$ is a metric on $\tilde{X}$. Then one shows that

- $(\tilde{X}, \tilde{d})$ is complete
- the map $j : X \to \tilde{X}$ which maps a to the (equivalence class of) the constant sequence with value a is an isometry
- $j(X)$ is dense in $(\tilde{X}, \tilde{d})$.

□

Lemma 1.29. *Let (X, d) be a complete metric space. Then the space $(\mathcal{C}, d_H)$ of closed, bounded, non-empty sets with the Hausdorff metric, introduced in Lemma 1.18 is complete.*

Proof. Exercise. Hint: first show that a decreasing Cauchy sequence of sets B_j converges to $B^* := \cap_{j\in\mathbb{N}} B_j$ (and that in particular $B^* \neq \emptyset$). By passage to a subsequence you can assume without loss of generality that $d_H(B_j, B_{j+1}) < 2^{-j}$ (explain why). For a general Cauchy sequence A_j define $B_j := \overline{\cup_{k\geq j} A_k}$ and show that $d_H(A_j, B_j) \to 0$. □

One can also introduce a metric on set of (equivalence classes of) bounded and complete metric spaces, the so called Gromov-Hausdorff metric. Here we say that $(X, d_X) \sim (Y, d_Y)$ if there exist an isometry $I : X \to Y$. Then one defines

$$\begin{aligned} &d_{GH}((X, d_x), (Y, d_y)) \\ :=& \inf\{d_{H,Z}(I(X), J(Y)) : (Z, d_Z) \text{ metric space}, \\ &\quad I : X \to Z,\ J : Y \to Z \text{ isometric immersion}\}. \end{aligned} \tag{1.25}$$

Here $d_{H,Z}$ denotes the Hausdorff distance in the metric space (Z, d_Z). The GH metric plays an important role in geometry as well as in application to image processing.

1.3 Normed spaces

Let X be a $\mathbb{K}$ vector space. Recall our convention that always

$$\mathbb{K} = \mathbb{R} \quad \text{or } \mathbb{K} = \mathbb{C}. \tag{1.26}$$

Definition 1.30 (Normed space)**.** *A pair $(X, \|\cdot\|)$ is called a normed space if X is a $\mathbb{K}$ vector space and $\|\cdot\| : X \to [0, \infty)$ is a map with the following properties*

(i) (Definiteness) $\|x\| = 0 \Longrightarrow x = 0$.

(ii) (Homogeneity) $\|\alpha x\| = |\alpha|\,\|x\| \quad \forall \alpha \in \mathbb{K}, x \in X$.

(iii) (Triangle inequality) $\|x + y\| \leq \|x\| + \|y\|$.

The map $\|\cdot\|$ is called a norm. Note also that the homogeneity condition (ii) implies that $\|0\| = 0$. Thus condition (i) can also be written as $\|x\| = 0 \Leftrightarrow x = 0$.

A map $\|\cdot\| : X \to [0, \infty)$ which satisfies the second and third condition is called a seminorm. One can pass from a seminorm to a norm by considering the quotient space $\tilde{X} = X/\sim$, where $x \sim y$ if $\|x - y\| = 0$ with the norm $\|[x]\|_\sim := \|x\|$ (see the corresponding discussion for metric spaces).

If $(X, \|\cdot\|)$ is a normed space then $d(x, y) = \|x - y\|$ is a metric on X. The notions of convergence, continuity and completeness on a normed space are defined using this metric.

It follows from the triangle inequality that

$$|\,\|x\| - \|y\|\,| \leq \|x - y\| \tag{1.27}$$

(write $x = y + (x - y)$ and $y = x + (y - x)$). In particular the map $x \mapsto \|x\|$ is continuous from X to $\mathbb{R}$ (use the ε-δ definition with $\delta = \varepsilon$).

Definition 1.31 (Banach space)**.** *A normed space is called a Banach space if it is complete under the above metric d.*

Example. For $1 \leq p \leq \infty$ the sequence spaces l_p are Banach spaces with norms $\|x\|_p := (\sum_{k=0}^{\infty} |x_k|^p)^{1/p}$ (for $p < \infty$) and $\|x\|_\infty := \sup_{k \in \mathbb{N}} |x_k|$, respectively.

Definition 1.32. *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on the $\mathbb{K}$ vector space X. We say that $\|\cdot\|_1$ is stronger than $\|\cdot\|_2$ if the corresponding metric d_1 is stronger than d_2. We say that the two norms are equivalent if the corresponding metrics are equivalent (i.e., if the induced topologies are the same).*

Proposition 1.33. *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on the $\mathbb{K}$ vector space X. The norm $\|\cdot\|_1$ is stronger than $\|\cdot\|_2$ if and only if there exists a constant $C > 0$ such that*

$$\|x\|_2 \leq C\|x\|_1 \quad \forall x \in X. \tag{1.28}$$

The two norms are equivalent if and only if there exists constant $c > 0$ and $C > 0$ such that

$$c\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1 \quad \forall x \in X. \tag{1.29}$$

Proof. It suffices to prove the first statement. For the 'if' statement use the characterization (iii) in Proposition 1.23. For the 'only if' part apply the characterization (iv) in Proposition 1.23 with $\varepsilon = 1$ and $y = 0$ and let

$C > \frac{1}{\delta}$. Let $x \neq 0$. Then $y = \frac{1}{C\|x\|_1} x$ satisfies $\|y\|_1 < \delta$ and hence $\|y\|_2 < 1$. By the homogeneity of the norm this implies that

$$\|x\|_2 = C\|x\|_1 \, \|y\|_2 < C\|x\|_1.$$

□

[18.10. 2017, Lecture 3]

[20.10. 2017, Lecture 4]

1.4 Hilbert spaces

For $\alpha \in \mathbb{C}$ we denote by $\bar{\alpha}$ the complex conjugate.

Definition 1.34. *Let X be a $\mathbb{K}$ vector space. A map $(x_1, x_2) \mapsto (x_1, x_2)_X$ from $X \times X$ to $\mathbb{K}$ is called a sesquilinear form if for all $x, y, x_1, x_2, y_1, y_2 \in X$ and all $\alpha \in \mathbb{K}$ we have*

(i) $(\alpha x, y)_X = \alpha(x, y) \quad (x, \alpha y)_X = \bar{\alpha}(x, y)_X$,

(ii) $(x_1 + x_2, y)_X = (x_1, y)_X + (x_2, y)_X, \quad (x, y_1 + y_2)_X = (x, y_1)_X + (x, y_2)_X$.

A sesquilinear form is called symmetric if for all $x, y \in X$

$$(y, x)_X = \overline{(x, y)_X}. \tag{1.30}$$

A sesquilinear form is called positive semidefinite if

$$\forall x \in X \quad (x, x)_X \geq 0 \tag{1.31}$$

A symmetric sesquilinear form is called positive definite if

$$\forall x \in X \quad (x, x)_X \geq 0 \quad \text{and } (x, x) = 0 \Leftrightarrow x = 0. \tag{1.32}$$

Remark. If $\mathbb{K} = \mathbb{R}$ a sesquilinear form is bilinear.
One often writes only (x_1, x_2) instead of $(x_1, x_2)_X$. Another common notation is $\langle x_1, x_2 \rangle$.

Definition 1.35. *A positive definite symmetric sesquilinear form on a $\mathbb{K}$ vector space is called a scalar product. If $(\cdot, \cdot)_X$ is a scalar product then the pair $(X, (\cdot, \cdot)_X)$ is called a pre-Hilbert space.*

Lemma 1.36. *Let X be a $\mathbb{K}$ vector space and let $(\cdot, \cdot)$ be a positive semidefinite symmetric sesquilinear form and set*

$$\|x\| := \sqrt{(x, x)}. \tag{1.33}$$

Then

(i) (Homogeneity)

$$\|\alpha x\| = |\alpha|\|x\| \quad \forall \alpha \in \mathbb{K}, x \in X. \tag{1.34}$$

(ii) (Cauchy-Schwarz inequality)

$$|(x,y)| \le \|x\|\|y\| \quad \forall x, y \in X. \tag{1.35}$$

(iii) (Triangle inequality)

$$\|x+y\| \le \|x\| + \|y\| \quad \forall x, y \in X. \tag{1.36}$$

(iv) (Parallelogram identity)

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \forall x, y \in X. \tag{1.37}$$

In particular $\|\cdot\|$ is a seminorm on X. If, in addition, $(\cdot,\cdot)_X$ is positive definite then $\|\cdot\|$ is a norm.

Proof. (i): This follows directly from Definition 1.34 (i).

(ii): We may assume that $(x,y) \neq 0$ since otherwise there is nothing to show. Moreover we may assume that

$$(x,y)_X \in \mathbb{R} \quad \text{and } (x,y)_X > 0. \tag{1.38}$$

Indeed if this condition does not hold we set $\alpha = (x,y)/|(x,y)|$. Then $|\alpha| = 1$ and $(\bar\alpha x, y) = |(x,y)|$. We then prove the result for $\tilde{x} := \bar\alpha x$ and y. This implies the assertion for x and y since by (i) we have $\|\bar\alpha x\| = \|x\|$.

Assume now (1.38) and assume in addition that If $\|x\| \neq 0$ and $\|y\| \neq 0$. Set $\xi = \frac{x}{\|x\|}$, $\eta = \frac{y}{\|y\|}$ and note that

$$0 \le (\xi - \eta, \xi - \eta) = \|\xi\|^2 + \|\eta\|^2 - 2\,\mathrm{Re}(\xi,\eta) \tag{1.39}$$

In view of (1.38) this implies

$$|(\xi,\eta)| \le \frac{1}{2}\|\xi\|^2 + \frac{1}{2}\|\eta\|^2 = 1. \tag{1.40}$$

Since $|(\xi,\eta)| = \frac{1}{\|x\|\,\|y\|}|(x,y)|$ the assertion follows.

If $\|x\| = 0$ apply (1.40) with $\xi = kx$ and $\eta = \frac{1}{k}y$. Then $|(x,y)| = (\xi,\eta)_X \le 0 + k^{-2}\|y\|^2$ for all $k > 0$ which implies $|(x,y)| = 0$. A similar argument applies if $y = 0$.

(iii): This follows from $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}(x,y)$ and (ii).

(iv): Exercise. □

Lemma 1.37. *Let* $(X, \|\cdot\|)$ *be a normed space. Then there exists a scalar product* $(\cdot,\cdot)$ *with*

$$\|x\| := \sqrt{(x,x)} \tag{1.41}$$

if and only if the parallelogram identity holds, i.e., if and only if

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \forall x, y \in X. \tag{1.42}$$

Proof. Exercise. The 'only if' part is just Lemma 1.36 (iv).

□

Definition 1.38. *A Hilbert space is pre-Hilbert space which is complete under the norm induced by the scalar product.*

2 Function spaces

2.1 Spaces of bounded, continuous and differentiable functions

Definition 2.1. *Let X be a set, let $(Y, \|\cdot\|)$ be a normed space Then the space of bounded functions $B(X;Y)$ is defined as*

$$B(X;Y) := \{f : X \to Y : \sup\{\|f(x)\| : x \in X\} < \infty. \tag{2.1}$$

Proposition 2.2. *Let Y be a Banach space and for $f \in B(X;Y)$ define*

$$\|f\| := \sup\{\|f(x)\| : x \in X\}. \tag{2.2}$$

Then $(B(X;Y), \|\cdot\|)$ is a Banach space.

Remark. An important special case is $Y = \mathbb{R}$ or $Y = \mathbb{C}$. In this case we usually write $B(X)$ instead of $B(X;\mathbb{K})$.

Proof. It is easy to see that $\|\cdot\|$ is a norm so we only need to verify completeness. Let $f : \mathbb{N} \to B(X;Y)$ be a Cauchy sequence, i.e.,

$$\forall \delta > 0 \ \exists k_0(\delta) \ \forall j,k \geq k_0 \quad \sup_{x \in X} \|f_j(x) - f_k(x)\| < \delta. \tag{2.3}$$

Taking $\delta = 1$ we deduce in particular that $\|f_j\| \leq M := \|f_{k_0(1)}\| + 1$ for all $j \geq k_0(1)$. Moreover for each $x \in X$ the sequence $j \mapsto f_j(x)$ is a Cauchy sequence. Since Y is complete this sequence has a limit which we call $f_*(x)$. We have $\|f_*(x)\| \leq M+1$ and thus the map $f_* : X \to Y$ belongs to $B(X;Y)$.

We finally show that $\lim_{j\to\infty} \|f_j - f_*\| = 0$. We have $\lim_{k\to\infty} \|f_k(x) - f_*(x)\| = 0$ and it thus follows from (2.3) that

$$\forall j \geq k_0(\delta) \quad \|f_j(x) - f_*(x)\| \leq \delta. \tag{2.4}$$

Since k_0 depends only on δ and not on x we get

$$\forall j \geq k_0(\delta) \quad \sup_{x \in X} \|f_j(x) - f_*(x)\| \leq \delta. \tag{2.5}$$

Now $\sup_{x\in X} \|f_j(x) - f_*(x)\| = \|f_j - f_*\|$. Thus $\limsup_{j\to\infty} \|f_j - f_*\| \leq \delta$. Since $\delta > 0$ was arbitrary this finishes the proof. □

We can now give a short proof of the fact for each metric space (X, d) there exist a complete metric space $(\tilde{X}, \tilde{d})$ and an isometry $j : X \to \tilde{X}$ such that $j(X)$ is dense in $\tilde{X}$.

Proof of Theorem 1.28. Let $Y = B(X)$. For each $x \in X$ we define a function $d_x : X \to \mathbb{R}$ by

$$d_x(z) = d(x, z).$$

We fix a point $x_0 \in X$ and define j by

$$j(x) = d_x - d_{x_0}.$$

Using the triangle inequality we can easily to show that $|j(x)(z)| \leq d(x, x_0)$. Thus $j(x)$ an element of $Y = B(X)$, Similarly one shows that $j : X \to Y$ is an isometry (exercise). Now we set $\tilde{X} = \overline{j(X)}$ where the bar denotes the closure in Y. Then $\tilde{X}$ is a closed subset of the complete metric space Y and hence by Proposition 1.26 itself a complete metric space. □

Proposition 2.3. *The space $B(X) = B(X, \mathbb{R})$ is separable if and only if X is a finite set. In particular the space of bounded sequences $l_\infty = B(\mathbb{N})$ is not separable.*

Proof. Exercise. Hint: Let $A \subset B(X)$ be the set of functions which only take values 0 or 1. This set can be mapped bijectively to $\{0, 1\}^X$ and if X is infinite then A is uncountable. Moreover

$$f, g \in A, \ , f \neq g \quad \Longrightarrow \quad \|f - g\| = 1. \tag{2.6}$$

Assume $D = \{h_1, h_2, \ldots\}$ is a countable and dense subset of X. Use (2.6) to show that there exists an injective map $j : A \to D$. This contradicts the fact that A is uncountable. □

Definition 2.4. *Let $(X, \mathcal{T})$ be a topological space and Y be a normed space. The space of continuous functions from X to Y is defined as*

$$C(X; Y) := \{f : X \to Y : f \text{ continuous}\} \tag{2.7}$$

and the space of bounded continuous functions is defined as

$$C_b(X; Y) := C(X; Y) \cap B(X; Y). \tag{2.8}$$

Theorem 2.5. *Let Y be a Banach space and for $f \in C_b(X; Y)$ define*

$$\|f\| := \sup\{\|f(x)\| : x \in X\}. \tag{2.9}$$

Then $(C_b(X; Y), \| \cdot \|)$ is a Banach space.

Proof. The space $C_b(X; Y)$ is a subset of $B(X; Y)$. In view of Proposition 1.26 it suffices to show that $C_b(X; Y)$ is a closed subset. Assume that $f_k \in C_b(X; Y)$, $f_* \in B(X; Y)$ and $\lim_{k\to\infty} \|f_k - f_*\| = 0$. We need to show that f_* is continuous. For this it suffices to show that f is continuous at every $x \in X$, i.e., that for every neighbourhood V of $f_*(x)$ there exists a neighbourhood U of x such that $f_*(U) \subset V$. Let x and V be given. Since V is open there exists an $\varepsilon > 0$ such that $B(f_*(x), \varepsilon) \subset V$. There exists k such that $\|f_k - f_*\| < \varepsilon/3$. Set $U = f_k^{-1}(B(f_k(x), \varepsilon/3))$. Then $U \in \mathcal{T}$ since f_k is continuous. Moreover $x \in U$. We claim that $f_*(U) \subset B(f_*(x), \varepsilon) \subset V$. To this see this note that the triangle inequality implies that for each $z \in U$

$$d(f_*(z), f_*(x)) \le d(f_*(z), f_k(z)) + d(f_k(z), f_k(x)) + d(f_k(x), f_*(x)) < \varepsilon. \tag{2.10}$$

□

Theorem 2.6. *Let $(X, \mathcal{T})$ be a compact topological space. Let Y be a normed space. If $f : X \to Y$ is continuous, then f is bounded. Thus $C(X; Y) = C_b(X; Y)$.*

Proof. The map $y \mapsto \|y\|$ is continuous as a map from Y to $\mathbb{R}$. Hence the map $h : X \to \mathbb{R}$ defined by $h(x) := \|f(x)\|$ is also continuous. By Theorem 1.12 it follows that h attains its maximum on X. In particular h is bounded on X and hence f is bounded on X. □

[20.10. 2017, Lecture 4]

[25.10. 2017, Lecture 5]

Theorem 2.7. *Let (X, d) be a compact metric space and let Y be a normed space, which is separable. Then the space $C(X; Y)$ is separable.*

Remark. (i) In particular $C(X) = C(X; \mathbb{R})$ is separable when X is compact.
(ii) Compare this with the result that $B(X; \mathbb{R})$ is separable if and only if X is a finite set.

Proof. Step 1: Uniform continuity.
Let $\varepsilon > 0$, $f \in C(X; Y)$. Then here exists a $\delta_{f,\varepsilon} > 0$ such that

$$d(x, y) < \delta_{f,\varepsilon} \quad \Longrightarrow \quad \|f(x) - f(y)\| < \frac{\varepsilon}{2}. \tag{2.11}$$

This argument is known from Analysis 1 ('continuous functions on a compact interval are uniformly continuous'). We recall the proof for the convenience of the reader. Since f is continuous for each $z \in X$ there exists an $r_z > 0$ such that $f(B(z, r_z)) \subset B(f(z), \varepsilon/4)$. Trivially $X \subset \bigcup_{z \in X} B(z, r_z/2)$. Since X is compact there exist finitely many balls $B(z_1, r_1), \ldots B(z_k, r_k)$ such that $X \subset \bigcup_{i=1}^k B(z_i, r_i/2)$. Let $\delta = \frac{1}{2} \min\{r_i : i = 1, \ldots k\}$. Let $x, y \in X$, with $|x - y| < \delta$. Then there exist $i \in \{1, \ldots, k\}$ such that $|x - z_i| < r_i/2$. Since $\delta \le r_i/2$ it follows that $|y - z_i| < r_i$ and thus

$$\|f(x) - f(y)\| \le \|f(x) - f(z_i)\| + \|f(z_i) - f(y)\| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \frac{\varepsilon}{2}. \tag{2.12}$$

Step 2: Partition of unity.
Let $\delta > 0$. We claim that there exist finitely many points $x_1, \dots x_k \in X$ and continuous functions $\eta_1, \dots, \eta_k : X \to \mathbb{R}$ such that

$$\sum_{i=1}^{k} \eta_i(x) = 1 \quad \forall x \in X, \qquad \eta_i = 0 \text{ on } \mathbb{R}^n \setminus B(x_i, \delta) \tag{2.13}$$

(such a family of functions is called a partition of unity).

Indeed $X = \cup_{x \in X} B(x, \delta/2)$ and hence by compactness there exist $x_1, \dots x_k$ such that $X = \cup_{i=1}^{k} B(x_i, \delta/2)$. Let $f : [0, \infty) \to [0, 1]$ be a continuous function such that $f \geq \frac{1}{2}$ on $[0, \frac{1}{2})$ and $f = 0$ on $[1, \infty)$ (we may take $f(t) = \max(0, 1-t)$). Set $f_i(x) = f(\frac{1}{\delta} d(x, x_i))$. Then $f_i \geq 1/2$ on $B(x_i, \delta/2)$ and thus $\sum_{i=1}^{k} f_i(x) \geq 1/2$. Moreover $f_i = 0$ on $\mathbb{R}^n \setminus B(x_i, \delta)$. Hence the functions

$$\eta_i(x) := \frac{f_i(x)}{\sum_{j=1}^{k} f_j(x)} \tag{2.14}$$

have the desired properties.

Step 3: Approximation by subspace isomorphic to Y^k.
Let $f \in C(X; Y)$, let $\varepsilon > 0$ and let $\delta_{f,\varepsilon}$ be as in Step 1. Assume that $0 < \delta < \delta_{f,\varepsilon}$ and let $B(x_i, \delta/2)$ and η_i be as in Step 2. Set

$$g(x) = \sum_{i=1}^{k} f(x_i) \eta_i(x). \tag{2.15}$$

We claim that $\|f - g\| \leq \varepsilon/2$. To see this recall that $\sum_i \eta_i(x) = 1$ and thus

$$f(x) - g(x) = \sum_{i=1}^{k} [f(x) - f(x_i)] \eta_i(x). \tag{2.16}$$

If $\eta_i(x) \neq 0$ then $|x - x_i| < \delta$ and thus $\|f(x) - f(x_i)\| < \varepsilon/2$. Thus

$$\|f(x) - g(x)\| < \sum_{i=1}^{k} \frac{\varepsilon}{2} \eta_i(x) \leq \frac{\varepsilon}{2} \tag{2.17}$$

Step 4: Approximation by a countable set.
Let ε, $\delta_{f,\varepsilon}$, δ and η_i be as in Step 3, let $D \subset Y$ be countable and dense and set

$$A_\delta := \{g : X \to \mathbb{R} : g(x) = \sum_{i=1}^{k} d_i \eta_i(x),\, d_i \in D\} \tag{2.18}$$

Then A_δ is countable since it can be mapped bijectively to D^k. There exist $d_i \in D$ such that $|f(x_i) - d_i| < \varepsilon/2$. Together with Step 3 it follows that there exists $g \in A_\delta$ with $\|f - g\| < \varepsilon$.

Step 5: Conclusion.
Finally set $A = \bigcup_{k=1}^{\infty} A_{\frac{1}{k}}$. Then A is countable. If $\varepsilon > 0$, $f \in C(X;Y)$ and $k > 1/\delta_{f,\varepsilon}$ then by Step 4 there exist $g \in A_k \subset A$ such that $\|f - g\| < \varepsilon$. Hence A is dense in $C(X;Y)$. □

If $U \subset \mathbb{R}^n$ is open, then continuous maps $f : U \to \mathbb{R}$ need not be bounded (example: $U = (0,1)$, $f(x) = \frac{1}{x}$). One can still define a metric on such continuous functions by exhausting U with compact sets.

Theorem 2.8. *Let $U \subset \mathbb{R}^n$ be open and let Y be a normed space. Let $K_i \subset U$ be an increasing sequence of compact sets with $\bigcup_{i=1}^{\infty} K_i = U$ and assume that for every $x \in U$ there exists $r > 0$ und i such that $B(x,r) \subset K_i$. For $f \in C(U;Y)$ define*

$$[f]_i := \sup_{x \in K_i} \|f(x)\|, \qquad d(f,g) := \sum_{i=1}^{\infty} 2^{-i} \frac{[f-g]_i}{1 + [f-g]_i}. \tag{2.19}$$

Then $[\cdot]_i$ is a seminorm on $C(U;Y)$ and d is a metric. If Y is a Banach space then $(C(U;Y), d)$ is complete. If Y is separable then $(C(U;Y), d)$ is separable.

Remark. (i) The procedure to use countably many seminorms (or semimetrics) to obtain a metric as in (2.19) is used in a number of other situations. The metric d is sometimes called the Frechet metric generated by the seminorms $[\cdot]_i$.
(ii) There exists no norm on $C(U;Y)$ such that the induced metric is equivalent to d (for a similar problem see homework sheet 2).
(iii) For each open set $U \subset \mathbb{R}^n$ there exists such a sequence K_i. One may take $K_i = \{x \in U : \operatorname{dist}(x, \mathbb{R}^n \setminus U) \geq 2^{-i}\ |x| \leq i\}$.
(iv) Different choices of the sequence K_i lead to equivalent metrics d (compare Proposition 2.9 below).

Proof. It is easy to see that $[\cdot]_i$ is a seminorm (note the $[f]_i < \infty$ since K_i is compact). Thus $d_i(f,g) = [f-g]_i$ is a semimetric. From this it easily follows that then $(f,g) \mapsto \frac{d_i(f,g)}{1+d_i(f,g)}$ and d are also semimetrics (see Lemma 1.16). To see that d is actually a metric assume that $d(f,g) = 0$. Then $d_i(f,g) = 0$ for all i and hence $f = g$ on K_i. Since $U = \bigcup_{i=1}^{\infty} K_i$ we get $f = g$.

To prove completeness let $j \mapsto f_j$ be a Cauchy sequence with respect to d. This implies that $j \mapsto f_j$ is a Cauchy sequence for each d_i. It follows from the completeness of $C(K_i;Y)$ that there exist functions $g_i \in C(K_i;Y)$ such that $\lim_{j\to\infty} \sup_{x \in K_i} \|f_j(x) - g_i(x)\| = 0$. Since $K_i \subset K_{i+1}$ one easily sees that for $k < i$ we have $g_k = g_{i|K_k}$. Thus we can define $g : X \to Y$ by $g(x) = g_i(x)$ if $x \in K_i$. By construction we have $\lim_{j\to\infty} d_i(f_j, g) = 0$

for each i. This implies that $d(f_j, g) \to 0$ (see Proposition 2.9). Finally we need to show that g is continuous. This follows from the fact that $g_{|K_i}$ is continuous that for each $x \in U$ there exists a ball $B(x, r)$ which is contained in one K_i.

To prove separability let $\varepsilon > 0$ and let i_0 be such that $2^{-i+2} < \varepsilon$. Then for any set $d(f, g) \leq d_{i_0}(f, g) + \varepsilon$ and the result follows easily from the proof of Theorem 2.7 (details: exercise, see also the proof of Proposition 2.9 below). □

Proposition 2.9. *Let $U \subset \mathbb{R}^n$ be open and let $(C(U;Y), d)$ be as in Theorem 2.8. Let $f : \mathbb{N} \to C(U;Y)$ be a sequence. Then the following statements are equivalent:*

(i) $\lim_{j\to\infty} d(f_j, g) = 0$*;*

(ii) $\forall i \quad \lim_{j\to\infty} d_i(f, g) = 0$*;*

(iii) for all compact sets $K \subset U$ $\quad \lim_{j\to\infty} \sup_{x\in K} \|f_j(x) - g(x)\| = 0.$

Remark. The convergence in (iii) is often called locally uniform convergence.

Proof. (iii) $\Longrightarrow$ (ii): obvious.

(ii) $\Longrightarrow$ (i): Exercise. Hint: let $2^{-i_0+2} < \varepsilon$ and split the sum into a finite part for $i \leq i_0$ and the rest.

(i) $\Longrightarrow$ (iii): For each $x \in K$ there exists an $r > 0$ and an i such that $B(x, r) \subset K_i$. Since K is compact, finitely many of these balls $B(x_j, r_j) \subset K_j$ cover K. Since the sets K_i are increasing in i there exists i_0 such that $K \subset K_{i_0}$. By assumption $d_{i_0}(f_j, g) \to 0$. This implies (iii). □

Let $U \subset \mathbb{R}^n$ be open. By $C^k(U;\mathbb{R}^m)$ we denote the space of k times differentiable functions $f : U \to \mathbb{R}^m$ whose derivatives are continuous. The derivative $D^l f(x)$ at a point x is an l-multilinear form on $\mathbb{R}^n$. On the finite-dimensional vector space of l-multilinear forms are a number of natural norms which are all equivalent (we will see shortly that all norms on a finite dimensional vector space are equivalent). If nothing else is said we will use the Euclidean norm

$$|D^l f(x)| := (\sum_{|\alpha|=l} |\partial^\alpha f(x)|^2)^{1/2}. \tag{2.20}$$

Here $\alpha \in \mathbb{N}^n$ is a multiindex, $|\alpha| = \sum_{i=1}^n \alpha_i$ and $\partial^\alpha f = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n} f$ with the convention that $\partial_i^0 f = f$.

Definition 2.10. *Let $U \subset \mathbb{R}^n$ be open and bounded. We define*

$$C^k(\overline{U};\mathbb{R}^m) := \{f \in C^k(U;\mathbb{R}^m) : D^l f \textit{ has a continuous extension to } \overline{U} \quad \forall l \leq k\} \tag{2.21}$$

and, for $0 \le l \le k$,

$$[f]_{C^l(\overline{U})} := \sup_{x \in \overline{U}} |D^l f(x)|, \quad \|f\|_{C^k} := \sum_{l=0}^{k} [f]_{C^l}. \tag{2.22}$$

Moreover we set $C^\infty(\overline{U};\mathbb{R}^m) = \bigcap_{k=0}^\infty C^k(\overline{U};\mathbb{R}^m)$ *and*

$$d(f,g) = \sum_{l=0}^{\infty} 2^{-l} \frac{[f]_{C^l}}{1 + [f]_{C^l}} \tag{2.23}$$

Theorem 2.11. *For* $0 \le l \le k$ *the expression* $[\cdot]_{C^l}$ *is a seminorm on* $C^k(\overline{U};\mathbb{R}^m)$*, the expression* $\|\cdot\|_{C^k}$ *is a norm and* $(C^k(\overline{U};\mathbb{R}^m), \|\cdot\|_{C^k})$ *is a separable Banach space. Moreover* $(C^\infty(\overline{U};\mathbb{R}^m), d)$ *is a separable and complete metric space.*

Remark. One can combine this result and Theorem 2.8 to introduce metrics on $C^k(U;\mathbb{R}^m)$ and $C^\infty(U;\mathbb{R}^m)$ so that these spaces become complete and separable metric space. Moreover $d(f_j, g) \to 0$ if and only if $D^l f_j$ converges uniformly to $D^l g$ on all compact subsets of U and all $l \le k$ or all $l \in \mathbb{N}$, respectively.

Proof. Exercise. Hints: For completeness of $C^k(U;\mathbb{R}^m)$ consider a Cauchy sequence $f : \mathbb{N} \to C^k(U;\mathbb{R}^m)$ and first deduce from Theorem 2.5 that $\partial^\alpha f_j \to g^\alpha$ in $C(\overline{U};\mathbb{R}^m)$ for $|\alpha| \le k$. Assume first $k = 1$. For x and y sufficiently close one has $f_j(y) - f_j(x) = \int_0^1 Df_j(x + t(y-x))(y-x)\,dt$ and passing to the limit one easily sees that g is differentiable and $\partial_i g = g^i$. This shows that $g \in C^1(U;\mathbb{R}^m)$. For general k one argues by induction.

To establish separability on can proceed in a similary way as in the proof of Theorem 2.5 as long as one chooses in addition $f \in C^\infty(\mathbb{R}^n)$ and one uses the k-th order Taylor expansion in the definition of the approximating functions.

The corresponding results for $C^\infty(\overline{U};\mathbb{R}^m)$ then follow by using the relation between the Frechet metric d and the seminorms $[\cdot]_{C^l}$ as in the proof of Theorem 2.8. □

Functions with compact support.

Definition 2.12 (Functions with compact support)**.** *(i) Let* $(X, \mathcal{T})$ *be a topological space, let* Y *be a normed space and let* $f : X \to Y$*. The support of* f *is defined as*

$$\operatorname{supp} f := \overline{\{x \in X : f(x) \neq 0\}}. \tag{2.24}$$

(ii) The space of continuous functions with compact support is defined as

$$C_c(X;Y) := \{f \in C(X;Y) : \operatorname{supp} f \text{ compact}\}. \tag{2.25}$$

(iii) If $U \subset \mathbb{R}^n$ is open and $Y = \mathbb{R}^m$ we define

$$C_c^k(U;\mathbb{R}^m) := C^k(U;\mathbb{R}^m) \cap C_c(U;\mathbb{R}^m), \tag{2.26}$$
$$\mathcal{D}(U;\mathbb{R}^m) := C_c^\infty(U;\mathbb{R}^m) := C^\infty(U;\mathbb{R}^m) \cap C_c(U;\mathbb{R}^m). \tag{2.27}$$

Remark. (i) For $m = 1$ we often write $C_c^k(U) := C_c^k(U,\mathbb{R})$. Some authors (e.g. H.W. Alt) write $C_0^k(U;\mathbb{R}^m)$ instead of $C_c^k(U;\mathbb{R}^m)$ etc.
(ii) In part (iii) of the definition the relative topology on U is used to define 'closure' and 'compact'. Equivalently we can take the closure of the set $\{x : f(x) \neq 0\}$ in $\mathbb{R}^n$ and require that this closure is contained in U and compact. Note that the closure of $\{x : f(x) \neq 0\}$ in $\mathbb{R}^n$ is compact if and only if $\{x : f(x) \neq 0\}$ is bounded.
(iii) Define $f(x) = 1 - x$ if $x \in [0, 1)$ and $f(x) = 0$ if $x \in [1, \infty)$. Then $f \in C_c([0,\infty))$ and $f \in C_c([0, 1])$ but $f \notin C_c((0, 1))$ and $f \notin C_c([0, 1))$.
(iv) The space $C_c(U;\mathbb{R}^m)$ is dense in $C(U;\mathbb{R}^m)$ (equipped with the Frechet metric defined above).

[25.10. 2017, Lecture 5]

[27.10. 2017, Lecture 6]

Interesting subspaces of the space of continuous functions arise when we consider functions with a given modulus of continuity $\rho : [0,\infty) \to [0,\infty)$, i.e., functions for which $\|f(x) - f(y)\| \leq C_f\, \rho(d(x, y))$ where $\lim_{t\downarrow 0} \rho(t) = 0$. The most important examples are Hölder continuous functions, which correspond to $\rho(t) = t^\alpha$. For simplicity we focus on functions on $A \subset \mathbb{R}^n$.

Definition 2.13. *Let $\alpha \in (0, 1]$. Let $A \subset \mathbb{R}^n$. We say that $f : A \to \mathbb{R}^m$ is Hölder continuous with exponent α if*

$$[f]_{\alpha,A} := \sup_{x,y\in A,\ x\neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty. \tag{2.28}$$

We define

$$C^{0,\alpha}(A;\mathbb{R}^m) := \{f \in C_b(A;\mathbb{R}^m) : [f]_{\alpha,A} < \infty\}, \quad \|f\|_{\alpha,A} := \sup_{x\in A} \|f(x)\| + [f]_{\alpha,A}. \tag{2.29}$$

If U is open and bounded then we define

$$C^{k,\alpha}(U;\mathbb{R}^m) := \{f \in C^k(\bar{U};\mathbb{R}^m) : [D^k f]_{\alpha,U} < \infty\}, \quad \|f\|_{k,\alpha,A} := \|f(x)\|_{C^k(\overline{U};\mathbb{R}^m)} + [f]_{\alpha,U} \tag{2.30}$$

Remark. (not discussed in class)
(i) Similarly one can define $C^{0,\alpha}(A;Y)$ where Y is a normed space.
(ii) Hölder continuous functions with exponent $\alpha = 1$ are called Lipschitz

continuous and we write $\mathrm{Lip}(f, A) := [f]_{1,A}$. Often the set A is dropped from the notation.
(iii) If $U \subset \mathbb{R}^n$ is open, $f : U \to \mathbb{R}^m$ and $[f]_{\alpha,U} < \infty$ for some $\alpha > 1$ then f is differentiable at each point $x \in U$ and $Df(x) = 0$. If, in addition U is connected then f is constant in U. This is why we only consider Hölder exponents $\alpha \leq 1$.
(iv) In analogy with Theorem 2.8 we can define a metric on the set $C^{k,\alpha}_{loc}(U) := \{f \in C^{k,\alpha}(U) : [D^k f]_{\alpha,K_i} < \infty \ \forall i\}$.

Proposition 2.14. *The expression $[\cdot]_{\alpha,A}$ is a seminorm on $C^{0,\alpha}(A;\mathbb{R}^m)$ and the spaces $C^{0,\alpha}(A;\mathbb{R}^m)$ and $C^{k,\alpha}(U;\mathbb{R}^m)$ are Banach spaces with the norms $\|\cdot\|_{\alpha,A}$ and $\|\cdot\|_{k,\alpha,U}$.*

Remark. The space $C^{0,\alpha}(A;\mathbb{R})$ is *not* separable (unless A is finite).

Proof. The assertions about the seminorms and norms are easy to prove.
Completness of $C^{0,\alpha}(A;\mathbb{R}^m)$: see homework sheet 3.
Completeness of $C^{k,\alpha}(\overline{U};\mathbb{R}^m)$ is shown similarly. □

2.2 L^p spaces and the Lebesgue integral

Here we very quickly recall from Analysis 3 important features of the Lebesgue integral and the L^p spaces .

Definition 2.15. *Let X be a set. Then $\mathcal{S} \subset 2^X$ is a σ-algebra if*

(i) $\emptyset \in \mathcal{S}, X \in \mathcal{S}$;

(ii) $A \in \mathcal{S} \implies X \setminus A \in \mathcal{S}$;

(iii) $\forall k \in \mathbb{N} \quad A_k \in \mathcal{S} \implies \bigcup_{k\in\mathbb{N}} A_k \in \mathcal{S}$.

An arbitrary intersection of σ-algebras is a σ-algebra and 2^X is always a σ-algebra. Hence the smallest σ-algebra which contains a given subset of 2^X is well defined. If $(X, \mathcal{T})$ is a topological space then the Borel-algebra $\mathcal{B}(X)$ is defined as the smallest σ-algebra which contains $\mathcal{T}$.

It is interesting to compare the definition of a σ-algebra and a topology. A σ-algebra is closed under *countable union* and complement and thus under *countable intersection.* A topology is closed under *arbitrary union* and *finite intersection.*

Definition 2.16. *Let $\mathcal{S}$ be a σ-algebra on X, $\mu : \mathcal{S} \to [0, \infty]$. The map μ is called a measure on $\mathcal{S}$ if :*

(i) $\mu(\emptyset) = 0$;

(ii) μ is σ-additive, i.e., if $A : \mathbb{N} \to \mathcal{S}$ and $A_h \cap A_k = \emptyset$ for all $h \neq k$ then

$$\mu\left(\bigcup_{h\in\mathbb{N}} A_h\right) = \sum_{h\in\mathbb{N}} \mu(A_h)\,. \tag{2.31}$$

In this case the triple $(X, \mathcal{S}, \mu)$ is called a measure space.

The elements of $\mathcal{S}$ are called measurable sets. An element $A \in \mathcal{S}$ is called null set if $\mu(A) = 0$. A measure space is called complete if every subset B of a null set belongs to $\mathcal{S}$ (then necessarily B is also a null set).

The measure μ is called σ-finite if there exist countably many sets $A_k \in \mathcal{S}$ such that $X = \bigcup A_k$ and $\mu(A_k) < \infty$.

Here we use the usual extended arithmetic on $[0, \infty]$. We say the a properties holds almost everywhere (or a.e., for short) in X if there exists a null set N, such that the property holds in $X \setminus N$.

Examples. (i) Let $\mathcal{M}_n$ denote the Lebesgue measurable subsets of $\mathbb{R}^n$ and let $\mathcal{L}^n$ denote the Lebesgue measure. Then $(\mathbb{R}^n, \mathcal{M}_n, \mathcal{L}^n)$ is a complete measure space.
(ii) Let # denote the counting measure. Then $(\mathbb{N}, 2^{\mathbb{N}}, \#)$ is a measure space.

In the following we will always assume that μ is σ-finite.

Let $(X, \mathcal{S}, \mu)$ be a measure space and let $(Y, \mathcal{T})$ be a topological space. We say that $f : X \to Y$ is measurable if $f^{-1}(U) \in \mathcal{S}$ for all $U \in \mathcal{T}$ ('preimages of open sets are measurable') [2]. One can easily check that a map $f : X \to \mathbb{R}$ or $f : X \to [-\infty, \infty]$ is measurable if and only if the sets $\{x : f(x) > a\}$ are measurable for all $a \in (-\infty, \infty)$ and that a map $f : X \to \mathbb{R}^m$ or $f : X \to [-\infty, \infty]^m$ is measurable if and only if all the component maps are measurable

If $(X, \mathcal{S}, \mu)$ is a measure space and $E \in \mathcal{S}$ then we define the characteristic function by

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E \end{cases} \tag{2.32}$$

and we define $\int_X \chi_E \, d\mu = \mu(E)$. One can see easily that any measurable function $f : X \to [0, \infty)$ can be uniformly approximated by functions of the form $\sum_{i=0}^{\infty} a_i \chi_{E_i}$ with $E_i \in \mathcal{S}$. This allows one to define $\int f \, d\mu$ as a number in $[0, \infty]$ and this definition can be extended to measurable functions

[2]Here I follow Def. 2.3.2. in H. Federer, *Geometric measure theory* and not H.W. Alt, *Lineare Funktionalanalysis*. H.W. Alt requires in addition that there exists a μ null set N such that $f(X \setminus N)$ is separable. For us the difference does not matter since $\mathbb{R}^m$ and $[-\infty, \infty]^m$ are separable, so the extra condition is empty.

$f : X \to [0, \infty]$. We say that a function $f : X \to [-\infty, \infty]$ is integrable if it is measurable and if its positive and negative part, $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$, both have finite integral. Then one defines $\int_X f\,d\mu = \int_X f^+\,d\mu - \int_X f^-\,d\mu$. For measurable functions $f : X \to \mathbb{R}^m$ one can define the integral componentwise. Then the integral has the usual properties. In addition for this (Lebesgue) integral one has the following three powerful convergence theorems.

Theorem 2.17 (Beppo Levi, monotone convergence)**.** *Let* $(X, \mathcal{S}, \mu)$ *be a measure space,* $E \in \mathcal{S}$ *and assume that for all* $k \in \mathbb{N}$ $f_k : E \to [0, \infty]$ *is measurable and* $f_k \le f_{k+1}$*. Then*

$$\lim_{k\to\infty} \int_E f_k\,d\mu = \int_E \lim_{k\to\infty} f_k\,d\mu\,. \tag{2.33}$$

The assumption $f_k \ge 0$ can be replaced by $f_k \ge g$ for an integrable function g (proof: consider $f_k - g$).

Theorem 2.18 (Fatou, lower semicontinuity of the integral)**.** *Let* $(X, \mathcal{S}, \mu)$ *be a measure space. Assume that* $E \in \mathcal{S}$ *and for all* $k \in \mathbb{N}$ *the functions* $f_k : E \to [0, \infty]$ *are measurable. Then*

$$\int_E \liminf_{k\to\infty} f_k\,d\mu \le \liminf_{k\to\infty} \int_E f_k\,d\mu\,. \tag{2.34}$$

Again it suffices to assume $f_k \ge g$, for some integrable g.

Theorem 2.19 (Lebesgue, dominated convergence theorem)**.** *Let* $(X, \mathcal{S}, \mu)$ *be a measure space. Assume that* $E \in \mathcal{S}$ *and that for all* $k \in \mathbb{N}$ *the functions* $f_k : E \to [-\infty, \infty]$ *are measurable. Suppose that there exists a null set* N *and* $f : E \setminus N \to \mathbb{R}$ *such that*

$$f_k(x) \to f(x) \quad \forall x \in E \setminus N. \tag{2.35}$$

Suppose further that there exists $g : E \to [0, \infty]$ *integrable, such that*

$$|f_k(x)| \le g(x) \quad \forall k \in \mathbb{N}\ \forall x \in E. \tag{2.36}$$

Then

$$\int_E f\,d\mu = \lim_{k\to\infty} \int_E f_k\,d\mu \tag{2.37}$$

and

$$\lim_{k\to\infty} \int_E |f_k - f|\,d\mu = 0\,. \tag{2.38}$$

The function g is often called an integrable majorant.

Another fundamental result is Fubini's theorem. In Analysis 3 we proved this for Lebesgue measure on $\mathbb{R}^n$.

Theorem 2.20 (Fubini)**.** *Let $f : \mathbb{R}^{n+m} \to [-\infty, \infty]$ be integrable and for $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ define*

$$g_x : \mathbb{R}^m \to [-\infty, \infty], \quad h_y : \mathbb{R}^n \to [-\infty, \infty] \tag{2.39}$$

by

$$g_x(y) = h_y(x) = f(x, y). \tag{2.40}$$

Then there exists an $\mathcal{L}^n$ null set $N_1 \subset \mathbb{R}^n$ and an $\mathcal{L}^m$ null set $N_2 \subset \mathbb{R}^m$ such that

(i) The function g_x is $\mathcal{L}^m$-integrable for $x \in \mathbb{R}^n \setminus N_1$ and

$$x \mapsto \int_{\mathbb{R}^m} f(x, y)\, d\mathcal{L}^m(y) := \int_{\mathbb{R}^m} g_x\, d\mathcal{L}^m \tag{2.41}$$

is $\mathcal{L}^n$ integrable;

(ii) the function h_y is $\mathcal{L}^n$-integrable for $y \in \mathbb{R}^m \setminus N_2$ and

$$y \mapsto \int_{\mathbb{R}^n} f(x, y)\, d\mathcal{L}^n(x) := \int_{\mathbb{R}^n} h_y\, d\mathcal{L}^n \tag{2.42}$$

is $\mathcal{L}^m$ integrable;

(iii)

$$\int_{\mathbb{R}^n \times \mathbb{R}^m} f\, d\mathcal{L}^{n+m} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} f(x, y)\, d\mathcal{L}^m(y)\ d\mathcal{L}^n(x) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} f(x, y)\, d\mathcal{L}^n(x)\ d\mathcal{L}^m(y). \tag{2.43}$$

There exists a partial converse: if f is $\mathcal{L}^{n+m}$ measurable and one of the iterated integrals $\int_{\mathbb{R}^n} \int_{\mathbb{R}^m} |f(x, y)|\, d\mathcal{L}^m(y)\ \mathcal{L}^n(x)$ or $\int_{\mathbb{R}^m} \int_{\mathbb{R}^n} |f(x, y)|\, d\mathcal{L}^n(x)\ \mathcal{L}^m(y)$ is finite then all the integrals in (2.43) exist and equality holds. It is not sufficient that only the maps g_x and h_y are measurable (see Analysis 3, Satz 3.39 and the warning after Satz 3.25).

Definition 2.21. *Let $(X, \mathcal{S}, \mu)$ be a measure space and let $E \in \mathcal{S}$. For a measurable function $f : E \to \mathbb{R}^m$ we define*

$$\|f\|_{L^p} := \left(\int_E |f|^p\, d\mu \right)^{1/p} \qquad \text{if } p \in [1, \infty), \tag{2.44}$$

$$\|f\|_{L^\infty} := \operatorname{ess\,sup}_E |f| := \inf\{M \in [0, \infty) : \mu\{x \in E : |f(x)| > M\} = 0\} \tag{2.45}$$

For $p \in [1, \infty]$ we set

$$L_p(E) := \{f : E \to \mathbb{R}^m : f \text{ measurable }, \|f\|_p < \infty\} \tag{2.46}$$

and we denote by $L^p(E)$ the corresponding equivalence classes of functions with respect to the equivalence relation

$$f \sim g \quad \Leftrightarrow \quad f = g \text{ a.e.} \tag{2.47}$$

A more precise notation is $L_p(E, \mathcal{S}, \mu; \mathbb{R}^m)$ since measurability depends on the σ-algebra S and the norm depends on μ. It will, however, usually be clear which σ-algebra, which measure and which target space we consider. Note that $\|f\|_{L^p} = \|g\|_{L^p}$ if $f \sim g$. Hence we can define $\|[f]\|_{L^p} := \|f\|_{L^p}$ The elements of L_p are called p-integrable functions.

If μ is the counting measure, defined on all subsets of $\mathbb{N}$ then we see that $L_p(\mathbb{N}) = l_p$.

Theorem 2.22. *Let $p \in [1, \infty]$. The expression $\| \cdot \|_p$ is a seminorm on $L_p(E)$ and a norm on $L^p(E)$. If the measure space $(X, \mathcal{S}, \mu)$ is complete then the spaces $L^p(E)$ are complete, i.e., Banach spaces.*

In particular the spaces $L^p(E)$ of Lebesgue measurable, p-integrable functions are Banach spaces. The completeness of the L^p spaces is known as the Fischer-Riesz theorem.

A fundamental estimate (which can be used to prove that $\| \cdot \|_p$ satisfies the triangle inequality) is Hölder's inequality.

Theorem 2.23 (Hölder). *Let $p, q \in (1, \infty)$ with*

$$\frac{1}{p} + \frac{1}{q} = 1\,, \tag{2.48}$$

or $\{p, q\} = \{1, \infty\}$. Let $f \in L_p(E)$, $g \in L_q(E)$. Then $fg \in L_1(E)$ and

$$\|fg\|_1 \leq \|f\|_p \|g\|_q\,. \tag{2.49}$$

We will often use the fact that L^p functions on (subsets of) $\mathbb{R}^n$ can be approximated in the L^p norm by continuous or smooth functions.

Theorem 2.24. *Let $p \in [1, \infty)$ and let $U \subset \mathbb{R}^n$ be open. Then $C_c^0(U)$ is dense in $L_p(U)$.*

Proof. See Analysis 3, Satz 4.20.. The main idea is to approximate f by linear combinations of characteristic functions and Lebesgue measurable sets by open and compact sets. □

Remark. $C(U) \cap L_\infty(U)$ is *not* dense in $L_\infty(U)$. Example: take $U = (-1, 1)$ and $f(x) = \operatorname{sgn} x$. If $g \in C(U)$ and $\sup g \geq 1/2$ and $\inf g \leq -1/2$ then there exists a non-empty open set V (and hence a set of positive measure) such that $g(V) \subset (-1/2, 1/2)$. Hence $\|f - g\|_\infty \geq 1/2$ for each continuous g.

For the approximation by smooth functions the notion of convolution is crucial.

Definition 2.25 (Convolution). *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be measurable. Let*

$$N = \{x \in \mathbb{R}^n : y \mapsto f(y)g(x - y) \text{ is not integrable }\}\,. \tag{2.50}$$

Then the convolution $f * g : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$(f * g)(x) = \begin{cases} \int_{\mathbb{R}^n} f(y)g(x-y)d\mathcal{L}^n(y) & \text{if } x \notin N \\ 0 & \text{if } x \in N. \end{cases} \tag{2.51}$$

One easily sees that $f * g = g * f$.
Moreover $\tilde{f} = f$ a.e. and $\tilde{g} = g$ a.e. implies that $(\tilde{f} * \tilde{g})(x) = (f * g)(x)$ for all $x \in \mathbb{R}^n$.

Theorem 2.26. *Let $p \in [1, \infty]$ and $f \in L_1(\mathbb{R}^n)$, $g \in L_p(\mathbb{R}^n)$. Then the set N in (2.50) is an $\mathcal{L}^n$ null set, $f * g \in L_p(\mathbb{R}^n)$ and*

$$\|f * g\|_{L_p(\mathbb{R}^n)} \leq \|f\|_{L_1(\mathbb{R}^n)} \|g\|_{L_p(\mathbb{R}^n)} . \tag{2.52}$$

Proof. For $p = 1$ see Analysis 3, Satz 4.24., for general p, see Analysis 3, Satz 4.25. The proof of Theorem 2.26 for $p = 1$ relies on a linear change of variable $(x, y) \mapsto (x, y - x)$ and Fubini's theorem. □

Lemma 2.27. *Let $f, g, h \in L_1(\mathbb{R}^n)$. Then*

(i) $(f * g) * h = f * (g * h)$ *a.e.*

(ii) If, in addition $h \in L_\infty(\mathbb{R}^n)$ and $(Sf)(x) := f(-x)$ then

$$\int_{\mathbb{R}^n} (f * g)h \, d\mathcal{L}^n = \int_{\mathbb{R}^n} f(Sg * h) \, d\mathcal{L}^n = \int_{\mathbb{R}^n} g(Sf * h) \, d\mathcal{L}^n. \tag{2.53}$$

(iii) If $g \in C_c^k(\mathbb{R}^n)$ then $N = \emptyset$, $f * g \in C^k(\mathbb{R}^n)$ and for $k \geq 1$ we have

$$\partial^\alpha (f * g) = f * \partial^\alpha g \tag{2.54}$$

for all multiindices α with $|\alpha| \leq k$.

Proof. See Analysis 3, Lemma 4.26.
For the first equality in (ii) one uses the identity $f(y)g(x-y)h(x) = f(y)Sg(y-x)h(x)$ and Fubini's theorem and the second identity is proved similarly. □

[27.10. 2017, Lecture 6]

[3.11. 2017, Lecture 7]

Lemma 2.28. *(i) $\exists\, \eta \in C_c^\infty(B(0,1))$ with $\eta \geq 0$ and*

$$\int_{\mathbb{R}^n} \eta \, dx = \int_{B(0,1)} \eta \, dx = 1. \tag{2.55}$$

(ii) $\exists\, \eta \in C_c^\infty(B(0,1))$ with $0 \leq \eta \leq 1$ and $\eta(x) = 1$ if $|x| \leq \frac{1}{2}$.
We may in addition assume that η is radially symmetric.

Proof. Analysis 3, Lemma 4.27. □

Theorem 2.29. *Let $p \in [1, \infty)$.*

(i) Let $\varphi \in L_1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \varphi \, d\mathcal{L}^n = 1$. Set $\psi_k(x) := k^n \varphi(kx)$. Then

$$\psi_k * f \to f \quad \text{in} \quad L_p(\mathbb{R}^n) \quad \forall f \in L_p(\mathbb{R}^n)\,. \tag{2.56}$$

(ii) Let $U \subset \mathbb{R}^n$ be open and let $p \in [1, \infty)$. Then $C_c^\infty(U)$ is dense in $L_p(U)$.

Proof. See Analysis 3, Lemma 4.28.
The idea is to prove (i) first for $\tilde{f} \in C_c(\mathbb{R}^n)$.Then (i) follows for $f \in L_p(\mathbb{R}^n)$ from Theorem 2.24 since one can bound the error terms $\psi_k * (f - \tilde{f})$ using Theorem 2.26.
For (ii) one first approximates $f \in L_p(U)$ by $\tilde{f} \in C_c(U)$ using Theorem 2.24. Then we take $\varphi \in C_c^\infty(B(0,1))$. It now suffices to note that $\psi_k * \tilde{f} \in C_c^\infty(U)$ if $k > 1/\delta$, where $\delta = \operatorname{dist}(\operatorname{supp} f, \mathbb{R}^n \setminus U) = \min_{x \in \operatorname{supp} f} \operatorname{dist}(x, \mathbb{R}^n \setminus U) > 0$. □

Let $U \subset \mathbb{R}^n$ be open. We say that $f \in L_{1,loc}(U)$ if $f \in L_1(K)$ for all compact subsets $K \subset U$.

Lemma 2.30. *Let $U \subset \mathbb{R}^n$ be open and let $f \in L_{1,loc}(U)$.*

(i) If ψ_k is as in Theorem 2.29 then $\psi_k * f \to f$ in $L_{1,loc}(U)$. More precisely for every compact set $K \subset U$ there exists $k_0(K)$ such that for $k \geq k_0(K)$ the function $y \mapsto \psi_k(y) f(x - y)$ is integrable for $x \in K$ and

$$\psi_k * f \to f \quad \text{in } L_1(K). \tag{2.57}$$

(ii) If

$$\int_U f \varphi \, d\mathcal{L}^n = 0 \quad \forall \varphi \in C_c^\infty(U) \tag{2.58}$$

then $f = 0$ a.e. in U.

Proof. (i) Let $\delta := \operatorname{dist}(K, \mathbb{R}^n \setminus U)$. Then $\delta > 0$ since K is compact. Set $K' := \{x \in \mathbb{R}^n : \operatorname{dist}(x, K) \leq \delta/2\}$. Then K' is compact and $K' \subset U$. Set $g = f$ in K' and $g = 0$ in $\mathbb{R}^n \setminus K'$. Then $g \in L_1(\mathbb{R}^n)$. If $k \geq 2/\delta$ then $\psi_k * f(x) = \psi_k * g(x)$ for all $x \in K$. Hence the assertion follows from Theorem 2.29.

(ii) Let $K \subset U$ be compact. If we set $\varphi(y) = \psi_k(x - y)$ we see that $f * \psi_k = 0$ in K if k is large enough. Thus by (i) we have $f = 0$ a.e. in K. The assertion follows since U can be written as a countable union of compact subsets of U. □

2.3 Sobolev spaces

Motivation: Consider the space $C^1([0,1])$. On this space there is a natural scalar product given by

$$(f,g) = \int_0^1 f(x)g(x) + f'(x)g'(x)\,dx. \tag{2.59}$$

It is, however, easy to see that the space is not complete under the induced norm $\|f\|_{H^1} := (\|f\|_{L^2}^2 + \|f'\|_{L^2}^2)^{1/2}$, just as $C([0,1])$ is not complete in the L^2 norm.

To overcome this problem there are two possibilities. First one can consider the abstract completion of C^1 in $\|\cdot\|_{H^1}$ and then try to identify the objects in the completion with usual functions. The space obtained in this way was originally called $H^{1,2}$.

Secondly, one can weaken the notion of derivative to allow functions whose derivatives in the weak sense are only L^2 functions and show that the resulting space is complete in the H^1 norm. The space obtained in this way was originally called $W^{1,2}$.

We will see that both approaches actually lead to the same space, i.e. $W^{1,2} = H^{1,2}$. To prove this we will follow the second approach and then show that C^1 (in fact C^∞) is dense in $W^{1,2}$.

2.3.1 Definition and completeness

In the following we always assume

$$U \subset \mathbb{R}^n \quad \text{is open.} \tag{2.60}$$

If other conditions, e.g., boundedness are imposed on U we will state this explicitly.

Definition 2.31. *We say that $f \in L_{1,loc}(U)$ is weakly differentiable if there exist functions $g_1, \ldots, g_n$ in $L_{1,loc}$ such that*

$$\int_U f \partial_i \varphi\,dx = -\int_U g_i \varphi\,dx \quad \forall \varphi \in C_c^\infty(U). \tag{2.61}$$

We say that f is k times weakly differentiable if for all multiindices $\alpha \in \mathbb{N}^n$ with $|\alpha| \le k$ there exist $g^\alpha \in L^1_{loc}(U)$ such that

$$\int_U f \partial^\alpha \varphi\,dx = (-1)^{|\alpha|} \int_U g^\alpha \varphi\,dx \quad \forall \varphi \in C_c^\infty(U). \tag{2.62}$$

Remark. (i) The functions g_i and g^α are called weak derivatives and are still denoted by $\partial_i f$ and $\partial^\alpha f$, respectively. If they exist they are unique by Lemma 2.30 up to sets of measure zero, i.e. they correspond to a unique equivalence class in L^1_{loc}. In particular the weak derivative and the usual derivative agree if $g \in C^k(U)$.
(ii) The weak derivatives depend only on the equivalence class of f.

In view of (i) and (ii) it makes sense that an equivalence class $[f] \in L^1_{loc}$ is weakly differentiable and has weak derivatives in $[g^\alpha] \in L^1_{loc}$. In the following we will usually make no distinction in notion between functions and their equivalence classes.

Examples. (i) Let $U = (-1,1)$ and $f(x) = |x|$. Then f is weakly differentiable and the weak derivative is $f'(x) = \operatorname{sgn}(x)$.
(ii) Let $U = (-1,1)$ and $f(x) = \operatorname{sgn}(x)$. Then f is *not* weakly differentiable. Indeed $\int_U f\varphi'\,dx = -2\varphi(0)$ for all $\varphi \in C_c^\infty(U)$, but there is no $g \in L^1_{loc}(U)$ such that $\int_U g\varphi\,dx = \varphi(0)$ for all $\varphi \in C_c^\infty(U)$.
(iii) Let $U = \mathbb{R}^n$ and $f = \chi_{B(0,1)}$. Then f is not weakly differentiable. Indeed $\int_{B(0,1)} f\partial_i\varphi\,dx = \int_{\partial B(0,1)} \varphi\nu_i\,d\mathcal{H}^{n-1}$.
(iv) Let $U = B(0,1) \subset \mathbb{R}^n$, $n \geq 2$, $\alpha \in \mathbb{R} \setminus \{0\}$ and $f(x) = |x|^\alpha$ for $x \neq 0$, $f(0) = 0$. Then f is weakly differentiable if and only if $\alpha > -(n-1)$. To see this consider first $\int_{B(0,1)\setminus B(0,\varepsilon)} f\partial_i\varphi\,dx$ and then pass to the limit $\varepsilon \to 0$.

¿From these examples we see that weakly differentiable functions cannot jump across (smooth) hypersurfaces but they may have singularities on lower dimensional sets. For $n \geq 2$ they can be discontinuous and unbounded.

Definition 2.32. *Let $1 \leq p \leq \infty$ and $k \in \mathbb{N} \setminus \{0\}$. The Sobolev space $W^{k,p}(U)$ consists of all $f \in L^p(U)$ which are k times weakly differentiable with all weak derivatives in $L^p(U)$. We define for $1 \leq p < \infty$*

$$\|f\|_{W^{k,p}(U)} := \left(\sum_{l=0}^{k} \int_U |D^l f|^p\right)^{1/p} = \left(\sum_{l=0}^{k} \|\,|D^l f|\,\|^p_{L^p(U)}\right)^{1/p} \tag{2.63}$$

and for $p = \infty$

$$\|f\|_{W^{k,\infty}(U)} := \max_{l=0,\dots,k} \|\,|D^l f|\,\|_{L^\infty(U)}. \tag{2.64}$$

Recall that $|D^l f(x)|^2 := \sum_{|\alpha|=l} |\partial^\alpha f(x)|^2$.
H.W. Alt uses the equivalent norm $\|f\|^p_{\sim,k,p} := \sum_{|\alpha|\leq k} \int_U |\partial^\alpha f|^p\,d\mathcal{L}^n$.

Theorem 2.33. *The pair $(W^{k,p}(U), \|\cdot\|)$ is a Banach space. The space $W^{k,2}(U)$ is a Hilbert space with scalar product*

$$(f,g)_{W^{k,2}(U)} := \sum_{|\alpha|\leq k} (\partial^\alpha f, \partial^\alpha g)_{L^2(U)}. \tag{2.65}$$

Proof. It is easy to see that $\|\cdot\|_{W^{k,p}}$ is a norm and $(\cdot,\cdot)_{W^{k,2}}$ is a scalar product. Let $f : \mathbb{N} \to W^{k,p}(U)$ be a Cauchy sequence. Then for any multiindex α with $|\alpha| \le k$ the sequence $\partial^\alpha f$ is a Cauchy sequence in $L^p(U)$ and hence has a limit $g^\alpha \in L^p(U)$. In particular $f_j \to g$ in $L^p(U)$. To show completeness we only need to show that g is weakly differentiable and the weak derivatives of g are given by g^α.

By the definition of the weak derivative we have

$$\int_U f_j \partial^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_U \partial^\alpha f_j \varphi \, dx. \tag{2.66}$$

Since φ and $\partial^\alpha \varphi$ are bounded and have compact support they are in particular in $L^{p'}(U)$. Thus using Hölder's inequality we can pass to the limit $j \to \infty$ and get

$$\int_U g \partial^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_U g^\alpha \varphi \, dx. \tag{2.67}$$

This shows that g is k times weakly differentiable and the derivatives are given by g^α. □

[3.11. 2017, Lecture 7]

[8.11. 2017, Lecture 8]

2.3.2 Approximation by smooth functions and calculus rules

We next want to show that element of $W^{k,p}$ can be approximated in the $W^{k,p}$ norm by smooth functions. The key observation if the following: if f is k times weakly differentiable in $\mathbb{R}^n$ and $\eta \in C_c^\infty(\mathbb{R}^n)$ then $\eta * f$ is in $C^\infty(\mathbb{R}^n)$ and

$$\partial^\alpha(\eta * f) = \eta * \partial^\alpha f \quad \text{for all } \alpha \text{ with } |\alpha| \le k.$$

where $\partial^\alpha f$ denotes the *weak* derivative of f. We state a slightly more general version for weakly differentiable functions defined in a general open subset U of $\mathbb{R}^n$.

Lemma 2.34. *Let $k \ge 1$, let $p \in [1, \infty]$, let $f \in W^{k,p}(U)$ and denote by $f\chi_U$ the function which agrees with f in U and is zero in $\mathbb{R}^n \setminus U$.*

Let $\eta \in C_c^\infty(B(0,1))$ with $\eta \ge 0$ and $\int_{\mathbb{R}^n} \eta \, d\mathcal{L}^n = 1$ and set $\eta_j(x) := j^n \eta(jx)$ and

$$U_j := \{x \in U : \operatorname{dist}(x, \mathbb{R}^n \setminus U > \frac{1}{j})\}. \tag{2.68}$$

Then $\eta_j * (f\chi_U) \in C^\infty(\mathbb{R}^n)$ and

$$\partial^\alpha(\eta * (f\chi_U)) = \eta * \partial^\alpha f \quad \text{in } U_j \quad \text{if } |\alpha| \le k, \tag{2.69}$$

where $\partial^\alpha f$ denotes the weak derivative of f.
If $U = \mathbb{R}^n$ then $f\chi_U = f$ and (2.68) holds with $U_j = \mathbb{R}^n$.

The functions η_j are sometimes called mollifiers (or mollifying kernels) and the sequence $j \mapsto \eta_j$ is sometimes called a Dirac sequence.

Proof. By Lemma 2.27 we have $\eta_j * (f\chi_U) \in C^\infty(\mathbb{R}^n)$ and

$$\partial_\alpha(\eta_j * (f\chi_U))(x) = \partial^\alpha((f\chi_U) * \eta_j)(x) \tag{2.70}$$

$$= \int_U f(y)\partial^\alpha \eta_j(x-y)\, dy \tag{2.71}$$

Now fix $x \in U_j$ and set $\varphi(y) := \eta_j(x-y)$. Then $\varphi \in C_c^\infty(U)$ and $\partial^\alpha \varphi = (-1)^{|\alpha|}(\partial^\alpha \eta_j)(x-y)$. Thus the definition of the weak derivative implies that

$$\int_U f(y)\partial^\alpha \eta_j(x-y)\, dy = \int_U \partial^\alpha f(y)\partial^\alpha \eta_j(x-y)\, dy = \eta_j * ((\partial^\alpha f)\chi_U)(x). \tag{2.72}$$

□

Theorem 2.35. *Let $p \in [1,\infty)$, $k \in \mathbb{N}$. Then $C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$;*

Proof. For $k = 0$ this follows from Theorem 2.29. Let $k \geq 1$ and let η_j be as in Lemma 2.34 and set $f_j := \eta_j * f$. By Lemma 2.34 and Theorem 2.29 we have $f_j \in C^\infty(\mathbb{R}^n)$ and

$$\partial^\alpha f_j = \eta_j * \partial^\alpha f \to \partial^\alpha f \quad \text{in } L^p(\mathbb{R}^n). \tag{2.73}$$

Thus $f_j \to f$ in $W^{k,p}(\mathbb{R}^n)$. □

Theorem 2.36. *Let $p \in [1,\infty)$, $k \in \mathbb{N}$. Then $C^\infty(U) \cap W^{k,p}(U)$ is dense in $W^{k,p}(U)$.*

Proof. The proof combines the previous argument with a smooth partition of unity. We only give a sketch of the argument.

Step 1: Partition of unity.
Define for $i \in \mathbb{N}$, $i \geq 1$,

$$U_i := \{x \in U : |x| < i,\ \operatorname{dist}(x, \mathbb{R}^n \setminus U) > \frac{1}{i}\}. \tag{2.74}$$

Then U_i is open and the closure $\overline{U_i}$ is given by $\{x \in U : |x| \leq i,\ \operatorname{dist}(x, \mathbb{R}^n \setminus U) \geq \frac{1}{i}\}$. Hence $\overline{U_i} \subset U_{i+1} \subset U$ and $\overline{U_i}$ is compact. Moreover $\bigcup_{i=1}^\infty U_i = U$.

Consider now the open sets $V_i := U_{i+3} \setminus \overline{U_i}$ and the compact sets $K_i := \overline{U_{i+2}} \setminus U_{i+1}$. Set $K_0 := \overline{U_2}$ and $V_0 := U_3$. Then $\bigcup_{i=0}^\infty K_i = U$ and each point $x \in U$ is contained in at most three of the sets V_i.

Now there exist $h_i \in C_c^\infty(V_i)$ with $h_i = 1$ on K_i and $h_i \geq 0$ (see Analysis 3, Lemma 5.10). We define

$$\psi_i(x) := \frac{h_i(x)}{\sum_{j=0}^\infty h_j(x)} \tag{2.75}$$

For $x \in U_k$ and $k \geq 3$. we have $h_j(x) = 0$ if $j \geq k$. Thus the sum in the denominator converges for each $x \in U$ and is a smooth function in each set U_k and hence in U. In addition the sum in the denominator is always ≥ 1 in U since each $x \in U$ lies in at least one set K_j. Thus

$$\psi_i \in C_c^\infty(V_i), \quad \psi_i \geq 0, \quad \sum_{i=0}^{\infty} \psi_i(x) = 1 \quad \forall x \in U. \tag{2.76}$$

Step 2: Local approximation.
Now let $\varepsilon > 0$ and $f \in W^{k,p}(U)$. We set

$$f_i(x) := \psi_i(x) f(x) \quad \text{for } x \in V_i, \qquad f_i(x) = 0 \quad \text{for } x \in \mathbb{R}^n \setminus V_i. \tag{2.77}$$

Then it is easy to see that $f_i \in W^{k,p}(\mathbb{R}^n)$. Let η_j be as in the proof of Theorem 2.35. Then $\eta_j * f_i \to f_i$ in $W^{k,p}(\mathbb{R}^n)$ as $j \to \infty$. Since $\operatorname{supp} \psi \subset V_i$ is compact it follows that $\eta_j * f_i$ has compact support in V_i if $j \geq j_i$. Thus there exist

$$\exists g_i \in C^\infty(V_i), \quad \|f_i - g_i\|_{W^{k,p}(\mathbb{R}^n)} \leq 2^{-i-2}\varepsilon. \tag{2.78}$$

Step 3: Conclusion. Set

$$g(x) := \sum_{j=0}^{\infty} g_j(x). \tag{2.79}$$

If $x \in U_i$ and $i \geq 3$ then $g_j(x) = 0$ for all $j \geq i$ since $V_j \cap U_i = \emptyset$ for $j \geq i$. Hence the sum converges for all $x \in U_i$ and g is C^∞ in the open set U_i as a finite sum of C^∞ functions. Thus $g \in C^\infty(U)$. Similarly we see that $f = \sum_{j=0}^{i} f_j$ in U_i. This yields (for $i \geq 3$)

$$\|f - g\|_{W^{k,p}(U_i)} \leq \sum_{j=0}^{i} \|f_j - g_j\|_{W^{k,p}} \leq \frac{\varepsilon}{2} \tag{2.80}$$

Thus can be rewritten as

$$\sum_{l=0}^{k} \int_{U_i} |D^l(f - g)|^p \, d\mathcal{L}^n \leq \frac{\varepsilon^p}{2^p}. \tag{2.81}$$

It follows from the monotone convergence theorem that

$$\|f - g\|_{W^{k,p}(U)}^p = \sum_{l=0}^{k} \int_{U} |D^l(f - g)|^p \, d\mathcal{L}^n \leq \frac{\varepsilon^p}{2^p}. \tag{2.82}$$

This finishes the proof. □

Lemma 2.37 (Product rule). *Let $p, q \in [1, \infty]$ with*

$$\frac{1}{p} + \frac{1}{q} = 1 \tag{2.83}$$

(with the convention $\frac{1}{\infty} = 0$). Let $f \in W^{1,p}(U)$, $g \in W^{1,q}(U)$. Then $fg \in W^{1,1}(U)$ and the weak derivatives satisfy

$$\partial_i(fg) = \partial_i f\, g + f \partial_i g. \tag{2.84}$$

Remark. (i) The analogous assertion holds for higher derivatives.
(ii) The assumptions can be a bit weakened. For example, if U is a bounded interval $(a, b) \subset \mathbb{R}$ then it suffices that $f, g \in W^{1,1}((a, b))$. More generally, it suffices that f and g are weakly differentiable, $fg \in L^1(U)$ and $\partial_i f\, g + f \partial_i g \in L^1(U)$ for all i (this can be proved using the one dimensional result and the characterization of $W^{1,1}$ using restrictions of the function to a.e. line segment, see the remark after Theorem 2.42).

Proof. We have $p \neq \infty$ or $q \neq \infty$. We may assume $p < \infty$, since otherwise we can exchange f and g.

Assume first that in addition $f \in C^\infty(U)$. Then the usual product an the definition of the weak derivative imply the assertion since for all $\varphi \in C_c^\infty(U)$

$$\int_U fg \partial_i \varphi \, d\mathcal{L}^n = \int_U g \partial_i (f\varphi) \, d\mathcal{L}^n - \int_U g \partial_i f \, \varphi \, d\mathcal{L}^n \tag{2.85}$$

$$= -\int_U \partial_i g \, f\varphi \, d\mathcal{L}^n - \int_U g \partial_i f \, \varphi \, d\mathcal{L}^n. \tag{2.86}$$

If $f \in W^{1,p}(U)$ set $f_j := \eta_j * (f\chi_U)$. Then it follows from Lemma 2.34 and Theorem 2.29 that $f_j \to f$ and $\partial_i f_j \to \partial_i f$ in $L^p(K)$ for every compact set $K \subset U$. Thus

$$\int_U f_j g \partial_i \varphi \, d\mathcal{L}^n \to \int_U fg \partial_i \varphi \, d\mathcal{L}^n, \tag{2.87}$$

$$\int_U f_j \partial_i g \varphi \, d\mathcal{L}^n \to \int_U f \partial_i g \varphi \, d\mathcal{L}^n, \tag{2.88}$$

$$\int_U \partial_i f_j \, g\varphi \, d\mathcal{L}^n \to \int_U \partial_i f \, g\varphi \, d\mathcal{L}^n. \tag{2.89}$$

This finishes the proof. □

Lemma 2.38 (Chain rule). *Let $f \in C^1(\mathbb{R})$ and assume that f' is bounded. Let $p \in [1, \infty]$ and $g \in W^{1,p}(U)$. Then $f \circ g$ is in $L^p_{loc}(U)$, is weakly differentiable, the weak derivatives belong to L^p and are given by*

$$\partial_i(f \circ g) = (f' \circ g)\, \partial_i g. \tag{2.90}$$

Moreover, if U has finite Lebesgue measure or if $f(0) = 0$ of if $p = \infty$ then $f \circ g \in W^{1,p}(U)$.

Remark. The analogous results holds for $g : U \to \mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}^d$.

Proof. Let $M := \sup_{x\in\mathbb{R}} |f'|(x)$. Then $|f(x)-f(y)| \le M|x-y|$ and $|f(x)| \le |f(0)| + M|x|$. This shows that $f \circ g$ is in $L^p_{loc}(U)$ and moreover in $L^p(U)$ if $f(0) = 0$ or if U has finite Lebesgue measure or if $p = \infty$.

Let $g_j := \eta_j * (g\chi_U)$. Let $\varphi \in C^\infty_c(U)$ and $K = \operatorname{supp}\varphi$. By Lemma 2.34 and Lemma 2.30 we have for each compact set $K \subset U$

$$g_j \to g, \quad \partial_i g_j \to \partial_i g \quad \text{in } L^1(K). \tag{2.91}$$

By the usual chain rule we get for each $\varphi \in C^\infty_c(U)$

$$\int_U f \circ g_j \, \partial_i\varphi \, d\mathcal{L}^n = -\int_U (f' \circ g_j)\, \partial_i g_j \, \varphi \, d\mathcal{L}^n. \tag{2.92}$$

Let $K = \operatorname{supp}\varphi$. Then there exists a subsequence such that $g_{j_k} \to g$ a.e. in K. Thus

$$f' \circ g_{j_k}\, \partial_i g \, \varphi \to f' \circ g \, \partial_i g \, \varphi \quad \text{a.e.}, \quad |f' \circ g_{j_k}\, \partial_i g| \, \varphi \le M \sup|\varphi||g'|. \tag{2.93}$$

Hence by the dominated convergence theorem

$$\int_U f' \circ g_{j_k}\, \partial_i g \, \varphi \, d\mathcal{L}^n \to \int_U f' \circ g \, \partial_i g \, \varphi \, d\mathcal{L}^n. \tag{2.94}$$

On the other hand

$$\left| \int_U f' \circ g_{j_k} \, (\partial_i g_{j_k} - \partial_i g) \, \varphi \, d\mathcal{L}^n \right| \le M \sup|\varphi| \int_K |\partial_i g_{j_k} - \partial_i g| \, d\mathcal{L}^n \to 0. \tag{2.95}$$

Hence

$$\int_U f' \circ g_{j_k}\, \partial_i g_{j_k} \, \varphi \, d\mathcal{L}^n \to \int_U f' \circ g \, \partial_i g \, \varphi \, d\mathcal{L}^n. \tag{2.96}$$

On the other hand $|f \circ g_{j_k} - f \circ g| \le M|g_{j_k} - g|$ and therefore

$$\int_U f \circ g_{j_k}\, \partial_i\varphi \, d\mathcal{L}^n \to \int_U f \circ g \, \partial_i \varphi \, d\mathcal{L}^n. \tag{2.97}$$

Thus $f \circ g$ is weakly differentiable with weak derivative $(f' \circ g)\partial_i g$. Since f' is bounded the weak derivative is in $L^p(U)$ □

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Corollary 2.39. *Let $f \in W^{1,p}(U)$. Then the functions*

$$f^+ := \max(f, 0), \quad f^- := \max(-f, 0) \quad \textit{and } |f| = f^+ - f^- \tag{2.98}$$

are also in $W^{1,p}$ and the weak derivatives are given by

$$\partial_i f^+ = \chi_{E^+}\partial_i f, \quad \partial_i f^- = -\chi_{E^-}\partial_i f, \quad \partial_i |f| = \chi_{E^+}\partial_i f - \chi_{E^-}\partial_i f, \tag{2.99}$$

where

$$E^+ := \{x \in U : f(x) > 0\}, \quad E^- := \{x \in U : f(x) < 0\}. \tag{2.100}$$

Proof. Exercise. Hint: if suffices to show the result for f^+. For this show first that there exist functions $h_k \in C^1(\mathbb{R})$ with $|h_k(t) - t^+| \le \frac{1}{k}$, $|h_k'| \le 1$ and $h_k'(t) \to 1$ if $t > 0$ and $h_k'(t) \to 0$ if $t \le 0$. □

Corollary 2.40. *Let $p \in [1, \infty]$, $f \in W^{1,p}(U)$. Let $E \subset U$ be measurable and let $a \in \mathbb{R}$. Then*

$$f = a \quad \text{a.e. on } E \quad \Longrightarrow \quad \partial_i f = 0 \quad \text{a.e. on } E. \tag{2.101}$$

Proof. Exercise. Hint: use Corollary 2.39. □

Remark. The map $f \mapsto f^+$ is continuous in $W^{1,p}(U)$, i.e. $f_j \to f$ in $W^{1,p}(U)$ implies that $f_j^+ \to f^+$ in $W^{1,p}(U)$ (exercise). Hint: use that $\lim_{j\to 0} \mathcal{L}^n\{x \in U : f_j(x) > 0,\ f(x) \le -\frac{1}{k}\} = 0$ and show that$\lim_{k\to\infty} \int_{A_k} |\partial_i f| \, d\mathcal{L}^n = 0$, where $A_k = \{x \in U : \frac{1}{k} \le f \le 0\}$.
It follows that the maps $(f, g) \mapsto \max(f, g)$ and $(f, g) \mapsto \min(f, g)$ are also continuous in $W^{1,p}(U)$.

2.3.3 Sobolev functions in one dimension

We finally show that in one dimension every element on $W^{1,1}$ has a unique continuous representative. In fact elements of $W^{1,1}$ are absolutely continuous in the following sense.

Definition 2.41. *Let $(a, b) \subset \mathbb{R}$ be a bounded interval. A map $f : [a, b] \to \mathbb{R}$ is called absolutely continuous if there exists $g \in L^1((a, b))$ such that*

$$F(x) = F(a) + \int_a^x g(y)\, dy \quad \forall x \in [a, b]. \tag{2.102}$$

Theorem 2.42. *Let $I = (a, b) \subset \mathbb{R}$ be a bounded interval. Then $f \in W^{1,1}(I)$ if and only if f has an absolutely continuous representative $\tilde{f} : [a, b] \to \mathbb{R}$. In that case*

$$\tilde{f}(x) = c + \int_a^x f'(z)\, dz \quad \forall x \in [a, b] \tag{2.103}$$

Proof. Homework. Hint: If F satisfies (2.102) one can use Fubini's theorem to show that F is weakly differentiable with weak derivative g and hence belongs to $W^{1,1}(I)$. Conversely if $f \in W^{1,1}(I)$ is given one can define

$$F(x) := \int_a^x f'(y)\,dy.$$

Then $f - F$ is in $W^{1,1}(I)$ and has weak derivative zero. It only remains to show that this implies that $f - F$ equals a constant c a.e. (one can use mollification to prove this). Then one can set $\tilde{f} = c + F$. □

Remark. One can show by measure-theoretic methods that every absolutely continuous function is (classically) differentiable at a.e. point in (a, b). The classical derivative and the weak derivative agree a.e.

Remark. There exists a similar characterization for functions defined on an open subset $U \subset \mathbb{R}^n$. We have $f \in W^{1,1}(U)$ if and only if there exists a representative $\tilde{f}$ such that for every cube $Q = \prod(a_i, b_i) \subset U$ the maps $x_i \mapsto \tilde{f}(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$ belongs to $W^{1,1}((a_i, b_i))$ (and hence are absolutely continuous) for $\mathcal{L}^{n-1}$ a.e. choice of the other coordinates x_i'. Moreover for $\mathcal{L}^{n-1}$ a.e. x_i' and all $x_i, y_i \in (a, b)$ we have

$$\begin{aligned} &\tilde{f}(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) \\ &= \int_{y_i}^{x_i} \partial_i f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)\,dt, \end{aligned} \tag{2.104}$$

where $\partial_i f$ is (a representative of) the weak derivative.

Corollary 2.43. *Let $I = (a, b) \subset \mathbb{R}$ be a bounded interval. Let $p \in (1, \infty]$ and $\alpha := 1 - \frac{1}{p}$ (with the convention $\frac{1}{\infty} = 0$). Let $f \in W^{1,p}(I)$. Then the representative $\tilde{f}$ in Theorem 2.42 satisfies $\tilde{f} \in C^{0,\alpha}(\overline{I})$ and*

$$[\tilde{f}]_\alpha \leq \|f'\|_{L^p}. \tag{2.105}$$

Remark. Using Rademacher's theorem (Lipschitz functions are a.e. differentiable and absolutely continuous) one can show that $f \in W^{1,\infty}(I)$ if and only if f has a Lipschitz continuous representative.

Remark. (i) Using the usual identification of functions and representatives one often writes $C^{0,\alpha}(I) \subset W^{1,p}(I)$ and $C^{0,1}(I) = W^{1,\infty}(I)$.
(ii) The Cantor function (also known as the 'devil's staircase') is in $C^{0,\alpha}$ (with $\alpha = \frac{\ln 2}{\ln 3}$) and differentiable a.e. with derivative 0, but *not* absolutely continuous and hence *not* in $W^{1,1}$.

Proof. Homework. □

2.3.4 Boundary values of Sobolev functions

Definition 2.44. *Let $p \in [1, \infty)$. The space $W_0^{k,p}(U)$ is defined as the closure of $C_c^\infty(\mathbb{R}^n)$ in $W^{k,p}(U)$. For $p = \infty$ we define*

$$W_0^{k,\infty}(U) := \{f \in W^{k,\infty}(U) : \exists f_j \in C_c^\infty(U),\ \sup_j \|f_j\|_{W^{k,\infty}} < \infty,$$
$$\partial^\alpha f_j \to \partial^\alpha f \ \text{in}\ L^1_{loc}(U)\}. \tag{2.106}$$

We think of $W_0^{k,p}(U)$ as the Sobolev space of functions with zero boundary values.

Example.

(i) If $f \in W^{k,p}(U)$ and if there exists a compact set $K \subset U$ such that $f = 0$ a.e. in $U \setminus K$ then $f \in W_0^{k,p}(U)$. To see this denote by Ef the extension of f by zero outside U, consider $f_j := \eta_j * (Ef)$ and note that $f_j \in C_c^\infty(U)$ if j is large enough and $f_j \to Ef$ in $W^{k,p}(\mathbb{R}^n)$ (if $p < \infty$; for $p = \infty$ we have the convergence required in the definition of $W_0^{k,\infty}$).

(ii) Let $f \in W_0^{k,p}(U)$ and denote by Ef the extension of f by zero outside U. Then $f \in W^{k,p}(\mathbb{R}^n)$.
Proof: Clearly for $f \in C_c^\infty(U)$ we have $Ef \in C_c^\infty(\mathbb{R}^n)$ and $\|Ef\|_{W^{k,p}(\mathbb{R}^n)} = \|f\|_{W^{k,p}(U)}$. If $p < \infty$ let $f_j \in C_c^\infty(U)$ with $f_j \to f$ in $W^{k,p}(U)$. Then Ef_j is a Cauchy sequence in $W^{k,p}(\mathbb{R}^n)$ and thus $Ef_j \to g \in W^{k,p}(\mathbb{R}^n)$. On the other hand $Ef_j \to Ef$ in $L^p(\mathbb{R}^n)$. Thus $f = g$ (in the sense of equivalence classes) and hence $f \in W^{k,p}(\mathbb{R}^n)$.
Added Jan 18, 2013 A similar argument applies for $p = \infty$.

(iii) $W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$, i.e., $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$ (idea: consider $f_j(x) = \varphi(\frac{x}{j})f(x)$ with $\varphi \in C_c^\infty(B(0,1))$ and $\varphi = 1$ in $B(0, \frac{1}{2})$; details: exercise).

Remark. Let $I = (a, b)$ be a bounded interval, $p \in [1, \infty]$. Then

$$f \in W_0^{1,p}(I) \quad \Longleftrightarrow \quad \tilde{f}(a) = \tilde{f}(b) = 0, \tag{2.107}$$

where $\tilde{f}$ denotes the absolutely continuous representative of f (see Theorem 2.42).

3 Subsets of function spaces: convexity and compactness

3.1 Convexity and best approximation

Definition 3.1. *Let X be a $\mathbb{K}$ vector space.*

(i) A set $A \subset X$ is convex if

$$x, y \in A,\ \alpha \in (0,1) \quad \Longrightarrow (1-\alpha)x + \alpha y \in A. \tag{3.1}$$

(ii) Let $A \subset X$ be convex. A function $f : A \to \mathbb{R} \cup \{+\infty\}$ is convex if

$$\forall x, y \in A\ \forall \alpha \in (0,1) \quad f((1-\alpha)x + \alpha y) \le (1-\alpha)f(x) + \alpha f(y). \tag{3.2}$$

(iii) For an arbitrary set $A \subset X$ the convex hull $\operatorname{conv} A$ *is defined as*

$$\operatorname{conv} A := \{\sum_{i=1}^{k} \alpha_i x_i : k \in \mathbb{N} \setminus \{0\},\ x_i \in A,\ \alpha_i \ge 0, \sum_{i=1}^{k} \alpha_i = 1\}. \tag{3.3}$$

One easily sees that $\operatorname{conv} A$ is the smallest convex set which contains A.

Theorem 3.2 (Projection theorem)**.** *Let X be a Hilbert space, let $A \subset X$ be non-empty, convex and closed. Then there exists one and only one map $P : X \to A$ such that*

$$\|x - P(x)\|_X = \operatorname{dist}(x, A) := \inf_{y \in A} \|x - y\|_X \tag{3.4}$$

for all $x \in X$. The value $P(x)$ is equivalently characterized by the condition

$$\operatorname{Re}(x - P(x), a - P(x)) \le 0 \quad \forall a \in A. \tag{3.5}$$

The map $P : X \to A$ is called the orthogonal projection from X to A.

Proof. Step 1: Uniqueness of $P(x)$.
Set $m := \inf_{y \in A} \|x - y\|_X$ and assume that there exist $a, b \in A$ such that

$$\|a - x\| = \|b - x\| = m. \tag{3.6}$$

It follows from the triangle inequality and the homogeneity of the norm that $\|\frac{a+b}{2} - x\| \le m$. We now use the parallelogram identity to show that the inequality is strict if $a \ne b$. Since A is convex we get

$$m^2 \le \|\frac{a+b}{2} - x\|^2 = \|\frac{a-x}{2} + \frac{b-x}{2}\|^2 = \frac{1}{2}\|a-x\|^2 + \frac{1}{2}\|b-x\|^2 - \|\frac{a-b}{2}\|^2 \tag{3.7}$$

$$\le m^2 - \|\frac{a-b}{2}\|^2. \tag{3.8}$$

Thus $a = b$.

Step 2: Existence of $P(x)$.
By definition of m there exist $a_k \in A$ such that $\|a_k - x\|^2 \le m^2 + \frac{1}{k}$ for $k \in \mathbb{N} \setminus \{0\}$. Since A is convex we have $\|\frac{a_k+a_j}{2} - x\| \ge m$. Hence the parallelogram identity gives

$$m^2 \le m^2 + \frac{1}{2k} + \frac{1}{2j} - \|\frac{a_k - a_j}{2}\|^2. \tag{3.9}$$

Thus $j \mapsto a_j$ is a Cauchy sequence and since X is complete there exists $a \in X$ such that $a_j \to a_*$ in X. Since A is closed we have $a_* \in A$. Finally the continuity of the norm implies that $\|a_* - x\| \le m$. By the definition of m we must have equality and we set $P(x) = a_*$.

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Step 3: Characterization of $P(x)$.
To see that (3.4) implies (3.5) let $\lambda \in (0,1)$. Since A is convex we have $(1-\lambda)P(x) + \lambda a \in A$ and thus

$$\begin{aligned} &\|x - P(x)\|^2 \le \|x - [(1-\lambda)P(x) + \lambda a]\|^2 \; = \|x - P(x) - \lambda(a - P(x))\|^2 \\ =&\|x - P(x)\|^2 - 2\operatorname{Re}(x - P(x), \lambda(a - P(x))) + \|\lambda(a - P(x))\|^2. \end{aligned} \tag{3.10}$$

Subtract $\|x - P(x)\|^2$ on both sides, divide by $\lambda > 0$ and consider the limit $\lambda \downarrow 0$. This gives (3.5).

Conversely assume that (3.5) holds. Then for all $a \in A$

$$\begin{aligned} \|x - a\|^2 &= \|(x - P(x)) + (P(x) - a)\|^2 && (3.11)\\ &= \|x - P(x)\|^2 + 2\underbrace{\operatorname{Re}(x - P(x), P(x) - a)}_{\ge 0} + \|P(x) - a\|^2 && (3.12)\\ &\ge \|x - P(x)\|^2. && (3.13) \end{aligned}$$

Thus (3.4) holds. □

Corollary 3.3 (Projection onto a subspace)**.** *Let X be a Hilbert space and let $Y \subset X$ be a closed subspace. Then there exists one and only one map $P : X \to Y$ with*

$$\|x - P(x)\| = \operatorname{dist}(x, Y). \tag{3.14}$$

This map is linear and it is equivalently characterized by the condition

$$(x - P(x), y) = 0 \quad \forall y \in Y. \tag{3.15}$$

Notation: If Y is a subspace we define the orthogonal space by

$$Y^\perp := \{x \in X : (x, y) = 0 \ \forall y \in Y\}. \tag{3.16}$$

We have

$$Y \text{ closed} \quad \Longrightarrow \quad (Y^\perp)^\perp = Y. \tag{3.17}$$

Clearly $Y \subset (Y^\perp)^\perp$. Assume that $z \in (Y^\perp)^\perp \setminus Y$ and let P as in the Corollary. Then $z - Pz \in Y^\perp$ by (3.15). At the same time $z - Pz \in (Y^\perp)^\perp$ since $Y \subset (Y^\perp)^\perp$. Thus $z - Pz = 0$, i.e., $z \in Y$, a contradiction.

Proof. Theorem 3.2 implies the existence and uniqueness of $P(x)$. Moreover $P(x)$ is characterized by the condition

$$\mathrm{Re}(P(x) - x, a - P(x)) \le 0 \quad \forall a \in Y. \tag{3.18}$$

Let $y \in Y$. If X is a real Hilbert space application of (3.18) with $a = P(x) \pm y$ implies (3.15). If X is a complex Hilbert space application of (3.18) with $a = P(x) \pm y$ and $a = P(x) \pm iy$ implies (3.15). Conversely (3.15) always implies (3.18).

Finally linearity of P follows from (3.15) and uniqueness. Indeed if $x_1, x_2 \in X$ and $\lambda \in \mathbb{K}$ then (3.15) implies that

$$(x_1 + x_2 - P(x_1) + P(x_2), y) = 0 \quad , (\lambda x_1 - \lambda P(x_1), y) = 0 \quad \forall y \in Y \tag{3.19}$$

and by uniqueness we get $P(x_1 + x_2) = P(x_1) + P(x_2)$ and $P(\lambda x) = \lambda P(x)$. □

Example 3.4. *Let $U \subset \mathbb{R}^n$ be open and bounded, let $v \in W^{1,2}(U)$. Let*

$$A := v + W_0^{1,2}(U) := \{u \in W^{1,2}(U) : u = v + w,\ w \in W_0^{1,2}(U)\} \tag{3.20}$$

and let

$$(f, g) := \int_U \sum_{i=1}^n \partial_i f \partial_i g \, d\mathcal{L}^n, \quad [f]^2 := (f, f) = \int_U |\nabla f|^2 \, d\mathcal{L}^n. \tag{3.21}$$

Note that $(\cdot, \cdot)$ is a positive semidefinite symmetric bilinear form and $[\cdot]$ is a seminorm on $W^{1,2}$. We claim that there exists a unique $\bar{u} \in A$ such that

$$[\bar{u}] = \mathrm{dist}\,(0, A) = \inf_{u \in A} [u] \tag{3.22}$$

and

$$0 = (\bar{u}, w) = \int_U \sum_{i=1}^n \partial_i \bar{u} \, \partial_i w \, d\mathcal{L}^n \quad \forall w \in W_0^{1,2}(U). \tag{3.23}$$

This does not directly follow from Theorem 3.2 since $(\cdot, \cdot)$ is only positive semidefinite. The parallelogram identity, however, still holds and thus for any sequence $k \mapsto u_k \in A$ with $[u_k]^2 \le \mathrm{dist}^2(0, A) + \frac{1}{k}$ we get that

$$[\frac{u_j - u_k}{2}]^2 \le \frac{1}{2j} + \frac{1}{2k}. \tag{3.24}$$

Now $u_k - v \in W_0^{1,2}(U)$ *and hence it follows from the Poincaré inequality below that* $k \mapsto u_k - v$ *is a Cauchy sequence in* $W_0^{1,2}(U)$*. Thus* $u_j \to \bar{u}$ *in* $W^{1,2}(U)$ *and* $\bar{u} \in A$*. The condition* (3.23) *now follows as in Corollary 3.3.*

If $\bar{u} \in C^2(U) \cap C(\bar{U})$ *and* $v \in C(\bar{U})$ *then it follows from* (3.23) *and the definition of* A *that*

$$-\Delta u = 0 \quad \text{in } U, \qquad u = v \quad \text{on } \partial U, \tag{3.25}$$

i.e. u *is a classical solution of the Dirichlet problem.*

If $\bar{u} \in v + W_0^{1,2}(U)$ *and* (3.23) *holds we call* $\bar{u}$ *a* weak solution *of the Dirichlet problem. Note that in Einführung PDG we have shown that* (3.23) *implies that* $\bar{u} \in C^\infty(U)$ *and* $-\Delta \bar{u} = 0$ *in* U *(Weyl's lemma).*

Lemma 3.5 (Poincaré inequality)**.** *Let* $n \geq 1$*,* $p \in [1, \infty]$*. Let* $U \subset \mathbb{R}^n$ *be bounded and open and assume that* $U \subset \prod_{i=1}^n (a_i, a_i + L_i)$*. Then for* $f \in W_0^{1,p}(U)$ *one has*

$$\|f\|_{L^p(U)} \leq L_i \|\partial_i f\|_{L^p(U)}. \tag{3.26}$$

Proof. By density it suffices to prove this for $f \in C_c^\infty(U)$. By Fubini one can easily reduce the problem to the case $n = 1$ for which the assertion follows from the fundamental theorem of calculus. See Homework Sheet 4, Problem 3. □

Definition 3.6 (Uniformly convex spaces)**.** *A normed space* X *is called uniformly convex if*

$$\forall \varepsilon > 0 \; \exists \delta > 0 \quad \|x\| = 1, \|y\| = 1, \|\frac{x+y}{2}\| \geq 1 - \delta \quad \Longrightarrow \quad \|x - y\| \leq \varepsilon. \tag{3.27}$$

Example. (i) Let $p \in (1, \infty)$. Then $L^p(X, \mathcal{S}, \mu)$ is uniformly convex (Homework sheet 5, Problems 3 and 4). The spaces $L^1(X, \mathcal{S}, \mu)$ and $L^\infty(X, \mathcal{S}, \mu)$ are not uniformly convex (except in the trivial case when μ is concentrated on one point).
(ii) By the parallelogram identity every Hilbert space is uniformly convex (with $\delta = \varepsilon^2/2$).

Theorem 3.7. *Let* X *be a uniformly convex Banach space and let* $A \subset X$ *be non-empty, closed and convex. Then there exists one and only one map* $P : X \to A$ *such that*

$$\|x - P(x)\| = \operatorname{dist}(x, A). \tag{3.28}$$

Proof. Let $m = \operatorname{dist}(x, A)$. If $m = 0$ then $x \in A$ because A is closed. Thus $P(x) = x$. If $m > 0$ we proceed as in the Hilbert space case and use uniform convexity instead of the parallelogram identity to deduce convergence.

Let $a_k \in A$ be such that $m_k := \|a_k - x\| \le m + \frac{1}{k}$. Since A is convex we have $\|\frac{a_k+a_l}{2} - x\| \ge m$. Set

$$z_k = \frac{a_k - x}{m_k}. \tag{3.29}$$

Then

$$\frac{a_k + a_l}{2} - x = \frac{1}{2}(m_k z_k + m_l z_l) = m\frac{z_k + z_l}{2} + \frac{1}{2}(m_k - m)z_k + \frac{1}{2}(m_l - m)z_l. \tag{3.30}$$

Thus

$$\|\frac{z_k + z_l}{2}\| \ge 1 - \frac{1}{2km} - \frac{1}{2lm}. \tag{3.31}$$

It follows from the definition of uniform convexity that $k \mapsto z_k$ is a Cauchy sequence. Thus $z_k \to z$ in X and hence $a_k \to a$ in X and $a \in A$ since A is closed. Moreover $\|a - x\| = m$ since the norm is continuous. Uniqueness follows directly from strict convexity. □

Lemma 3.8 ('Almost orthogonal element')**.** *Let X be a normed space, let Y be a closed subspace with $Y \ne X$ and let $\theta > 0$. Then there exists $x_\theta \in X$ with*

$$\|x_\theta\| = 1, \quad \operatorname{dist}(x_\theta, Y) \ge 1 - \theta. \tag{3.32}$$

Remark. If X is a Hilbert space and x is orthogonal to Y, i.e., $(x, y) = 0$ for all $y \in Y$, then Corollary 3.3 implies that $\|x\| = \operatorname{dist}(x, Y)$ (since orthogonality implies $P(x) = 0$). In a general normed space there is no notion of orthogonality, but the element x_θ is almost as good as a vector orthogonal to Y in the sense that the ratio of the distance to Y to the norm is almost 1.

Proof. Let $z \in X \setminus Y$. Then $\operatorname{dist}(z, Y) > 0$ since Y is closed. Thus there exists $y \in Y$ such that $\|z - y\| \le \frac{1}{1-\theta}\operatorname{dist}(z, Y)$. Since Y is a linear space we have $\operatorname{dist}(z, Y) = \operatorname{dist}(z - y, Y)$. Now set $x_\theta = \frac{z-y}{\|z-y\|}$. □

3.2 Compactness

Theorem 3.9. *Let (X, d) be a metric space and let $A \subset X$. Then the following statements are equivalent.*

(i) A is compact, i.e. every cover of A by open sets contains a finite subcover.

(ii) A is sequentially compact, i.e. every sequence $x : \mathbb{N} \to A$ has a convergent subsequence whose limit is in A.

(iii) (A, d) is complete and A is precompact, i.e. for each $\varepsilon > 0$ there exists a finite number of ε balls which cover A.

Remark. The definition of 'precompact' in (iii) follows H.W. Alt's book. Some authors also call this property 'totally bounded'.

Proof. (i) $\Rightarrow$ (ii): If the sequence x contains no convergent subsequence with limit in A then for each $y \in A$ there exists an $r_y > 0$ such that the set $N_y := \{j \in \mathbb{N} : x_j \in B(y, r_y)\}$ is finite. The balls $B(y, r_y)$ form an open cover of A. Hence there exist finitely many balls such that

$$A \subset \bigcup_{i=1}^{k} B(y_i, r_{y_i}). \tag{3.33}$$

Thus $\mathbb{N} \subset \bigcup_{i=1}^{k} N_{y_i}$. This is a contradiction since the right hand side is a finite set.

(ii) $\Rightarrow$ (iii): We first show that (A, d) is complete. Let $x : \mathbb{N} \to A$ be a Cauchy sequence. By (ii) there exists a convergent subsequence whose limit x_* is in A. Since x is a Cauchy sequence, the whole sequence converges to x_*. Hence (A, d) is complete. Now let $\varepsilon > 0$ and assume that A cannot be covered by a finite number of ε balls. We inductively construct a sequence $x : \mathbb{N} \to A$ such that $d(x_j, x_k) \geq \varepsilon$ if $j \neq k$. Indeed if $x_1, \dots x_k$ are given with that property then

$$A \setminus \bigcup_{i=1}^{k} B(x_i, \varepsilon) \neq \emptyset. \tag{3.34}$$

Take $x_{k+1} \in A \setminus \bigcup_{i=1}^{k} B(x_i, \varepsilon)$. Then $d(x_{k+1}, x_j) \geq \varepsilon$ for $j \leq k$. The sequence x_k contains no convergent subsequence. This contradicts (ii).

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(iii) $\Rightarrow$ (i): Let Λ be an index set, let $U_\lambda \subset X$ be open for all $\lambda \in \Lambda$ and assume that $A \subset \bigcup_{\lambda \in \Lambda} U_\lambda$. Let $\mathcal{B}$ denote the set of all subsets of A which cannot be covered by finitely many of the sets U_λ, i.e.,

$$\mathcal{B} := \{B \subset A : \Lambda' \subset \Lambda,\, B \subset \bigcup_{\lambda \in \Lambda'} U_\lambda \implies \Lambda' \text{ infinite}\}. \tag{3.35}$$

We want to show $A \notin \mathcal{B}$.

Since A is precompact we have

$$B \in \mathcal{B} \quad \text{and } \varepsilon > 0 \quad \Longrightarrow \quad \exists x_1, \dots x_{k_\varepsilon} \quad A \subset \bigcup_{i=1}^{k_\varepsilon} B(x_i, \varepsilon) \tag{3.36}$$

$$\Longrightarrow \quad \exists i \quad B \cap B(x_i, \varepsilon) \in \mathcal{B} \tag{3.37}$$

Now suppose that $A \in \mathcal{B}$. Then application of the above implication with $\varepsilon = \frac{1}{k}$ shows that there exist $B_k \in \mathcal{B}$ such that $B_1 = A$ and $B_{k+1} =$

$B_k \cap B(x_k, \frac{1}{k+1})$. Now let $y_k \in B_k \subset A$. Then by construction $d(y_l, y_k) \leq \frac{2}{k}$ if $l \geq k$. Since A is complete there exists $y_* \in A$ such that $y_k \to y_*$ as $k \to \infty$. By assumption there exists $\overline{\lambda} \in \Lambda$ such that $y_* \in U_{\overline{\lambda}}$. Since $U_{\overline{\lambda}}$ is open there exists $\delta > 0$ such that $B(y_*, \delta) \subset U_{\overline{\lambda}}$. Thus for k large enough

$$B_k \subset B(x_k, \frac{1}{k}) \subset B(y_k, \frac{2}{k}) \subset B(y_*, \delta) \subset U_{\overline{\lambda}}. \tag{3.38}$$

This contradicts the fact that $B_k \in \mathcal{B}$. □

Proposition 3.10. *Let (X, d) be a metric space and let $A \subset X$. Then the following assertions holds.*

(i) Subsets of precompact sets are precompact.

(ii) A precompact $\implies$ A bounded

(iii) A precompact $\implies$ $\overline{A}$ precompact and closed.

(iv) A compact $\implies$ A closed.

(v) If (X, d) is complete then

$$A \text{ precompact} \iff \overline{A} \text{ compact.} \tag{3.39}$$

(vi) If $X = \mathbb{K}^n$ with the standard norm then

$$A \subset \mathbb{K}^n \text{ precompact} \iff \overline{A} \text{ bounded.} \tag{3.40}$$

(vii) (Heine-Borel property) If $X = \mathbb{K}^n$ with the standard norm then

$$A \subset \mathbb{K}^n \text{ compact} \iff A \text{ bounded and closed.} \tag{3.41}$$

(viii) If $A, A_i \subset X$ and $\delta_i > 0$ for $i \in \mathbb{N}$ with $\lim_{i\to\infty} \delta_i \to 0$ then

$$\forall i \in \mathbb{N} \quad A \subset B_{\delta_i}(A_i) \text{ and } A_i \text{ precompact} \implies A \text{ precompact.} \tag{3.42}$$

(ix) Let (X, d_X) and (Y, d_Y) be metric spaces and assume that $f : X \to Y$ is continuous. Then

$$A \subset X \quad \text{compact} \implies f(A) \subset Y \quad \text{compact.} \tag{3.43}$$

(x) Let (X, d_X) and (Y, d_Y) be metric spaces and assume that $f : X \to Y$ is uniformly continuous. Then

$$A \subset X \quad \text{precompact} \implies f(A) \subset Y \quad \text{precompact.} \tag{3.44}$$

Remark. In (x) it is not enough to assume that f is continuous. Example $f(x) = \frac{x}{1+|x|}$ is a continuous map from $(-1,1)$ to $\mathbb{R}$ and $(-1,1)$ is precompact, but $\mathbb{R}$ is not.

Proof. Properties (i)–(iv) follow directly from the definition of precompactness and Theorem 3.9. Properties (vi) and (vii) were proved in Analysis I.
(v) $\Longleftarrow$: by Theorem 3.9 $\overline{A}$ is precompact and the assertion follows from (i).
(v) $\Longrightarrow$: By (iii) $\overline{A}$ is precompact and closed. Thus $(\overline{A}, d)$ is complete by Proposition 1.26. Now use Theorem 3.9.
(viii): Let $\varepsilon > 0$. Let $i \in \mathbb{N}$ such that $\delta_i \leq \varepsilon/2$. Since A_i is precompact there exist finitely many $x_1, \ldots, x_m \in X$ such that

$$A_i \subset \bigcup_{j=1}^{m} B(x_i, \varepsilon/2), \qquad \text{and hence } A \subset \bigcup_{j=1}^{m} B(x_i, \varepsilon). \tag{3.45}$$

(ix): Assume that $f(A) \subset \bigcup_\lambda V_\lambda$ and V_λ open. Then $U_\lambda := f^{-1}(V_\lambda)$ is open and $A \subset \bigcup_\lambda U_\lambda$. Since A is compact there exist finitely many λ_i with $A \subset \bigcup_{i=1}^k U_{\lambda_i}$. Thus $f(A) \subset \bigcup_{i=1}^k V_{\lambda_i}$.
(x): Let $\varepsilon > 0$. By uniform continuity there exists $\delta > 0$ such that $d(x, x') < \delta$ implies $d(f(x), f(x')) < \varepsilon$. Since A is precompact there exist finitely many balls $B(x_i, \delta)$ which cover A. Thus the balls $B(f(x_i), \varepsilon)$ cover $f(A)$. □

Lemma 3.11. *Let X be a finite dimensional $\mathbb{K}$ vector space. Then all norms of on X are equivalent. In particular if $\|\cdot\|$ is any norm on X, then $(X, \|\cdot\|)$ is complete.*

Proof. Let $n = \dim X$ and let $\{e_1, \ldots, e_n\}$ be a basis of X. Then the map

$$x \mapsto h(x) = \sum_{i=1}^{n} x_i e_i \tag{3.46}$$

is a bijective linear map from $\mathbb{K}^n$ to X. If $\|\cdot\|$ is a norm on X then

$$|x| := \|h(x)\| \tag{3.47}$$

is a norm in $\mathbb{K}^n$ and the map h is an isometry from $(\mathbb{K}^n, |\cdot|)$ to $(X, \|\cdot\|)$. Hence it suffices to show all norms on $\mathbb{K}^n$ are equivalent and that $(\mathbb{K}^n, |\cdot|)$ is complete.

Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{K}^n$, let $|\cdot|$ be an arbitrary norm on $\mathbb{K}^n$ and let $|x|_2 = (\sum |x_i|^2)^{1/2}$ denote the Euclidean norm. By the triangle inequality and the Cauchy Schwarz inequality in $\mathbb{R}^n$

$$|x| \leq \sum_{i=1}^{n} |x_i||e_i| \leq C|x|_2, \quad \text{with } C = (\sum_{i=1}^{n} |e_i|^2)^{1/2}. \tag{3.48}$$

From this it follows that the map $x \mapsto |x|$ is continuous (and even Lipschitz continuous) as a map from $(\mathbb{K}^n, |\cdot|_2)$ to $\mathbb{R}$. Let $S = \{x \in \mathbb{K}^n : |x|_2 = 1\}$. Then S is compact in $(\mathbb{K}^n, |\cdot|_2)$ and hence $|x|$ attains its minimum on S by Theorem 1.12. Thus

$$c = \inf\{|x| : x \in S\} = \min\{|x| : x \in S\} > 0 \tag{3.49}$$

since $|\cdot|$ is a norm. By the homogeneity of the norm this shows that

$$|x| \geq c|x|_2. \tag{3.50}$$

and thus $|\cdot|$ and $|\cdot|_2$ are equivalent. In particular $(\mathbb{K}^n, |\cdot|)$ is complete since $(\mathbb{K}^n, |\cdot|_2)$ is complete. □

Lemma 3.12. *Let $(X, \|\cdot\|)$ be a normed space and Y be a finite dimensional subspace. Then Y is complete and hence closed.*

Proof. Apply Lemma 3.11 to $(Y, \|\cdot\|)$. □

Theorem 3.13. *Let X be a normed space. Then*

$$\overline{B}(0,1) \quad \textit{compact} \quad \Longleftrightarrow \quad \dim X < \infty. \tag{3.51}$$

Remark. The assertion holds also for $\overline{B}(0,R)$ for any $R > 0$.

Proof. '$\Longleftarrow$': Let $n = \dim X$, let $\{e_1, \dots, e_n\}$ be a basis of X and let $h(x) := \sum_{i=1}^n x_i e_i$ and $|x| := \|h(x)\|$. Since $|\cdot|$ is equivalent to the Euclidean norm $|\cdot|_2$ on $\mathbb{K}^n$ the map $h : (\mathbb{K}^n, |\cdot|_2) \to (X, \|\cdot\|)$ is continuous and $K' := h^{-1}(\overline{B}(0,1))$ is closed and bounded in $(\mathbb{K}^n, |\cdot|_2)$. Thus K' is compact and therefore $\overline{B}(0,1) = h(K)$ is compact by Proposition 3.10 (ix).

'$\Longrightarrow$': Assume that $\dim X = \infty$. For $k \in \mathbb{N}$ we inductively construct $x_k \in \overline{B}(0,1)$ with $\|x_k\| = 1$ and $\|x_k - x_j\| \geq \frac{1}{2}$ if $k \neq j$. Thus $\overline{B}(0,1)$ cannot be covered by finitely many balls of radius $\frac{1}{4}$ and hence is not compact. Assume that $x_0, \dots x_K$ have already been constructed and let

$$Y := \operatorname{span}\{x_0, \dots, x_K\}. \tag{3.52}$$

Then Y is a finite dimensional subspace of X and hence closed by Lemma 3.12. Moreover $Y \neq X$ since $\dim X = \infty$. By Lemma 3.8 there exists $x_{K+1} \in X$ with $\|x_{K+1}\| = 1$ and

$$\operatorname{dist}(x_{K+1}, Y) \geq \frac{1}{2}. \tag{3.53}$$

Thus in particular $\|x_{K+1} - x_k\| \geq \frac{1}{2}$ for all $k \leq K$. □

Lemma 3.14. *Let (X, d) be a metric space and let $A \subset X$ be compact. Then for every $x \in X$ there exists an $a \in A$ such that*

$$d(x,a) = \operatorname{dist}(x, A) := \inf\{d(x,y) : y \in A\}. \tag{3.54}$$

Proof. Let $x \in X$. Then the map $y \mapsto d(x,y)$ is continuous. The assertion follows from Theorem 1.12. □

3.3 Compact sets in $C(S;Y)$ and $L^p(\mathbb{R}^n)$: the Arzela-Ascoli and Frechet-Kolmogorov-Riesz theorem

Theorem 3.15 (Arzela-Ascoli)**.** *Let (X,d) be a metric space and let $S \subset X$ be compact. Let Y be a Banach space. Then*

$$A \subset C(S;Y) \quad \textit{is precompact if and only if} \tag{3.55}$$

(i) (pointwise precompactness)

$$\forall x \in S \quad K_x := \{f(x) : f \in A\} \quad \textit{is precompact in } Y \textit{ and} \tag{3.56}$$

(ii) (equicontinuity)

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall f \in A \quad d(x,x') < \delta \quad \Longrightarrow \quad \|f(x) - f(x')\| < \varepsilon. \tag{3.57}$$

Example. Bounded sets in $C^{0,\alpha}(S;\mathbb{K}^m)$ are equicontinuous and precompact in $C(S;\mathbb{K}^m)$ (but in general not precompact in $C^{0,\alpha}(S;\mathbb{K}^m)$).

Remark. (i) Condition (3.56) and (3.57) together with the compactness of S imply that

$$K := \{f(x) : f \in A,\, x \in S\} = \bigcup_{x \in S} K_x \quad \text{is precompact in } Y. \tag{3.58}$$

(ii) The most frequently used case is $Y = \mathbb{K}^m$. In this case precompactness of K_x or K is the same as boundedness.
(iii) The assumption that S is compact is needed even if we use the stronger condition (3.58) and $Y = \mathbb{R}$. Let $\varphi \in C_c(-\frac{1}{2}, \frac{1}{2}))$ with $0 \le \varphi \le 1$ and $\varphi(0) = 1$ and let A be the set of integer translates of φ, i.e., $A = \{\varphi_k : \varphi_k(x) = \varphi(x-k),\, k \in \mathbb{Z}\}$. Then $\|\varphi_j - \varphi_k\| = 1$ if $j \ne k$. Hence A is not precompact.

Proof. '$\Longleftarrow$': Let $\varepsilon > 0$. Let $\delta > 0$ be as in the definition of equicontinuity. Since S is compact there exist finitely many balls $B(x_1,\delta), \ldots B(x_l,\delta)$ which cover S. Let $K' = \cup_{j=1}^{l} K_{x_j}$. Then K' is precompact as a finite union of precompact sets. Thus there exist finitely many balls $B(y_i,\varepsilon) \subset Y$, $i = 1, \ldots k$ which cover K'.

For each map $\sigma : \{1,\ldots,l\} \to \{1,\ldots,k\}$ define

$$A_\sigma := \{f \in A : \|f(x_j) - y_{\sigma(j)}\| < \varepsilon \quad \forall j = 1,\ldots,l\} \tag{3.59}$$

Then $\bigcup_\sigma A_\sigma = A$. For each A_σ which is not empty choose $f_\sigma \in A_\sigma$. Note that the number of maps σ is k^l and in particular finite.

Let $f \in A_\sigma$. Let $x \in S$. Then $x \in B(x_j, \delta)$ for some $j \in \{1, \dots, l\}$ and equicontinuity yields

$$\begin{aligned} \|f(x) - f_\sigma(x)\| &< \|f(x_j) - f_\sigma(x_j)\| + 2\varepsilon \\ &\le \|f(x_j) - y_{\sigma(x_j)}\| + \|y_{\sigma(x_j)} - f_\sigma(x_j)\| + 2\varepsilon \le 4\varepsilon. \end{aligned} \tag{3.60}$$

Thus

$$\forall f \in A_\sigma \quad \|f - f_\sigma\| := \sup_{x \in S} \|f(x) - f_\sigma(x)\| \le 4\varepsilon \tag{3.61}$$

and therefore $A_\sigma \subset B(f_\sigma, 5\varepsilon)$. Since $A = \bigcup_\sigma A_\sigma$ it follows that A can be covered by finitely many balls of radius 5ε. Since $\varepsilon > 0$ was arbitrary this finishes the proof.

'$\Longrightarrow$': Pointwise precompactness follows from the fact the map $f \mapsto f_x$ from $C(S; Y) \to Y$ is Lipschitz continuous and Proposition 3.10 (x).
To prove equicontinuity let $\varepsilon > 0$. Then there exists finitely many $f_1, \dots, f_k$ in A such that $A \subset \bigcup_{i=1}^k B(f_i, \varepsilon/3)$. Each f_i is a continuous function on a compact set S and hence uniformly continuous (see (2.11)). Thus there exist $\delta_i > 0$ such that

$$d(x, x') < \delta_i \quad \Longrightarrow \quad \|f_i(x) - f_i(x')\| < \varepsilon/3. \tag{3.62}$$

Let $\delta = \min_{i=1,\dots,k} \delta_i$ and assume that $d(x, x') < \delta$. If $f \in A$ there exists an i such that $\|f - f_i\| < \varepsilon/3$. Thus

$$\|f(x) - f(x')\| < \|f_i(x) - f_i(x')\| + \frac{2}{3}\varepsilon < \varepsilon. \tag{3.63}$$

This finishes the proof of (3.57). □

[17.11. 2017, Lecture 11]

[22.11. 2017, Lecture 12]

Theorem 3.16 (Frechet-Kolmogorov-M. Riesz)**.** *Let $p \in [1, \infty)$ and let $A \subset L^p(\mathbb{R}^n)$. Then A is precompact if and only if the following three conditions hold.*

(i) (L^p boundedness) $\quad \sup_{f \in A} \|f\|_{L^p(\mathbb{R}^n)} < \infty$;

(ii) (L^p equicontinuity)

$$\lim_{h \to 0} \sup_{f \in A} \|f(\cdot + h) - f(\cdot)\|_{L^p(\mathbb{R}^n)} = 0; \tag{3.64}$$

(iii) (tightness/ no escape to infinity)

$$\lim_{R \to \infty} \sup_{f \in A} \int_{\mathbb{R}^n \setminus B(0,R)} |f|^p \, d\mathcal{L}^n = 0. \tag{3.65}$$

Lemma 3.17. *Let $F \subset \mathbb{R}^n$ be measurable and $A \subset L^p(F)$. Define the extension operator E by*

$$(Ef)(x) := \begin{cases} f(x) & x \in F, \\ 0 & x \in \mathbb{R}^n \setminus F. \end{cases} \tag{3.66}$$

Then A is precompact in $L^p(F)$ if and only if EA is precompact in $L^p(\mathbb{R}^n)$.

Proof. This follows from Proposition 3.10 (x) and the fact that the extension operator $E : L^p(F) \to L^p(\mathbb{R}^n)$ and the restriction operator $R : L^p(\mathbb{R}^n) \to L^p(F)$ given by $f \mapsto f_{|F}$ are Lipschitz continuous and $REA = A$. $\square$

Example. Let $p \in [1, \infty)$ and let $U \subset \mathbb{R}^n$ be bounded. Then bounded sets in $W_0^{1,p}(U)$ are precompact in $L^p(U)$.

Proof: Let $A := \{f \in W_0^{1,p} : \|f\|_{W^{1,p}} \leq M\}$ and let E denote the extension operator $L^p(U) \to L^p(\mathbb{R}^n)$ (see Lemma 3.17). If $f \in W_0^{1,p}(U)$ then $Ef \in W^{1,p}(\mathbb{R}^n)$ (see Example (ii) after Definition 2.44). By Lemma 3.17 it suffices to show that EA is precompact in $L^p(\mathbb{R}^n)$. Now we have for every $g \in W^{1,p}(\mathbb{R}^n)$

$$\|g(\cdot + h) - g(\cdot)\|_{L^p(\mathbb{R}^n)} \leq |h| \, \|\nabla g\|_{L^p(\mathbb{R}^n)}. \tag{3.67}$$

Indeed, for $g \in C_c^1(\mathbb{R}^n)$ this follows from the identity

$$g(x+h) - g(x) = \int_0^1 Dg(x+th)h \, dt,$$

Jensen's inequality and Fubini's theorem applied to $\int_{\mathbb{R}^n} \int_0^1 |Dg(x+th)|^p \, dt \, dx$. For general $g \in W^{1,p}(\mathbb{R}^n)$ the assertion follows by density of $C_c^1(\mathbb{R}^n)$. Applying (3.67) to Ef we see that condition (ii) in Theorem 3.16 is satisfied. Moreover (i) holds since $\|Ef\|_{L^p(\mathbb{R}^n)} \leq M$ for all f and (iii) is trivially satisfied since U is bounded and $Ef = 0$ on $\mathbb{R}^n \setminus U$.

Proof of Theorem 3.16, general strategy. The main point is to verify that properties (i), (ii) and (iii) imply precompactness of A. To do so we show that for each $\delta > 0$ there exists

$$A_\delta \subset L^p(\mathbb{R}^n) \quad \text{precompact} \qquad \text{with } A \subset B_\delta(A_\delta). \tag{3.68}$$

Then the precompactness of A follows from Proposition 3.10 (viii).

To construct A_δ we modify f by truncation and convolution. Then we will use the Arzela-Ascoli theorem to show that the modified functions form a precompact set in $C^0(\overline{B}(0,R))$ and hence in $L^p(B(0,R))$. Conditions (ii) and (iii) guarantee that f and its modification differ only by δ in the L^p norm. $\square$

We will use the following result.

Lemma 3.18. *Let $p \in [1,\infty]$ and $f \in L^p(\mathbb{R}^n)$. Let $\varphi \in L^1(\mathbb{R}^n)$ with* $\operatorname{supp}\varphi \subset B(0,r)$*, $\varphi \geq 0$ and $\int_{\mathbb{R}^n} \varphi \, d\mathcal{L}^n = 1$. Then*

$$\|\varphi * f - f\|_{L^p(\mathbb{R}^n)} \leq \sup_{|h|\leq r} \|f(\cdot + h) - f(\cdot)\|_{L^p(\mathbb{R}^n)}. \tag{3.69}$$

Proof. Since $\int_{\mathbb{R}^n} \varphi \, d\mathcal{L}^n = 1$ we have

$$\varphi * f(x) - f(x) = \int_{\mathbb{R}^n} \varphi(y)(f(x-y) - f(x)) \, dy \tag{3.70}$$

and thus

$$|\varphi * f(x) - f(x)| \leq \int_{\mathbb{R}^n} \varphi(y)|(f(x-y) - f(x))| \, dy. \tag{3.71}$$

For $p = 1$ the assertion now follows from Fubini's theorem. For $1 < p < \infty$ set $\frac{1}{q} = 1 - \frac{1}{p}$ and write $\varphi(y) = \varphi(y)^{\frac{1}{q}} \varphi(y)^{\frac{1}{p}}$ and apply Hölder's inequality. This gives[3]

$$|\varphi * f(x) - f(x)| \leq \left(\int_{\mathbb{R}^n} \varphi(y)|(f(x-y) - f(x))|^p \, dy \right)^{\frac{1}{p}}. \tag{3.72}$$

The assertion follows by raising this inequality to the power p and applying Fubini's theorem.

Finally, let $p = \infty$ and let ω denote the right hand side of (3.69). Note that the map $(x,y) \mapsto |f(x-y) - f(x)|$ is measurable with respect to $\mathcal{L}^{2n}$. Hence the set $E := \{(x,y) : |f(x-y) - f(x)| > \omega\}$ is $\mathcal{L}^{2n}$ measurable. By assumption $\mathcal{L}^n(E \cap (\mathbb{R}^n \times \{y\})) = 0$ for all $y \in B(0,r)$. By Fubini's theorem $\mathcal{L}^{2n}(E \cap (\mathbb{R}^n \times B(0,r))) = 0$ and thus $\mathcal{L}^n(E \cap (\{x\} \times B(0,r)) = 0$ for a.e. $x \in \mathbb{R}^n$. Together with (3.71) this implies the assertion. □

Proof of the implication '$\Longleftarrow$' in Theorem 3.16, continued. We now construct the approximating sets A_δ by truncation and convolution.

Let $|h| \leq 1$. Since $(a+b)^p \leq 2^p(a^p + b^p)$ for all $a, b \geq 0$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} |(f\chi_{B(0,R)})(x+h) - (f\chi_{B(0,R)})(x)|^p \, dx \\
\leq &\int_{B(0,R-1)} |f(x+h) - f(x)|^p \, dx + \int_{\mathbb{R}^n \setminus B(0,R-1)} 2^p(|f(x+h)|^p + |f(x)|^p) \, dx \\
\leq &\int_{B(0,R-1)} |f(x+h) - f(x)|^p \, dx + 2\int_{\mathbb{R}^n \setminus B(0,R-2)} 2^p |f(x)|^p \, dx.
\end{aligned} \tag{3.73}$$

[3]alternatively the following estimate follows by applying Jensen's inequality with respect to the measure $\mu = \varphi\mathcal{L}^n$

Let $\delta > 0$. By conditions (ii) and (iii) there exist R and $j \in \mathbb{N}$ with $j \geq 1$ such that

$$\sup_{|h|\leq \frac{1}{j}} \|f\chi_{B(0,R)}(\cdot + h) - f\chi_{B(0,R)}(\cdot)\| \leq \frac{\delta}{4} \qquad \forall f \in A \tag{3.74}$$

and

$$\|f\chi_{B(0,R)} - f\|_{L^p(\mathbb{R}^n)} \leq \frac{\delta}{4} \qquad \forall f \in A. \tag{3.75}$$

For j and R and as above let $\eta_j(x) = j^n\eta(jx)$ be the standard mollifier and set

$$A_\delta := \{\eta_j * (f\chi_{B(0,R)}) : f \in A\}. \tag{3.76}$$

Then it follows from (3.74), Lemma 3.18 and (3.75) that

$$\|\eta_j * (f\chi_{B(0,R)}) - f\| \leq \frac{\delta}{2} \qquad \forall f \in A. \tag{3.77}$$

Thus $A \subset B_\delta(A_\delta)$.

To show that A_δ is precompact let $M := \sup_{f\in A} \|f\|_{L^p}$, let $\frac{1}{p'} = 1 - \frac{1}{p}$ and note that for all $f \in A$ we have

$$\eta_j * (f\chi_{B(0,R)}) = 0 \quad \text{in } \mathbb{R}^n \setminus \overline{B}(0, R+1), \tag{3.78}$$

$$\sup_{x\in\mathbb{R}^n} |\eta_j * (f\chi_{B(0,R)})(x)| \leq \|\eta_j\|_{L^{p'}} \|f\|_{L^p} \leq M\|\eta_j\|_{L^{p'}}, \tag{3.79}$$

$$\sup_{x\in\mathbb{R}^n} |D\ \eta_j * (f\chi_{B(0,R)})(x)| \leq \|D\eta_j\|_{L^{p'}} \|f\|_{L^p} \leq M\|D\eta_j\|_{L^{p'}}. \tag{3.80}$$

The last estimate implies that

$$|(\eta_j * f\chi_{B(0,R)})(x) - (\eta_j * f\chi_{B(0,R)})(x')| \leq M\|D\eta_j\|_{L^{p'}} |x - x'|. \tag{3.81}$$

It follows from (3.79) and (3.81) and the Arzela-Ascoli theorem that A_δ is precompact in $C(\overline{B}(0, R+1))$ and hence in $L^p(B(0, R+1))$. By (3.78) A_δ is precompact in $L^p(\mathbb{R}^n)$. More precisely the set of restrictions $\tilde{A}_\delta := \{g_{|\overline{B}(0,R)} : g \in A_\delta\}$ is precompact in $C(\overline{B}(0, R))$ and hence in $L^p(B(0, R))$. Then by Lemma3.17 the set $A_\delta = E\tilde{A}_\delta$ is precompact in $L^p(\mathbb{R}^n)$. This concludes the proof of that properties (i), (ii) and (iii) imply precompactness of A. □

To prove the converse implication '$\Longrightarrow$' in Theorem 3.16 we use the following result.

Proposition 3.19. *Let $p \in [1, \infty)$ and $f \in L^p(\mathbb{R}^n)$. Then*

$$\lim_{h\to 0} \|f(\cdot + h) - f(\cdot)\|_{L^p(\mathbb{R}^n)} = 0. \tag{3.82}$$

Proof. The assertion holds if $f \in C_c(\mathbb{R}^n)$. Since $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $p \in [1, \infty)$ the proposition follows.

Detailed proof: Let $\varepsilon > 0$. Then there exist $g \in C_c^\infty(\mathbb{R}^n)$ such that $\|f - g\| < \frac{\varepsilon}{3}$. Suppose that $\operatorname{supp} g \subset B(0, R-1)$. Then for $g(\cdot + h) \to g(\cdot)$ uniformly in $B(0, R)$ and $g(\cdot + h)$ and g are zero outside $B(0, R)$ if $|h| \le 1$. Hence

$$\lim_{h \to 0} \|g(\cdot + h) - g(\cdot)\|_{L^p(\mathbb{R}^n)} = 0 \tag{3.83}$$

and thus

$$\limsup_{h \to 0} \|f(\cdot + h) - f(\cdot)\|_{L^p(\mathbb{R}^n)} \le \|f - g\|_{L^p(\mathbb{R}^n)} + \|f(\cdot + h) - g(\cdot + h)\|_{L^p(\mathbb{R}^n)} < \varepsilon. \tag{3.84}$$

Since $\varepsilon > 0$ was arbitrary this finishes the proof. □

Proof of the implication '$\Longrightarrow$' in Theorem 3.16. Property (i) is obvious since precompact sets are bounded. To prove (ii) and (iii) let $\varepsilon > 0$. Then $A \subset \bigcup_{i=1}^{k_\varepsilon} B(f_i, \varepsilon)$, where $f_i \in L^p(\mathbb{R}^n)$. For $f \in B(f_i, \varepsilon)$ we have

$$\|f(\cdot + h) - f(\cdot)\|_{L^p} \le \|f_i(\cdot + h) - f_i(\cdot)\|_{L^p} + 2\varepsilon. \tag{3.85}$$

Thus

$$\sup_{f \in A} \|f(\cdot + h) - f(\cdot)\|_{L^p} \le \sup_{1 \le i \le k_\varepsilon} \|f_i(\cdot + h) - f_i(\cdot)\|_{L^p} + 2\varepsilon. \tag{3.86}$$

Hence Proposition 3.19 implies that

$$\limsup_{h \to 0} \sup_{f \in A} \|f(\cdot + h) - f(\cdot)\|_{L^p} \le 2\varepsilon. \tag{3.87}$$

Since $\varepsilon > 0$ was arbitrary this implies condition (ii).

Similarly for $f \in B(f_i, \varepsilon)$

$$\|f\|_{L^p(\mathbb{R}^n \setminus B(0,R))} \le \|f_i\|_{L^p(\mathbb{R}^n \setminus B(0,R))} + \varepsilon \tag{3.88}$$

and

$$\sup_{f \in A} \|f\|_{L^p(\mathbb{R}^n \setminus B(0,R))} \le \sup_{1 \le i \le k_\varepsilon} \|f_i\|_{L^p(\mathbb{R}^n \setminus B(0,R))} + \varepsilon. \tag{3.89}$$

Since $(a+b)^p \le 2^p(a^p + b^p)$ for $a, b \ge 0$ we get

$$\sup_{f \in A} \int_{\mathbb{R}^n \setminus B(0,R)} |f|^p \, d\mathcal{L}^n \le 2^p \left(\sup_{1 \le i \le k_\varepsilon} \int_{\mathbb{R}^n \setminus B(0,R)} |f_i|^p \, d\mathcal{L}^n + \varepsilon^p \right). \tag{3.90}$$

Since $f_i \in L^p(\mathbb{R}^n)$ and $\displaystyle\bigcap_{i=k}^{\infty} (\mathbb{R}^n \setminus B(0,k)) = \emptyset$ we have

$$\lim_{R \to \infty} \int_{\mathbb{R}^n \setminus B(0,R)} |f_i|^p \, d\mathcal{L}^n = \lim_{k \to \infty} \int_{\mathbb{R}^n \setminus B(0,k)} |f_i|^p \, d\mathcal{L}^n = 0. \tag{3.91}$$

Hence

$$\limsup_{R\to\infty} \sup_{f\in A} \int_{\mathbb{R}^n\setminus B(0,R)} |f|^p \, d\mathcal{L}^n \le \varepsilon^p. \tag{3.92}$$

Since $\varepsilon > 0$ was arbitrary this implies condition (iii). This concludes the proof of Theorem 3.16. □

4 Linear operators

In this chapter

$$X, Y, Z, \ldots\ldots \quad \text{denote normed } \mathbb{K} \text{ vector spaces.} \tag{4.1}$$

We will mention explicitly if these spaces are complete, i.e., Banach spaces. For a linear map $T : X \to Y$ we write

$$Tx := T(x) \tag{4.2}$$

and for the composition of two linear maps we write

$$ST := S \circ T. \tag{4.3}$$

Lemma 4.1. *Let $T : X \to Y$ be linear.Then the following three assertions are equivalent.*

(i) T is continuous.

(ii) T is continuous at 0.

(iii) T is bounded, i.e., there exists a constant C such that

$$\|Tx\| \leq C\|x\| \qquad \forall x \in X. \tag{4.4}$$

Proof. '(i) $\implies$ (ii)': obvious.
'(ii) $\implies$ (iii)': Apply the $\varepsilon - \delta$ characterization of continuity with $\varepsilon = 1$. Then there exist $\delta > 0$ such that

$$T(B(0, \delta)) \subset B(0, 1). \tag{4.5}$$

Thus

$$\|T \frac{\delta}{2} \frac{x}{\|x\|}\| \leq 1 \quad \text{and hence} \quad \|Tx\| \leq \frac{2}{\delta}\|x\|. \tag{4.6}$$

'(iii) $\implies$ (i)': For $x, x' \in X$ we have

$$\|Tx' - Tx\| \leq \|T(x - x')\| \leq C\|x - x'\|. \tag{4.7}$$

Thus T is Lipschitz continuous and hence continuous. □

[22.11. 2017, Lecture 12]

[24.11. 2017, Lecture 13]

It is easy to construct unbounded operators when X is not complete. One can, for example, take the identity map from $X = (l_1, \| \cdot \|_{l_\infty})$ to $Y = (l_1, \| \cdot \|_{l_1})$ or from $X = (C^0[0,1]; \| \cdot \|_{L^1})$ to $Y = (C^0[0,1]; \| \cdot \|_{C^0})$. To construct an unbounded operator from an (infinite dimensional) Banach X space to itself one can use the fact that every vector space has a basis B.

Then one takes a countable subset $B' = \{b_0, b_1, \ldots\}$ of B and set $Tb_i = ib_i$ and $Tb = 0$ if $b \in B \setminus B'$. Since B is a basis T has a unique extension to a linear map from X to itself and T is not bounded.

We set

$$\mathcal{L}(X, Y) := \{T : X \to Y : T \text{ linear and continuous}\} \tag{4.8}$$

and for $T \in \mathcal{L}(X, Y)$ we set

$$\|T\|_{\mathcal{L}(X,Y)} := \sup\{\|Tx\| : \|x\| \le 1\} = \sup\{\frac{\|Tx\|}{\|x\|} : x \neq 0\}. \tag{4.9}$$

Note that the equality follows from the homogeneity of the norm and that $\|T\|_{\mathcal{L}(X,Y)}$ is the smallest constant for which (4.4) holds. We call the elements of $\mathcal{L}(X, Y)$ bounded linear operators and we call $\| \cdot \|_{\mathcal{L}(X,Y)}$ the operator norm. Often we write $\|T\|$ instead of $\|T\|_{\mathcal{L}(X,Y)}$. We use the abbreviation

$$\mathcal{L}(X) := \mathcal{L}(X, X). \tag{4.10}$$

Theorem 4.2. *The space $(\mathcal{L}(X, Y), \| \cdot \|_{\mathcal{L}(X,Y)})$ is normed space. If Y is a Banach space then $\mathcal{L}(X, Y)$ is a Banach space. If $S, T \in \mathcal{L}(X)$ then $ST \in \mathcal{L}(X)$ and*

$$\|ST\| \le \|S\| \, \|T\| \tag{4.11}$$

Proof. Let $S, T \in \mathcal{L}(X, Y)$ and let $\lambda \in \mathbb{K}$. Then λT and $S+T$ are continuous linear maps. Hence $\mathcal{L}(X, Y)$ is a $\mathbb{K}$ vector space. Moreover $\|\lambda T\| = |\lambda| \|T\|$ and $\|T\| = 0$ implies that $Tx = 0$ for all x with $\|x\| \le 1$ and hence $T = 0$. To prove that $T \mapsto \|T\|$ is a norm it remains to show that

$$\|S + T\| \le \|S\| + \|T\|. \tag{4.12}$$

Now

$$\forall x \in \overline{B}(0, 1) \subset X \quad \|(S + T)x\| = \|Sx + Tx\| \le \|Sx\| + \|Tx\| \le \|S\| + \|T\| \tag{4.13}$$

Taking the supremum over $x \in \overline{B}(0, 1)$ we get (4.12)

Now assume that Y is a Banach space and let $k \mapsto T_k$ be a Cauchy sequence in $\mathcal{L}(X, Y)$. Then

$$\forall \varepsilon > 0 \; \exists k_0 \; \forall k, l \ge k_0 \quad \|T_l - T_k\| < \varepsilon. \tag{4.14}$$

Thus

$$\forall \varepsilon > 0 \; \exists k_0 \; \forall k, l \ge k_0 \quad \|T_l x - T_k x\| \le \varepsilon \|x\| \tag{4.15}$$

Hence for each $x \in X$ the sequence $k \mapsto T_k x$ is a Cauchy sequence in Y. Since Y is complete we can define

$$T_* x := \lim_{l \to \infty} T_l x \tag{4.16}$$

and it is easy to see that T_* is linear. Moreover

$$\|(T_k - T_*)x\| = \lim_{l\to\infty} \|T_k x - T_l x\| \le \varepsilon\|x\| \quad \forall k \ge k_0. \tag{4.17}$$

Hence $\|T_k - T_*\| \le \varepsilon$ if $k \ge k_0$. Thus $T_k \to T_*$ in $\mathcal{L}(X,Y)$.

To prove (4.11) note that for all $x \in X$

$$\|STx\| \le \|S\|\,\|Tx\| \le \|S\|\,\|T\|\,\|x\|. \tag{4.18}$$

□

Definition 4.3. *(i) The space $X' := \mathcal{L}(X,\mathbb{K})$ is called the dual space of X. Its elements are called (bounded) linear functionals. Note that by Theorem 4.2 the space X' is always a Banach space (since $\mathbb{K}$ is complete), even if X is not a Banach space.*

(ii) The set of compact operators from X to Y is defined as

$$\mathcal{K}(X,Y) := \{T \in \mathcal{L}(X,Y) : \overline{T(B(0,1))} \text{ compact in } Y\}. \tag{4.19}$$

If Y is complete then $\overline{T(B(0,1))}$ is compact if and only if $T(B(0,1))$ precompact in Y (see Proposition 3.10 (vi)).

(iii) A linear map $P : X \to X$ is called a projection if $P^2 = P$. The set of continuous linear projections (or projectors) is defined as

$$\mathcal{P}(X) := \{P \in \mathcal{L}(X) : P^2 = P\}. \tag{4.20}$$

(iv) For $T \in \mathcal{L}(X,Y)$ we call

$$\mathcal{N}(T) := \ker T := \{x : Tx = 0\} \tag{4.21}$$

the null space (or kernel) of T. Since T is continuous $\mathcal{N}(T)$ is a closed subspace of X. We call

$$\mathcal{R}(T) := \{Tx : x \in X\} \tag{4.22}$$

the range of T. The range is in general not closed.

(v) A map $T \in \mathcal{L}(X,Y)$ is called an embedding if T is injective, i.e., if $\mathcal{N}(T) = \{0\}$.

(vi) If $T \in \mathcal{L}(X,Y)$ is bijective and $T^{-1} \in \mathcal{L}(Y,X)$ we call T an invertible operator (or a (linear) isomorphism). If X and Y are Banach spaces then a fundamental theorem of functional analysis (which we will prove later) states that $T \in \mathcal{L}(X,Y)$ and T bijective already implies $T^{-1} \in \mathcal{L}(Y,X)$.

(vii) An operator $T \in \mathcal{L}(X,Y)$ is called an isometry if (see Definition 1.27)

$$\|Tx\| = \|x\| \quad \forall x \in X. \tag{4.23}$$

(viii) If $T \in \mathcal{L}(X,Y)$ then we define a linear map $T' : Y' \to X'$ by

$$(T'y')(x) := y'(Tx). \tag{4.24}$$

The map T' is called the adjoint (or dual) operator to T and we have $T' \in \mathcal{L}(Y', X')$

To see that $T' \in \mathcal{L}(Y', X')$ it suffices to note that

$$|(T'y')(x)| = |y'(Tx)| \leq \|y'\| \, \|Tx\| \leq \|y'\| \, \|T\| \|x\|. \tag{4.25}$$

Thus $\|T'y\| \leq \|y'\| \, \|T\|$. Hence $T' \in \mathcal{L}(Y', X')$ and $\|T'\| \leq \|T\|$ (we will see later that $\|T'\| = \|T\|$).

Proposition 4.4. *Let X be finite dimensional. Then every linear map $T : X \to Y$ is continuous.*

Proof. Let $\dim X = n$ and let $\{e_1, \ldots, e_n\}$ be a basis of X. Then every $x \in X$ has a unique representation $x = \sum_{i=1}^n x_i e_i$ and $\|x\|_1 := \sum_{i=1}^n |x_i|$ defines a norm on X. Now

$$\|Tx\| = \| \sum_{i=1}^n x_i T e_i \| \leq \|x\|_1 \sup_{1 \leq i \leq n} \|Te_i\|. \tag{4.26}$$

Since by Lemma 3.11 all norms on X are equivalent this shows that T is bounded and hence continuous. □

Definition 4.5 (Frechet differentiability)**.** *Let $U \subset X$ be open.*

(i) A map $F : U \to Y$ is called Frechet differentiable at $x_0 \in U$ if there exists $T \in \mathcal{L}(X,Y)$ such that

$$\lim_{x \to x_0} \frac{\|F(x) - F(x_0) - T(x - x_0)\|}{\|x - x_0\|} = 0. \tag{4.27}$$

If such a T exists it is unique. We call T the differential of F at x_0 and write

$$DF(x_0) = T. \tag{4.28}$$

(ii) A map $F : U \to Y$ is called Frechet differentiable if it is Frechet differentiable at every $x \in U$.

(iii) *We say F is continuously differentiable in U (notation: $F \in C^1(U;Y)$) if F is Frechet differentiable in U and the map*

$$x \mapsto DF(x) \quad \text{is continuous as a map} \quad U \to \mathcal{L}(X,Y). \tag{4.29}$$

Theorem 4.6 (Inverse function theorem)**.** *Let X and Y be Banach spaces, let $U \subset X$ be open, let $F \in C^1(U;F)$ and $x_0 \in U$. If*

$$DF(x_0) \in \mathcal{L}(X,Y) \quad \text{is invertible} \tag{4.30}$$

then F is a C^1 diffeomorphism in a neighbourhood of x_0. More precisely there exist open sets $V \subset X$ and $W \subset Y$ such that $x_0 \in V$, $F(x_0) \in W$ such that

$$F : V \to W \quad \text{bijective,} \quad F^{-1} \in C^1(W,X) \tag{4.31}$$

and

$$DF^{-1}(y) = (DF(F^{-1}(y)))^{-1}. \tag{4.32}$$

Proof. This was proved in Analysis 2. To construct the inverse function $y \mapsto G(y)$ one applies the Banach fixed point theorem to the map $T_y(x) = L^{-1}y - L^{-1}(F(x) - L(x - x_0))$, where $L = DF(x_0)$ and shows that T_y is a contraction on $B(x_0, \delta)$ if $y \in B(x_0, \varepsilon)$ and if $\delta, \varepsilon > 0$ are sufficiently small. □

Example 4.7. (i) *Let $E \subset \mathbb{R}^n$ be Lebesgue measurable. Let $p \in [1,\infty]$ and let p' be the dual exponent, i.e., $\frac{1}{p'} = 1 - \frac{1}{p}$. Let $g \in L^{p'}(E)$ and define $T : L^p(E) \to \mathbb{R}$ by*

$$Tf := \int_E fg \, d\mathcal{L}^n. \tag{4.33}$$

By Hölder's inequality we have $T \in (L^p(E))'$ and $\|T\|_{(L^p(E))'} \leq \|g\|_{L^{p'}(E)}$. We shall see later that equality holds and that for $p < \infty$ every element in $(L^p(E))'$ can be written in this way.

(ii) *For $f \in C([0,1])$ define*

$$Tf(x) := \int_0^x f(y) \, dy. \tag{4.34}$$

Then $T \in \mathcal{L}(C([0,1]); C^1([0,1]))$ and $T \in \mathcal{K}(C([0,1]); C([0,1]))$. Moreover $\mathcal{R}(T)$ is not closed in $C([0,1])$ (exercise).

(iii) *Differential operators. Let $U \subset \mathbb{R}^n$ be open. For $|\alpha| \leq m$ let $a_\alpha : U \to \mathbb{R}$ and set $Tf = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha f$. Then*

(a) *If $a_\alpha \in C(\overline{U})$ and U is bounded then $T \in \mathcal{L}(C^m(\overline{U}); C(\overline{U}))$;*

(b) *if U is bounded and $a_\alpha \in C^{0,\beta}(U)$ then $T \in \mathcal{L}(C^{m,\beta}(U); C^{0,\beta}(U))$;*

(c) *if* $a_\alpha \in L^\infty(U)$ *then* $T \in \mathcal{L}(W^{m,p}(U); L^p(U))$.

One calls T *a linear partial differential operator with coefficients* a_α. *A fundamental question in the theory of partial differential equation is if and under which additional conditions* T *is invertible.*

Theorem 4.8 (Neumann series)**.** *Let* X *be a Banach space and* $T \in \mathcal{L}(X)$ *with* $\limsup_{m\to\infty} \|T^m\|^{\frac{1}{m}} < 1$ *(this hold in particular if* $\|T\| < 1$*). Then* $\mathrm{Id} - T$ *is bijective and* $(\mathrm{Id} - T)^{-1} \in \mathcal{L}(X)$. *Moreover the series* $\sum_{n=0}^{\infty} T^n$ *converges in* $\mathcal{L}(X)$ *and*

$$(\mathrm{Id} - T)^{-1} = \sum_{n=0}^{\infty} T^n. \tag{4.35}$$

Proof. For $k \in \mathbb{N}$ set $S_k := \sum_{n=0}^{k} T^n$. By assumption there exist $\theta < 1$ and $m \in \mathbb{N}$ such that $\|T^n\| \leq \theta^n$ for all $n \geq m$. Then we have for $m \leq k < l$

$$\|S_l - S_k\| \leq \| \sum_{n=k+1}^{l} T^n\| \leq \sum_{n=k+1}^{l} \|T^n\| \leq \sum_{k+1}^{\infty} \theta^n \leq \frac{\theta^{k+1}}{1-\theta}. \tag{4.36}$$

The right hand side goes to zero as $k \to \infty$. Since $\mathcal{L}(X)$ is complete there exists $S \in \mathcal{L}(X)$ such that

$$S = \lim_{k\to\infty} S_k \quad \text{in } \mathcal{L}(X). \tag{4.37}$$

Hence we have in the limit $k \to \infty$

$$(I-T)S \longleftarrow (I-T)S_k = \sum_{n=0}^{k} (T^n - T^{n+1}) = \mathrm{Id} - T^{k+1} \longrightarrow I \tag{4.38}$$

since $\|T^k\| \leq \theta^k \to 0$ for $k \to \infty$. Thus $(\mathrm{Id} - T)S = \mathrm{Id}$. In the same way one shows that $S(\mathrm{Id} - T) = \mathrm{Id}$. Thus $\mathrm{Id} - T$ is surjective and injective and S is its inverse. $\square$

Corollary 4.9. *Let* X *and* Y *be Banach spaces. Then the set of invertible operators is an open subset of* $\mathcal{L}(X, Y)$. *More precisely: if* $X \neq \{0\}$ *and* $Y \neq \{0\}$ *and* $S, T \in \mathcal{L}(X, Y)$ *then*

$$T \text{ invertible}, \quad \|S - T\| < \|T^{-1}\|^{-1} \quad \Longrightarrow \quad S \text{ invertible}. \tag{4.39}$$

Proof. Set $R := T - S$. Then $S = T(\mathrm{Id} - T^{-1}R)$ and $\|T^{-1}R\| \leq \|T^{-1}\|\|R\| < 1$. Thus $\mathrm{Id} - T^{-1}R$ is invertible by Theorem 4.8 and S is invertible as a product of invertible operators. $\square$

Proposition 4.10. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{4.40}$$

be a power series with radius of convergence R*. Let* X *be a* $\mathbb{K}$ *Banach space. If* T *in* $\mathcal{L}(X)$ *then*

$$\limsup_{m\to\infty} \|T^m\|^{\frac{1}{m}} < R \quad \Longrightarrow \quad f(T) := \sum_{n=0}^{\infty} a_n T^n \quad \textit{exists in } \mathcal{L}(X). \tag{4.41}$$

Proof. By assumption there exists $r < R$ and $m \in \mathbb{N}$ such that $\|T^n\| \leq r^n$ for all $n \geq m$. Thus for $m \leq k \leq l$

$$\left\| \sum_{n=k}^{l} a_n T^n \right\| \leq \sum_{n=k}^{l} |a_n| \, \|T^n\| \leq \sum_{n=k}^{\infty} |a_n| r^n \to 0 \quad \text{as } k \to \infty \tag{4.42}$$

because the power series has radius of converges $R > r$. □

Example. (i) For all $T \in \mathcal{L}(X)$ one defines the exponential function by

$$\exp(T) := e^T := \sum_{n=0}^{\infty} \frac{1}{n!} T^n \quad \in \mathcal{L}(X). \tag{4.43}$$

For $S, T \in \mathcal{L}(X)$ we have

$$ST = TS \quad \Longrightarrow \quad e^{T+S} = e^T e^S. \tag{4.44}$$

From this one easily deduces that

$$\frac{d}{dt}_{t=t_0} e^{tA} = A e^{t_0 A} = e^{t_0 A} A. \tag{4.45}$$

(ii) If $T \in \mathcal{T}$ and $\|\mathrm{Id} - T\| < 1$ defines

$$\log(T) = -\sum_{n=0}^{\infty} \frac{1}{n} (\mathrm{Id} - T)^n. \tag{4.46}$$

Then $\exp(\log T) = T$ (exercise).

[24.11. 2017, Lecture 13]

[29.11. 2017, Lecture 14]

5 Linear functionals on Hilbert spaces and weak solutions of PDE

5.1 The Riesz representation theorem and the Lax-Milgram theorem

Motivation: If X is a Hilbert space and $x \in X$ then the Cauchy-Schwarz inequality implies that $y \mapsto (y, x)$ is a continuous linear map from X to $\mathbb{K}$, i.e. an element of X'. The Riesz representation theorem states that every element of X' can be written in this way.

Theorem 5.1 (Riesz representation theorem)**.** *Let X be a Hilbert space. Then the map J given by*

$$J(x)(y) := (y, x) \tag{5.1}$$

defines a conjugately linear isomorphism from X to X', i.e., J is bijective and $\|J(x)\|_{X'} = \|x\|_X$.

Remark. It follows that X' is a Hilbert space with scalar product $(x', y') := (J^{-1}(y'), J^{-1}(x'))$.

Notation: we denote the isomorphism J by R_X.

Definition 5.2. *Let X and Y be $\mathbb{K}$ vector spaces. A map $J : X \to Y$ is called conjugately linear if for all $x, y \in X$ and $\alpha \in \mathbb{K}$*

$$J(\alpha x + y) = \overline{\alpha} J(x) + J(y). \tag{5.2}$$

For $\mathbb{K} = \mathbb{R}$ conjugately linear is the same as linear.

Proof of Theorem 5.1. By the Cauchy-Schwarz inequality

$$|J(x)(y)| = |(y, x)| \le \|y\|_X \|x\|_X, \quad \text{and thus } J(x) \in X' \quad \|J(x)\|_{X'} \le \|x\|_X. \tag{5.3}$$

If $x = 0$ then $J(x) = 0$. If $x \neq 0$ the choice $y = x/\|x\|_X$ yields

$$\|J(x)\|_{X'} \ge J(x)(\frac{x}{\|x\|_X}) = (\frac{x}{\|x\|_X}, x) = \|x\|_X. \tag{5.4}$$

Thus $\|J(x)\|_{X'} = \|x\|_X$. Hence J is an isometry and in particular injective. It follows from the property of the scalar product that J is conjugately linear.

The main point is to show that J is surjective, i.e., for every $T \in X'$ there exists a $x \in X$ such that $T = J(x)$. Let $T \in X'$ and assume that $T \neq 0$. The null space $\mathcal{N}(T) = \{y : Ty = 0\}$ is closed since T is continuous. Let $P : X \to \mathcal{N}(T)$ denote the linear projection in Corollary 3.3. Since $T \neq 0$ there exists $e \in X$ with $T(e) = 1$. Let $x_0 = e - P(e)$. Then

$$(y, x_0) = 0 \quad \forall y \in \mathcal{N}(T). \tag{5.5}$$

Now $T(x_0) = T(e) = 1$ since $P(e) \in \mathcal{N}(T)$. Thus for all $x \in X$

$$x = \underbrace{x - T(x)x_0}_{\in \mathcal{N}(T)} + T(x)x_0 \tag{5.6}$$

and hence

$$(x, x_0) = 0 + T(x)\|x_0\|^2 \quad \forall x \in X \tag{5.7}$$

so that

$$T(x) = J(\frac{x_0}{\|x_0\|^2})(x). \tag{5.8}$$

□

Variant of the proof of surjectivity. . Let $A = \{x : T(x) = 1\}$. Then the projection theorem shows that there exists $x_0 \in A$ such that

$$\|x_0\|^2 = \operatorname{dist}^2(0, A) \leq \|z\|^2 \quad \forall z \in A. \tag{5.9}$$

For $|t|$ small we have $T(x_0 + tx) \neq 0$ and $\frac{x_0+tx}{T(x_0+tx)} \in A$. Thus the function

$$h(t) := \left\| \frac{x_0 + tx}{T(x_0 + tx)} \right\|^2 = \frac{\|x_0 + tx\|^2}{|T(x_0 + tx)|^2} \tag{5.10}$$

is minimized at $t = 0$. Differentiation at $t = 0$ gives (with $T(x_0) = 1$)

$$0 = h'(0) = 2\operatorname{Re}(x, x_0) - 2\|x_0\|^2 \operatorname{Re} T(x). \tag{5.11}$$

Hence for $\mathbb{K} = \mathbb{R}$ we get $T = J(x_0/\|x_0\|^2)$. If $\mathbb{K} = \mathbb{C}$ we first apply (5.11) for x and ix and then reach the same conclusion. □

Theorem 5.3 (Lax-Milgram)**.** *Let X be a $\mathbb{K}$ Hilbert space. Let $a : X \times X \to \mathbb{K}$ be a sesquilinear form. Suppose that there exist constants $c_0, C_0 > 0$ such that*

(i) (continuity) $|a(x, y)| \leq C_0\|x\| \, \|y\| \quad \forall x, y \in X$;

(ii) (coercivity) $\operatorname{Re} a(x, x) \geq c_0\|x\|^2$.

Then there exists one and only one map $A : X \to X$ such that

$$a(y, x) = (y, Ax)_X \quad \forall x, y \in X. \tag{5.12}$$

Moreover $A \in \mathcal{L}(X)$ and A is an invertible operator with

$$\|A\| \leq C_0 \quad \text{and} \quad \|A^{-1}\| \leq \frac{1}{c_0}. \tag{5.13}$$

Remark. Note that we did not assume that a is symmetric.

Proof. Step 1: Existence of A.
For $x \in X$ define T_x by $T_x(y) = a(y,x)$. By the assumption (i) we have $T_x \in X'$ and $\|T_x\| \le C_0\|x\|$. By the Riesz representation theorem there exists one and only one element $A(x) \in X$ such that

$$a(y,x) = (y, A(x))_X \quad \forall y \in X \quad \text{and} \quad \|A(x)\| \le C_0\|x\|. \tag{5.14}$$

Since $A(x)$ is unique and since a and the scalar product are conjugately linear in the second argument it follows that A is linear. Thus $A \in \mathcal{L}(X)$ and $\|A\| \le C_0$.

Step 2: Lower bound for $\|Ax\|$.
Coercivity of a implies that

$$c_0\|x\|^2 \le \operatorname{Re} a(x,x) = \operatorname{Re}(x, Ax)_X \le \|x\|\|Ax\| \tag{5.15}$$

and thus

$$c_0\|x\| \le \|Ax\|. \tag{5.16}$$

In particular A is injective.

Step 3: A has closed range.
Assume that $y_k = Ax_k$ and $y_k \to y$. Then it follows from (5.16) that

$$\|x_k - x_l\| \le \frac{1}{c_0}\|y_k - y_l\|. \tag{5.17}$$

Hence $k \mapsto x_k$ is a Cauchy sequence and $x_k \to x_*$ as $k \to \infty$. Since A is continuous we get $y_k = Ax_k \to Ax_* \in \mathcal{R}(A)$.

Step 4: A is surjective and $\|A^{-1}\| \le \frac{1}{c_0}$.
Assume that $\mathcal{R}(A) \ne X$. Since $\mathcal{R}(A)$ is closed there exists an $x_0 \ne 0$ such that

$$(x_0, y) = 0 \quad \forall y \in \mathcal{R}(A). \tag{5.18}$$

To see this take $x \in X \setminus \mathcal{R}(A)$ and let $x_0 = x - P(x)$ where P is the orthogonal projection in Corollary 3.3. Now coercivity yields

$$c_0\|x_0\|^2 \le \operatorname{Re}(x_0, Ax_0) = 0, \tag{5.19}$$

where we used (5.18). Thus $x_0 = 0$, a contradiction. Hence $\mathcal{R}(A) = X$. Finally the estimate $\|A^{-1}y\| \le \frac{1}{c_0}\|y\|$ follows from (5.16) by taking $x = A^{-1}y$. □

Corollary 5.4. *Let X be a Hilbert space and let a be as in the Lax-Milgram theorem. Let $T \in X'$. Then there exists one and only one $x \in X$ such that*

$$a(y,x) = T(y) \quad \forall y \in X. \tag{5.20}$$

Moreover the map $T \mapsto x$ is conjugately linear and

$$\|x\| \le \frac{1}{c_0}\|T\|. \tag{5.21}$$

Proof. Let $R_X : X \to X'$ be the conjugately linear isomorphism in the Riesz representation theorem and let A be the operator in the Lax-Milgram theorem. Then (5.20) is equivalent to

$$(y, Ax)_X = (y, R_X^{-1}(T))_X \quad \forall y \in X \tag{5.22}$$

and this is equivalent to $Ax = R_X^{-1}T$ or

$$x = A^{-1}R_X^{-1}T \tag{5.23}$$

and the estimate for $\|x\|$ follow from the Lax-Milgram theorem since R_X is an isometry. □

Corollary 5.5. *Let X be a Hilbert space and let $A \in \mathcal{L}(X)$. If there exists $c_0 > 0$ such that*

$$Re(x, Ax)_X \geq c_0\|x\|_X^2 \tag{5.24}$$

then A is invertible and

$$\|A^{-1}\| \leq \frac{1}{c_0}. \tag{5.25}$$

Proof. Apply the Lax-Milgram theorem with $a(y, x) = (y, Ax)_X$. □

5.2 Weak solutions of elliptic partial differential equations

We now will use the Lax-Milgram theorem and its corollaries to establish the existence of weak solutions of elliptic partial differential equations of second order.

We first quickly review the notion of classical solution. Let $U \subset \mathbb{R}^n$ be open and bounded, for $i = 1, \ldots, n$ and $j = 1, \ldots n$ let $a_{ij} \in C^1(U)$, $h_i \in C^1(U)$, $b \in C(U)$ and $f \in C(\Omega)$. We seek a function $u \in C^2(U)$ such that

$$Lu = f - \sum_{i=1}^{n} \partial_i h_i, \quad \text{where} \quad Lu = -\sum_{i,j=1}^{n} \partial_i(a_{ij}\partial_j u) + bu \tag{5.26}$$

In addition we assume either Dirichlet boundary conditions, i.e., $u \in C^2(U) \cap C(\overline{U})$ and

$$u = g \quad \text{on } \partial U \tag{5.27}$$

for a given function $g \in C(\bar{U})$ or Neumann boundary condition, i.e., $u \in C^2(U) \cap C^1(\overline{U})$ and

$$\sum_{i=1}^{n} \nu_i \left(-\sum_{j=1}^{n} a_{ij}\partial_j u + h_i \right) = g \quad \text{on } \partial U, \tag{5.28}$$

where ν is the outward normal of U (and where we assume that U has C^1 boundary).

We now reduce the Dirichlet and the Neumann problem to the case of zero boundary conditions. To have any chance to solve the Dirichlet problem there must exist a function $u_0 \in C^2(U) \cap C(\bar{U})$ with $u = g$ on ∂u. Then $\tilde{u} := u - u_0$ solves the problem

$$-\sum_{i,j=1}^{n} \partial_i(a_{ij}\partial_j \tilde{u}) + b\tilde{u} = \tilde{f} - \sum_{i=1}^{n} \partial_i \tilde{h}_i \qquad \text{in } U \tag{5.29}$$

$$\tilde{u} = 0 \qquad \text{on } \partial U, \tag{5.30}$$

where

$$\tilde{f} := f - bu_0, \quad \tilde{h}_i := h_i - \sum_{j=1}^{n} a_{ij}\partial_j u_0. \tag{5.31}$$

We call this the Dirichlet problem with homogeneous boundary conditions.

Similarly to have any chance to solve the Neumann problem there must exist a $u_0 \in C^2(U) \cap C^1(\bar{U})$ such that $-\sum_{i=1}^n \nu_i(-\sum_{j=1}^n a_{ij}\partial_j u + h_i) = 0$. Then $\tilde{u} = u - u_0$ solves the homogeneous Neumann problem with f replaced by $\tilde{f}$ and h_i replace by $\tilde{h}_i$.

In the following we write again u instead of $\tilde{u}$ etc.

To pass to the weak formulation of the Dirichlet problem we multiply (5.29) by a test function $\zeta \in C_c^\infty(U)$ and integrate over U and integrate by parts. This gives

$$\int_U \sum_{i,j=1}^{n} a_{ij}\partial_i\zeta\partial_j u + b\zeta u \, d\mathcal{L}^n = \int_U \zeta f + \sum_{i=1}^{n} \partial_i\zeta h_i \, d\mathcal{L}^n \forall \zeta \in C_c^\infty(U). \tag{5.32}$$

Conversely, if (5.32) holds then integration by parts yields (5.26) since $\int_U w\zeta = 0$ for all $\zeta \in C_c^\infty(U)$ implies $w = 0$.

Definition 5.6 (Weak solution)**.** *Let $a_{ij}, b \in L^\infty(U)$, $f, h_i \in L^2(U)$. We say that u is a weak solution of the Dirichlet problem*

$$-\sum_{i,j=1}^{n} \partial_i(a_{ij}\partial_j u) + bu = f - \sum_{i=1}^{n} \partial_i h_i \qquad \text{in } U \tag{5.33}$$

$$u = 0 \qquad \text{on } \partial U, \tag{5.34}$$

if

$$\int_U \sum_{i,j=1}^{n} a_{ij}\partial_i\zeta\partial_j u + b\zeta u \, d\mathcal{L}^n = \int_U \zeta f + \sum_{i=1}^{n} \partial_i\zeta h_i \, d\mathcal{L}^n \quad \forall \zeta \in W_0^{1,2}(U) \tag{5.35}$$

$$u \in W_0^{1,2}(U) \tag{5.36}$$

Theorem 5.7. *Let $U \subset \mathbb{R}^n$ be bounded and open. Let $a_{ij}, b \in L^\infty(U)$, $f, h_i \in L^2(U)$ and assume that*

$$b(x) \geq 0 \quad \text{for a.e. } x \in U \tag{5.37}$$

and that the coefficients a_{ij} are elliptic, *i.e.,*

$$\exists c > 0 \; \forall \xi \in \mathbb{R}^n \quad \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq c|\xi|^2 \quad \text{for a.e. } x \in U. \tag{5.38}$$

Then there exists one and only one weak solution u of the Dirichlet problem.

Moreover there exists a constant C which only depends on the set U such that

$$\|u\|^2_{W_0^{1,2}} \leq \frac{C}{c}(\|f\|^2_{L^2} + \sum \|h_i\|^2_{L^2}). \tag{5.39}$$

Proof. Consider the bilinear form

$$a(\zeta, v) := \int_U \sum_{i,j=1}^{n} a_{ij}\partial_i\zeta\partial_j v + b\zeta v \, d\mathcal{L}^n \tag{5.40}$$

and the linear map

$$T(\zeta) := \int_U \zeta f + \sum_{i=1}^{n} \partial_i\zeta h_i \, d\mathcal{L}^n. \tag{5.41}$$

Then T is a bounded functional on $W_0^{1,2}$ and

$$\|T\|_{(W_0^{1,2})'} \leq (\|f\|^2_{L^2} + \sum \|h_i\|^2_{L^2})^{1/2}. \tag{5.42}$$

One also sees easily that $a(\zeta, v) \leq C'\|\zeta\|_{W_0^{1,2}}\|v\|_{W_0^{1,2}}$. Hence all assertions follow from Corollary 5.4 if we can show that

$$a(v, v) \geq \frac{c}{C}\|v\|^2_{W_0^{1,2}}, \tag{5.43}$$

where C only depends on U. From the condition $b \geq 0$ a.e. and the ellipticity condition we see that

$$a(v, v) \geq c\|\nabla v\|^2_{L^2}. \tag{5.44}$$

Now the Poincaré inequality, Lemma 3.5, yields

$$\|v\|^2_{L^2} \leq C\|\nabla v\|^2_{L^2} \quad \text{and hence} \quad \|\nabla v\|^2_{L^2} \geq \frac{1}{C+1}\|v\|^2_{W_0^{1,2}}. \tag{5.45}$$

□

We now pass to the homogeneous Neumann problem, i.e., (5.26), (5.28) with $g = 0$. Multiplication of (5.26) by $\zeta \in C^1(\bar{U})$ and integration by parts yields

$$\int_U \sum_{i,j=1}^n a_{ij}\partial_i\zeta\partial_j u + b\zeta u \, d\mathcal{L}^n = \int_U \sum_{i=1}^n \partial_i\zeta h_i + f\zeta \, d\mathcal{L}^n = 0 \quad \forall \zeta \in C^1(\bar{U}). \tag{5.46}$$

Conversely if (5.46) holds for all $\zeta \in C^1(\bar{U})$ (and if u is $C^2(U) \cap C^1(\bar{U})$ we can first use all $\zeta \in C_c^\infty(U)$ to deduce the partial differential equation (5.26). Then integration by parts yields that

$$\int_{\partial U} \zeta \left(\sum_{i,j=1}^n \nu_i a_{ij} \partial_j u - \sum_{i=1}^n \nu_i h_i \right) d\mathcal{H}^{n-1} = 0 \tag{5.47}$$

for all $\zeta \in C^1(\bar{U})$. This implies the boundary condition (5.28) with $g = 0$.

Definition 5.8 (Weak solution of the Neumann problem). *Let $a_{ij}, b \in L^\infty(U)$, $f, h_i \in L^2(U)$. We say that u is a weak solution of the Neumann problem*

$$-\sum_{i,j=1}^n \partial_i(a_{ij}\partial_j u) + bu = f - \sum_{i=1}^n \partial_i h_i \qquad \text{in } U \tag{5.48}$$

$$\sum_{i=1}^n \nu_i \left(-\sum_{j=1}^n a_{ij}\partial_j u + h_i \right) = 0 \qquad \text{on } \partial U, \tag{5.49}$$

if

$$\int_U \sum_{i,j=1}^n a_{ij}\partial_i\zeta\partial_j u + b\zeta u \, d\mathcal{L}^n = \int_U \zeta f + \sum_{i=1}^n \partial_i \zeta h_i \, d\mathcal{L}^n \quad \forall \zeta \in W^{1,2}(U) \tag{5.50}$$

$$u \in W^{1,2}(U) \tag{5.51}$$

Theorem 5.9. *Let $U \subset \mathbb{R}^n$ be bounded and open. Let $a_{ij}, b \in L^\infty(U)$, $f, h_i \in L^2(U)$ and assume that*

$$b(x) \geq c > 0 \quad \text{for a.e. } x \in U \tag{5.52}$$

and that the coefficients a_{ij} are elliptic, *i.e.,*

$$\exists c > 0 \; \forall \xi \in \mathbb{R}^n \quad \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq c|\xi|^2 \quad \text{for a.e. } x \in U. \tag{5.53}$$

Then there exists one and only one weak solution u of the Neumann problem. Moreover

$$\|u\|^2_{W^{1,2}} \le \frac{1}{c}(\|f\|^2_{L^2} + \sum \|h_i\|^2_{L^2}). \tag{5.54}$$

Proof. This is parallel to the proof of Theorem 5.7. The quadratic form a given by (5.40) satisfies

$$a(v,v) \ge \int_U c|\nabla v|^2 + c|v|^2 \, d\mathcal{L}^n = c\|v\|^2_{W^{1,2}}. \tag{5.55}$$

Now the assertion follows from the Lax-Milgram theorem. □

In the important special case $b = 0$ then the assumptions of Theorem 5.9 are not satisfied. Indeed, we do not have existence of a weak solution for all f and h_i. If u is a weak solution we get with $\zeta = 1$

$$0 = a(1,u) = T(1) = \int_U f \, d\mathcal{L}^n. \tag{5.56}$$

Hence the condition $\int_U f \, d\mathcal{L}^n = 0$ is necessary for the existence of a solution if $b = 0$. Indeed it is also sufficient.

Theorem 5.10. *Let U, a_{ij}, f, h_i be as in Theorem 5.9 and assume that $b = 0$. Assume in addition that U has Lipschitz boundary. Then there exists a weak solution of the Neumann problem if and only if*

$$\int_U f \, d\mathcal{L}^n = 0. \tag{5.57}$$

If (5.57) holds then the weak Neumann problem has a unique solution u in the space

$$X := \{v \in W^{1,2}(U) : \int_U v \, d\mathcal{L}^n = 0\}. \tag{5.58}$$

Any other solution is of the form $u + const.$

Sketch of the proof. The main point is that in X a Poincaré inequality holds, i.e., there exists a constant C which depends only on U such that

$$\|v\|_{L^2} \le C\|\nabla v\|_{L^2} \quad \forall v \in X. \tag{5.59}$$

Together with the ellipticity assumptions this yields

$$a(v,v) \ge c\|\nabla v\|^2_{L^2} \ge c'\|v\|_{W^{1,2}} \quad \forall v \in X. \tag{5.60}$$

Moreover X is a closed subspace of $W^{1,2}(U)$ and thus a Hilbert space. Hence we can apply Corollary 5.4 in X and this shows that there exists one and only one $u \in X$ such that

$$a(\zeta,u) = T(\zeta) \quad \forall \zeta \in X. \tag{5.61}$$

Now every $\tilde{\zeta} \in W^{1,2}(U)$ can be written as $\tilde{\zeta} = \zeta + \text{const}$, where $\zeta \in X$. Since $b = 0$ we have $a(1, u) = 0$ and by assumption $T(\text{const}) = \text{const}\, T(1) = 0$. Thus $a(u, \tilde{\zeta}) = T(\tilde{\zeta})$ for all $\tilde{\zeta} \in W^{1,2}(U)$. Hence u is a weak solution of the Neumann problem.

It is easy to see that every constant function is a weak solution, too. Hence $u + \text{const}$ is a solution. Conversely let v be a weak solution of the Neumann problem. Then there exist a constant a such that $v - a \in X$ and v is a weak solution of the Neumann problem. In particular $a(v, \zeta) = T(\zeta)$ for all $\zeta \in X$. Hence $v = u$ since we have shown that (5.61) has only one solution. □

The theory of weak solutions does not only give existence and uniqueness of solutions in a natural way. It also provides an easy and systematic way to compute approximate solutions and to estimate the error between the approximate and the exact solutions. We state an abstract result which can be applied both to the Dirichlet problem and the Neumann problem.

Theorem 5.11. *Let X be a Hilbert space, let $T \in X'$ and let $a : X \times X \to \mathbb{K}$ be as in the Lax-Milgram theorem, i.e., a is a sesquilinear form and there exist $C, c > 0$ such that*

$$a(\zeta, u) \leq C\|\zeta\|\,\|u\|, \quad a(u, u) \geq c\|u\|^2 \quad \forall \zeta, u \in X. \tag{5.62}$$

Let $u \in X$ be the solution of

$$a(\zeta, u) = T(\zeta) \quad \forall \zeta \in X \tag{5.63}$$

which exists by Corollary 5.4.

Let $Y \subset X$ be a finite dimensional subspace. Then there exists one and only one u_Y such that

$$a(\zeta, u_Y) = T(\zeta) \quad \forall \zeta \in Y. \tag{5.64}$$

Moreover

$$\|u - u_Y\| \leq \frac{C}{c} \inf\{\|u - v\| : v \in Y\}. \tag{5.65}$$

If in addition a is symmetric then $\|u\|_a := a(u, u)^{1/2}$ is a norm (often called the 'energy norm') and

$$\|u - u_Y\|_a \leq \inf\{\|u - v\|_a : v \in Y\}. \tag{5.66}$$

Thus the error between the true solution u and the approximate solution u_Y in the energy norm agrees with error made by looking at the best possible approximation of u in the subspace Y. In the given norm of X such an estimate holds up to a constant.

If $u_1, \dots u_N$ is a basis of Y the approximate solution u_Y has a unique representation $u_Y = \sum_{i=1}^N \alpha_i u_i$ and the coefficients α_i are the solutions of the linear system of equations

$$\sum_{j=1}^{n} A_{ij}\overline{\alpha_j} = b_i \quad \forall i = 1, \dots n, \qquad \text{where} \quad A_{ij} = a(u_i, u_j), \quad b_i = T(u_i). \tag{5.67}$$

In the context of elliptic PDE the sequilinear form a and the functional T are given by (5.40) and (5.41), respectively and one usually chooses the u_i such that many of the matrix entries A_{ij} are zero. One can, e.g., take u_i as piecewise affine and continuous 'hat functions' (linear finite elements).

Proof. See, Homework sheet 7, problem 4. □

6 Linear functionals on Banach spaces and the Hahn-Banach theorems

Theorem 6.1 (Dual space of L^p). *Let $p \in [1, \infty)$, let $\frac{1}{p} + \frac{1}{p'} = 1$, let $E \subset \mathbb{R}^n$ be measurable. Then*

$$J(g)(f) := \int_E f\bar{g}\, d\mathcal{L}^n \tag{6.1}$$

defines a conjugately linear and isometric isomorphism from $L^{p'}(E)$ to $(L^p(E))'$.

Remark The result for $1 < p < \infty$ holds also for $L^p(X, \mathcal{S}, \mu)$ where $(X, \mathcal{S}, \mu)$ is a general measure space. For $p = 1$ one needs to assume in addition that μ is σ-finite, i.e., that X can be write as a countable union of measurable subsets with finite measure.

Proof. Assume first that $p \in (1, \infty)$. By Hölder's inequality $J(g)(f) \le \|f\|_{L^p} \|g\|_{L^{p'}}$ and thus $J(g) \in (L^p(E))'$ and $\|J(g)\| \le \|g\|$. To see that equality holds we set

$$f(x) = \begin{cases} |g(x)|^{p'-2} g(x) & \text{if } g(x) \neq 0, \\ 0 & \text{if } g(x) = 0. \end{cases} \tag{6.2}$$

Since $p = \frac{p'}{p'-1}$ we have

$$\|f\|_{L^p}^p = \int_E |g|^{p'}\, d\mathcal{L}^n = \|g\|_{L^{p'}}^{p'} \tag{6.3}$$

and thus

$$\|g\|_{L^{p'}}^{p'} = J(g)(f) \le \|J(g)\|\, \|f\|_{L^p} \le \|J(g)\|\, \|g\|_{L^{p'}}^{\frac{p'}{p}}. \tag{6.4}$$

Now $p' - \frac{p'}{p} = 1$ and hence $\|J(g)\| \ge \|g\|_{L^{p'}}$.

The main point is to show that J is surjective, i.e., every $T \in (L^p(E))'$ is of the above form. We first assume that $p \in (1, \infty)$. Let $T \in (L^p(E))'$, assume that $T \neq 0$ and set

$$A = \{f \in L^p(E) : T(f) = 1\}. \tag{6.5}$$

Then A is non empty, closed and convex (in fact affine). Since $L^p(E)$ is uniformly convex it follows from Theorem 3.7 that there exists $f_* \in A$ such that

$$\|f_*\|_{L^p} \leq \|g\|_{L^p} \quad \forall g \in A. \tag{6.6}$$

Let $h \in L^p(E)$. Then for sufficiently small $|t|$ we have $T(f_* + th) \neq 0$ and thus (6.6) implies that

$$\|f_*\|^p_{L^p} \leq \left\| \frac{f_* + th}{T(f_* + th)} \right\|^p_{L^p} = \frac{\|f_* + th\|^p_{L^p}}{|T(f_* + th)|^p}. \tag{6.7}$$

Hence the expression on the right hand side has a minimum at $t = 0$. Now $\frac{d}{dt}_{t=0}|a + tb|^p = \mathrm{Re}(p|a|^{p-2}\bar{a}b)$ and $T(f_*) = 1$. Thus differentiation with respect to t at $t = 0$ yields

$$0 = p\,\mathrm{Re}\left(\int_E |f_*|^{p-2}\bar{f}_* h \, d\mathcal{L}^n\right) - p\|f_*\|^p_{L^p}\,\mathrm{Re}(T(h)). \tag{6.8}$$

Using the above equation for h and ih we get

$$T(h) = \int_E g_T h \, d\mathcal{L}^n, \quad \text{with } g_T = \|f_*\|^{-p}_{L^p}\,|f_*|^{p-2}\bar{f}_* \tag{6.9}$$

This finishes the proof for $p \in (1, \infty)$.

For $p = 1$ consider first the case that $\mathcal{L}^n(E) < \infty$. Then $L^2(E) \subset L^1(E)$. Hence by the result for $p = 2$, for $T \in (L^1(E))'$ there exists $g \in L^2(E)$ such that

$$T(f) = \int f\bar{g} \, d\mathcal{L}^n \quad \forall f \in L^2(E). \tag{6.10}$$

We now show that

$$g \in L^\infty(E) \quad \text{and} \quad \|g\|_{L^\infty} \leq \|T\|. \tag{6.11}$$

Let

$$S_M := \{x \in E : |g(x)| \geq M\}, \quad f = \chi_{S_M} g. \tag{6.12}$$

Then

$$M\int_{S_M} |g| \, d\mathcal{L}^n \leq \int_{S_M} |g|^2 \, d\mathcal{L}^n = \int_E f\bar{g} \, d\mathcal{L}^n \leq \|T\|\,\|f\|_{L^1} \leq \|T\| \int_{S_M} |g| \, d\mathcal{L}^n. \tag{6.13}$$

For $M > \|T\|$ we deduce that $\int_{S_M} |g| \, d\mathcal{L}^n = 0$. Since $|g| \geq M$ in S_M this implies that S_M is a null set. This proves (6.11). To see that (6.10) holds for $f \in L^1(E)$, define for $K \in \mathbb{N}$ the truncated function f_K by

$$f_K(x) := \begin{cases} f(x) & \text{if } |f(x)| \leq K \\ 0 & \text{otherwise.} \end{cases}$$

Then by dominated convergence $f_K \to f$ in $L^1(E)$ as $K \to \infty$, and (again by dominated convergence)

$$\int_E f_K \bar{g} \, d\mathcal{L}^n \longrightarrow \int_E f \bar{g} \, d\mathcal{L}^n.$$

Thus by continuity of T,

$$\int_E f \bar{g} \, d\mathcal{L}^n = \lim_{K \to \infty} \int_E f_K \bar{g} \, d\mathcal{L}^n = \lim_{K \to \infty} T(f_K) = T(f).$$

[1.12. 2017, Lecture 15]

[8.12. 2017, Lecture 16]

Finally consider the case $\mathcal{L}(E) = \infty$ and let $k \mapsto E_k$ be an increasing sequence of measurable sets with $\mathcal{L}^n(E_k) < \infty$ and $E = \bigcup_{k \in \mathbb{N}} E_k$. Let

$$T_k(f) := T(f \chi_{E_k}). \tag{6.14}$$

Then $T_k \in (L^1(E))'$ with $\|T_k\|_{(L^1)'} \leq \|T\|_{(L^1)'}$ and also $T_k \in (L^2(E))'$ since

$$T_k(f) \leq \|T\| \, \|f \chi_{E_k}\|_{L^1} \leq \|T\| \, \|f\|_{L^2} \mathcal{L}(E_k)^{1/2}. \tag{6.15}$$

Thus there exist $g_k \in L^2(E)$ such that

$$T_k(f) = \int_E f \bar{g}_k \, d\mathcal{L}^n \quad \forall f \in L^2(E). \tag{6.16}$$

By the argument above this implies that $g_k \in L^\infty(E)$ and $\|g_k\|_{L^\infty} \leq \|T\|_{(L^1)'}$. The choice $f = \chi_{E_j \setminus E_k} g_k$ shows that $g_k = 0$ a.e. on $E \setminus E_k$. Now we have

$$\int_E f \chi_{E_k} \bar{g}_k \, d\mathcal{L}^n = T_k(f \chi_{E_k}) = T(f \chi_{E_k}) = T_{k+1}(f \chi_{E_k}) = \int_E f \chi_{E_k} \bar{g}_{k+1} \, d\mathcal{L}^n. \tag{6.17}$$

Thus $g_{k+1} = g_k$ a.e. in E_k and we can define $g \in L^\infty$ with $\|g\|_{L^\infty} \leq \|T\|$ by

$$g := g_k \quad \text{on} \quad E_k \tag{6.18}$$

Moreover

$$T(f \chi_{E_k}) = \int_E f \bar{g}_k \, d\mathcal{L}^n = \int_E f \chi_{E_k} \bar{g}_k \, d\mathcal{L}^n = \int_E f \chi_{E_k} \bar{g} \, d\mathcal{L}^n. \tag{6.19}$$

As above, this equality holds for $f \in L^1(E)$. If $f \in L^1(E)$ then the dominated convergence theorem implies that $f\chi_{E_k} \to f$ in $L^1(E)$. Since T is continuous on $L^1(E)$ and $\bar{g} \in L^\infty(E)$ passing to he limit $k \to \infty$ we get

$$T(f) = \int_E f\bar{g}\, d\mathcal{L}^n. \tag{6.20}$$

To show that J is isometric, let $g \in L^\infty(E)$. Then again by Hölder's inequality $J(g) \in (L^1(E))'$ and $\|J(g)\|_{(L^1)'} \le \|g\|_{L^\infty}$. Equality follows by the reasoning of (6.13) with S_M replaced by $S_M \cap E_k$. □

Theorem 6.2 (Hahn-Banach I)**.** *Let X be an $\mathbb{R}$-vector space. Suppose that*

(i) $p : X \to \mathbb{R}$ is sublinear, i.e., for all x, $y \in X$

$$p(x + y) \le p(x) + p(y) \quad \textit{and} \quad p(\alpha x) = \alpha p(x) \quad \textit{for all } \alpha \ge 0,$$

(ii) $f : Y \to \mathbb{R}$ is linear, where Y is a subspace of X, and

(iii) $f(x) \le p(x)$ for all $x \in Y$.

Then there is $F : X \to \mathbb{R}$ linear with

$$F(x) = f(x) \textit{ for all } x \in Y \quad \textit{and} \quad F(x) \le p(x) \textit{ for all } x \in X.$$

Note that p is sublinear if and only if p is convex and $p(\alpha x) = \alpha p(x)$ for all $\alpha \ge 0$.

Note also that the statement of the theorem is purely algebraic/ geometric. No norms or topologies appear in the statement.

First part of the proof. We show that if $Z \subset X$ is a subspace, if $g : Z \to \mathbb{R}$ is linear with $g \le p$ and $z_0 \notin Z$ then g can be extended to a linear map g_0 on $Z_0 := Z \oplus \operatorname{span}\{z_0\}$ with $g_0 \le p$. To do so we make the ansatz

$$g_0(z + \alpha z_0) = g(z) + c\alpha. \tag{6.21}$$

Clearly g_0 is linear on Z_0 and $g_{0|Z} = g$. We have to show that $c \in \mathbb{R}$ can be chosen so that

$$g(z) + c\alpha \le p(z + \alpha z_0) \quad \forall z \in Z \;\forall \alpha \in \mathbb{R}. \tag{6.22}$$

Since $g \le p$ on Z the condition holds for $\alpha = 0$. For $\alpha > 0$ the condition becomes

$$c \le \frac{1}{\alpha}(p(z + \alpha z_0) - g(z)) = p(\frac{z}{\alpha} + z_0) - g(\frac{z}{\alpha}). \tag{6.23}$$

For $\alpha < 0$ the condition becomes

$$c \ge \frac{1}{\alpha}(p(z + \alpha z_0) - g(z)) = g(-\frac{z}{\alpha}) - p(-\frac{z}{\alpha} - z_0). \tag{6.24}$$

We thus need to satisfy

$$\sup_{z\in Z} g(z) - p(z-z_0) \le c \le \inf_{z'\in Z} p(z'+z_0) - g(z') \tag{6.25}$$

This is possible since for all $z, z' \in Z$

$$g(z)+g(z') = g(z+z') \le p(z+z') = p((z-z_0)+(z'+z_0)) \le p(z-z_0)+p(z'+z_0) \tag{6.26}$$

which implies that

$$g(z) - p(z-z_0) \le p(z'+z_0) - g(z'). \tag{6.27}$$

If X is finite dimensional we can now deduce the theorem by induction. For general X induction is replaced by Zorn's lemma. □

To state Zorn's lemma we recall some basic notions for (partially) ordered sets. Let P be a set. A (partial) order $\le$ is a relation on P which is reflexive, antisymmetric and transitive, i.e., for all $a, b, c \in P$ we have

(i) (reflexivity) $a \le a$;

(ii) (antisymmetry) if $a \le b$ and $b \le a$ then $a = b$;

(iii) (transitivity) if $a \le b$ and $b \le c$ then $a \le c$.

We call the pair $(P, \le)$ an ordered set. A subset $Q \subset P$ is *totally ordered* if for any pair $(a, b) \in Q \times Q$ we have $a \le b$ or $b \le a$. For an arbitrary subset $Q \subset P$ we say that $c \in P$ is an *upper bound* for Q if $a \le c$ for all $a \in Q$. We say that an $m \in P$ is a *maximal element* of P if the relation $m \le x$ holds only for $x = m$. Note that a maximal element need not be an upper bound for P (e.g. if $a \le b$ if and only if $a = b$ then every element is a maximal element but P has no upper bound, if P contains more than one element).

We say that P is *inductive* if every totally ordered subset $Q \subset P$ has an upper bound.

Theorem 6.3 (Zorn's lemma)**.** *Every non empty ordered set that is inductive has a maximal element.*

Proof of the Hahn-Banach Theorem, second part. We set

$$P := \{(Z, g) : Z \subset X \text{ subspace},\ Y \subset Z, \tag{6.28}$$

$$g : Z \to \mathbb{R} \text{ linear},\ g_{|Y} = f,\ g \le p \text{ on } Z\} \tag{6.29}$$

and we define an order on P by

$$(Z_1, g_1) \le (Z_2, g_2) \iff Z_1 \subset Z_2, \quad g_2 = g_1 \quad \text{on } Z_1. \tag{6.30}$$

We now verify the assumptions of Zorn's lemma. Let $Q \subset P$ be totally ordered. Define

$$Z_* := \bigcup_{(Z,g)\in Q} Z, \tag{6.31}$$

$$g_*(x) := g(x) \quad \text{if } x \in Z \quad \text{and } (Z,g) \in Q. \tag{6.32}$$

We have to show that $(Z_*, g_*) \in P$. Then it follows that (Z_*, g_*) is an upper bound for Q. To see that $(Z_*, g_*) \in P$ first note that $Y \subset Z_* \subset X$. Moreover g_* is well defined. Indeed if

$$z \in Z_1 \cap Z_2 \quad \text{and} \quad (Z_1, g_1) \in Q, \quad (Z_2, g_2) \in Q \tag{6.33}$$

then

$$(Z_1, g_1) \le (Z_2, g_2) \quad \text{or} \quad (Z_2, g_2) \le (Z_1, g_1) \tag{6.34}$$

since Q is totally ordered. Assume the first case. Then

$$Z_1 \subset Z_2 \quad \text{and} \quad g_1 = g_2 \quad \text{on } Z_1 \tag{6.35}$$

and thus

$$g_1(z) = g_2(z). \tag{6.36}$$

In the second case we arrive at the same conclusion. A similar argument shows that Z_* is a linear space and $g_* : Z_* \to \mathbb{R}$ is linear. Finally the definition of P and of g_* yields $g_{*|Y} = f$ and $g_* \le p$ on Z_*. Thus every totally ordered subset of P has an upper bound.

By Zorn's lemma P has a maximal element (Z, g). If $Z = X$ we are done. If $Z \neq X$ then the first step yields (Z_0, g_0) with

$$(Z, g) \le (Z_0, g_0) \quad \text{and} \quad Z_0 \neq Z. \tag{6.37}$$

This contradicts the maximality of (Z, g). □

[8.12. 2017, Lecture 16]

[13.12. 2017, Lecture 17]

Theorem 6.4 (Hahn-Banach II). *Let X be a normed $\mathbb{K}$ vector space and let Y be a subspace (with the norm induced by X). Then for every $y' \in Y'$ there exists $x' \in X'$ such that*

$$x' = y' \quad \textit{on} \quad Y, \qquad \|x'\|_{X'} = \|y'\|_{Y'}. \tag{6.38}$$

Proof. For $\mathbb{K} = \mathbb{R}$ apply Theorem 6.2 with $p(x) = \|y'\| \, \|x\|$. Thus there exists a linear map $x' : X \to \mathbb{R}$ with $x'(y) = y'(y)$ for $y \in Y$ and for all x we have

$$x'(x) \le p(x) = \|y'\| \, \|x\|, \tag{6.39}$$

$$-x'(x) = x'(-x) \le p(-x) = \|y'\| \, \|x\|. \tag{6.40}$$

Thus $x' \in X'$ and $\|x'\| \le \|y'\|$. We have equality since by definition of $\|y'\|$ for each $\varepsilon > 0$ there exists $y \in Y \setminus \{0\}$ such that

$$x'(y) = y'(y) \ge (1-\varepsilon)\|y'\|. \tag{6.41}$$

For $\mathbb{K} = \mathbb{C}$ consider X and Y as normed $\mathbb{R}$ vector spaces $X_{\mathbb{R}}$ and $Y_{\mathbb{R}}$, i.e., the scalar multiplication is only carried out for real numbers and the norms are the same as before). If $y' \in Y'$ then

$$y'_{\mathrm{re}}(x) := \operatorname{Re} y'(x) \tag{6.42}$$

defines a bounded linear functional on $Y_{\mathbb{R}}$ and

$$\|y'_{\mathrm{re}}\|_{Y'_{\mathbb{R}}} \le \|y'\|_{Y'}. \tag{6.43}$$

Moreover

$$y'(x) = \operatorname{Re} y'(x) + i \operatorname{Im} y'(x) = y'_{\mathrm{re}}(x) - i y'_{\mathrm{re}}(ix). \tag{6.44}$$

Extend y'_{re} to x'_{re} and define

$$x'(x) := x'_{\mathrm{re}}(x) - i x'_{\mathrm{re}}(ix). \tag{6.45}$$

Then one sees easily that x' is $\mathbb{C}$ linear and a short calculation shows that $\|x'\| = \|x_{\mathrm{re}}\| = \|y_{\mathrm{re}}\| = \|y'\|$. □

Theorem 6.5. *Let X be a normed space, let $Y \subset X$ be a closed subspace and let $x_0 \in X \setminus Y$. Then there exist and $x' \in X'$ such that*

$$x' = 0 \quad \text{on} \quad Y, \qquad \|x'\| = 1, \qquad x'(x_0) = \operatorname{dist}(x_0, Y). \tag{6.46}$$

Remark. If X is a Hilbert space we can take $x'(x) = (x, \frac{x_0 - Px_0}{\|x_0 - Px_0\|})$ where P is the orthogonal projection onto Y. Theorem 6.5 can often be used as a substitute for the orthogonal projection.

Proof. On

$$Y_0 := Y \oplus \operatorname{span}\{x_0\} \tag{6.47}$$

define

$$y'_0(y + \alpha x_0) := \alpha \operatorname{dist}(x_0, Y). \tag{6.48}$$

Then $y'_0 : Y_0 \to \mathbb{K}$ is linear and $y'_0 = 0$ on Y. We only need to show that y'_0 is bounded and $\|y'_0\| = 1$. Then the assertion follow from Theorem 6.4.

For $y \in Y$ and $\alpha \ne 0$ we have

$$\operatorname{dist}(x_0, Y) \le \left\| x_0 - \frac{-y}{\alpha} \right\| \tag{6.49}$$

and thus

$$|y_0'(y+\alpha x_0)| \le |\alpha| \left\| x_0 - \frac{-y}{\alpha} \right\| = \|\alpha x_0 + y\|. \tag{6.50}$$

It follows that $y_0' \in Y_0'$ and $\|y_0'\| \le 1$.

On the other hand we have $\operatorname{dist}(x_0, Y) > 0$ since Y is closed. Let $\varepsilon > 0$. Then there exist $y_\varepsilon \in Y$ such that

$$\|x_0 - y_\varepsilon\| \le (1+\varepsilon)\operatorname{dist}(x_0, Y). \tag{6.51}$$

Then

$$y_0'(x_0 - y_\varepsilon) = y_0'(x_0) = \operatorname{dist}(x_0, Y) \ge \frac{1}{1+\varepsilon} \|x_0 - y_\varepsilon\|. \tag{6.52}$$

Since $x_0 - y_\varepsilon \ne 0$ this implies that $\|y_0'\| \ge \frac{1}{1+\varepsilon}$. Since $\varepsilon > 0$ was arbitrary we finally conclude that $\|y_0'\| = 1$. □

Corollary 6.6. *Let X be a normed space and $x_0 \in X$.*

(i) If $x_0 \ne 0$ then there exists $x_0' \in X'$ such that

$$\|x_0'\| = 1 \quad \textit{and} \quad x_0'(x_0) = \|x_0\|. \tag{6.53}$$

(ii) If $x'(x_0)$ for all $x' \in X'$ then $x_0 = 0$.

Proof. For the first assertion apply Theorem 6.5 with $Y = \{0\}$. The second assertion follows from the first. □

We now show that a point outside a closed convex sets can be separated from the set by a half space. We will see later that geometrically intuitive fact has also profound consequences for functional analysis.

Lemma 6.7. *Let X be a normed space and let $M \subset X$ be a closed convex set with $0 \in M^0$. Let*

$$p(x) := \inf\{r > 0 : \frac{x}{r} \in M\} \tag{6.54}$$

Then $p(x) < \infty$, $p : X \to \mathbb{R}$ is sublinear and

$$p(x) \le 1 \iff x \in M. \tag{6.55}$$

Theorem 6.8 (Separation of convex sets)**.** *Let X be a normed space. Let $M \subset X$ be non empty, closed and convex and let $x_0 \in X \setminus M$. Then there exists $x' \in X'$ and $\alpha \in \mathbb{R}$ such that*

$$\operatorname{Re} x'(x) \le \alpha \quad \forall x \in M \quad \textit{and} \quad \operatorname{Re} x'(x_0) > \alpha. \tag{6.56}$$

Proof. We first consider $\mathbb{K} = \mathbb{R}$. We may assume that $0 \in M$ (otherwise let $\tilde{x} \in M$ and consider $-\tilde{x} + M$ instead of M and $\tilde{x}_0 = x_0 - \tilde{x}$ instead of x_0).

Indeed we may assume that $0 \in M^0$. Otherwise let $0 < r < \text{dist}\,(x_0, M)$ and consider $\tilde{M} = \overline{B_r(M)}$. Then $\tilde{M}$ is closed and convex and $0 \in \tilde{M}^0$. If the result holds for $\tilde{M}$ it also holds for M.

Thus assume $0 \in M^0$ and let p be the sublinear map defined in Lemma 6.7. On span $\{x_0\}$ define f by

$$f(ax_0) := ap(x_0). \tag{6.57}$$

Then for $a \geq 0$ we have $f(ax_0) = p(ax_0)$ and for $a < 0$ we have $f(ax_0) \leq 0 \leq p(ax_0)$. Thus by the Hahn-Banach theorem, Theorem 6.2, there exists a linear extension $F : X \to \mathbb{R}$ with $F \leq p$. Thus

$$F \leq p \leq 1 \quad \text{on } M, \qquad F(x_0) = f(x_0) = p(x_0) > 1. \tag{6.58}$$

It remains to show that $F \in X'$. Then the assertion follows with $x' = F$ and $\alpha = 1$. Since $0 \in M^0$ there exist $\rho > 0$ with $\overline{B}(0, \rho) \subset M$. Thus

$$x \in X \quad \Longrightarrow \quad \frac{x}{\frac{1}{\rho}\|x\|} \in M \quad \Longrightarrow \quad p(x) \leq \frac{1}{\rho}\|x\| \quad \Longrightarrow \quad F(x) \leq \frac{1}{\rho}\|x\|. \tag{6.59}$$

We also have $-F(x) = F(-x) \leq \frac{1}{\rho}\|x\|$ and thus $F \in X'$.

For $\mathbb{K} = \mathbb{C}$ consider the real vector space $X_{\mathbb{R}}$ and obtain $F_{\mathbb{R}} \in X'_{\mathbb{R}}$ with the desired properties. Then consider $F(x) = F_{\mathbb{R}}(x) - iF_{\mathbb{R}}(ix)$ as in the proof proof of Theorem 6.4. □

We will now use the Hahn-Banach theorem to find the dual space of continuous functions. The arguments below were only sketched briefly in the lecture.

Let $K \subset \mathbb{R}^n$ be a compact set, let $\mathcal{B}(K)$ denote the Borel subsets of K and let $\mu : \mathcal{B}(K) \to \mathbb{R}$ be a measure with $\mu(K) < \infty$. Let $g \in C(K)$. Then

$$T(g) := \int_K g\, d\mu \tag{6.60}$$

is well-defined and $T : C(K) \to \mathbb{R}$ is linear and continuous with $\|T\| = \mu(K)$ (for equality consider $g \equiv 1$).

Similarly, if $\sigma : K \to \{-1, 1\}$ is Borel measurable then

$$T(g) := \int_K g\, \sigma\, d\mu \tag{6.61}$$

defines and element of $C(K)'$ with $\|T\| = \mu(K)$.

We now want to show that all elements of $C(K)'$ are of this form. There are several approaches to this. We will use the Hahn-Banach theorem to first construct a finitely additive signed measure which represents T.

Indeed if $T : C(K) \to \mathbb{R}$ is continuous and $B(K)$ denotes the space of bounded functions on the compact set K then T can be extended to a continuous linear functional on $\tilde{T} : B(K) \to \mathbb{R}$. Now we can define

$$\lambda(E) := \tilde{T}(\chi_E) \quad \forall E \subset K. \tag{6.62}$$

Then λ is a map from the subsets of K to $\mathbb{R}$ and $|\lambda(E)| \le \|T\|$. Moreover the linearity of $\tilde{T}$ yields

$$E_1 \cap E_2 = \emptyset \quad \Longrightarrow \quad \lambda(E_1 \cup E_2) = \lambda(E_1) + \lambda(E_2). \tag{6.63}$$

We call such maps λ finitely additive signed measures (or shorter: additive measures).

Definition 6.9. *Let X be a set and $\mathcal{S} \subset 2^X$. We say that $\mathcal{S}$ is a Boolean algebra if*

(i) $\emptyset, X \in \mathcal{S}$,

(ii) $E \in \mathcal{S} \quad \Longrightarrow \quad X \setminus E \in \mathcal{S}$,

(iii) $E, F \in \mathcal{S} \quad \Longrightarrow \quad E \cup F \in \mathcal{S}$.

If $\mathcal{S}$ is a Boolean algebra we say that $\lambda : \mathcal{S} \to \mathbb{R}^d$ is (finitely) additive measure if

$$E, F \in \mathcal{S}, E \cap F = \emptyset \quad \Longrightarrow \quad \lambda(E \cup F) = \lambda(E) + \lambda(F). \tag{6.64}$$

We define the variation measure $\|\lambda\|_{\mathrm{var}} : \mathcal{S} \to [0, \infty]$ associated to λ by

$$\|\lambda\|_{\mathrm{var}}(E) := \sup\{\sum_{i=1}^{k} |\lambda(E_i)| : E_i \subset E \in \mathcal{S},\ E_i \textit{ disjoint}\}. \tag{6.65}$$

It is easy to see that the arbitrary intersection of Boolean algebras is a Boolean algebra.

Definition 6.10. *Let $S \subset \mathbb{R}^n$ be bounded and open or closed. We set*

$$\mathcal{U} := \{E \subset S : E \textit{ relatively open}\} \tag{6.66}$$

and

$$\mathcal{B}_0(S) := \textit{ smallest Boolean algebra which contains } \mathcal{U} \tag{6.67}$$

and we recall that the Borel σ-algebra was defined as

$$\mathcal{B}(S) := \textit{ smallest } \sigma\textit{-algebra which contains } \mathcal{U}. \tag{6.68}$$

We set

$$\mathrm{ba}\,(S) := \{\lambda : \mathcal{B}_0(S) \to \mathbb{R}^d : \lambda \textit{ finitely additive},\ \|\lambda\|_{\mathrm{var}}(S) < \infty\}, \tag{6.69}$$

$$\mathrm{ca}\,(S) := \{\lambda : \mathcal{B}(S) \to \mathbb{R}^d : \lambda\ \sigma\textit{-additive},\ \|\lambda\|_{\mathrm{var}}(S) < \infty\}. \tag{6.70}$$

An element $\lambda \in \mathrm{ba}\,(S)$ *or* $\lambda \in \mathrm{ca}\,(S)$ *is called regular if*

$$\inf\{\|\lambda_{\mathrm{var}}\|(U \setminus K) : K \subset E \subset U : K \text{ compact}, U \text{ relatively open}\} = 0 \tag{6.71}$$

for all E *in* $\mathcal{B}_0(S)$ *or all* $E \in \mathcal{B}(S)$, *respectively. We set*

$$\mathrm{rba}\,(S) := \{\lambda \in \mathrm{ba}\,(S) : \lambda \text{ regular}\}, \quad \mathrm{rca}\,(S) := \{\lambda \in \mathrm{ca}\,(S) : \lambda \text{ regular}\}. \tag{6.72}$$

Remark. (i) The abbreviations can be read as 'bounded and (finitely) additive', 'countably additive', 'regular bounded additive' and 'regular countably additive'[4] The element of $\mathrm{rca}\,(S)$ are often also called regular Borel measures on S.
(ii) Note that the assumption $\|\lambda\|_{\mathrm{var}}(S) < \infty$ implies that for any sequence $i \mapsto E_i$ of disjoint sets in $\mathcal{B}(S)$ the sum $\sum_{i\in\mathbb{N}} \lambda(E_i)$ converges absolutely because all the partial sums $\sum_{i=1}^k |\lambda(E_i)|$ are bounded by $\|\lambda\|_{\mathrm{var}}(S)$.

Lemma 6.11. *The spaces* $\mathrm{ba}\,(S)$, $\mathrm{ca}\,(S)$, $\mathrm{rba}\,(S)$ *and* $\mathrm{rca}\,(S)$ *are Banach spaces with norm* $\|\lambda\| = \|\lambda\|_{\mathrm{var}}(S)$.

Proof. Exercise. □

For K compact and $\lambda \in \mathrm{ba}\,(K)$ one can define an integral in the usual way. First let $f : K \to \mathbb{R}$ be a simple function, i.e. $f = \sum_{i=1}^k \alpha_i \chi_{E_i}$ with $E_i \in \mathcal{B}_0(K)$. Then we define

$$\int_S f\, d\lambda := \sum \alpha_i \lambda(E_i) \tag{6.73}$$

and one can show that the sum on the right side depends only on f, i.e. if also $f = \sum_{j=1}^l \beta_j \chi_{F_j}$ then $\sum_i \alpha_i \lambda(E_i) = \sum_j \beta_j \lambda(F_j)$. Now every $f \in C(K)$ is a uniform limit of simple functions and using that $\|\lambda\| < \infty$ one can define the integral of a continuous function uniquely through approximation by simple functions.

Theorem 6.12. *(Riesz-Radon) Let* $K \subset \mathbb{R}^n$ *be compact. Then the map* $J : \mathrm{rca}\,(K) \to C(K)'$ *given by*

$$J(\nu)(f) := \int_K f\, d\nu \tag{6.74}$$

is an isometric isomorphism.

[4] This terminology comes from N.Dunford, J.T. Schwartz, Linear operators, IV 2.15–2.17; finitely additive measures are quite different from σ-additive measures and have some strange properties, see K. Yosida, E. Hewitt, Finitely additive measures, Trans. Amer. Math. Soc. vol. 72 (1952), pp. 46–66.

Remark. The assumption $K \subset \mathbb{R}^n$ is not essential here. The result holds for a general compact Hausdorff space K[5]

Sketch of proof. Isometry: The estimate $\|T\| \le \|\nu\|$ follows directly from the definition of the integral. To prove the reverse estimate let $E_i \in \mathcal{B}(K)$ be disjoint. If the test function $f := \sum \sigma_i \chi_{E_i}$ with $\sigma_i = \operatorname{sgn}(E_i)$ was admissible then one would conclude immediately $\|\nu\|_{\mathrm{var}}(K) \le \|T\|\|f\| = \|T\|$. For a complete proof one approximates χ_{E_i} by continuous functions and uses the regularity of ν.

Surjectivity: This is the hard part. One first uses the Hahn-Banach theorem to extend T to a bounded linear functional $\tilde{T}$ on the space of bounded function $B(K)$ and defines

$$\lambda(E) := \tilde{T}(\chi_E). \tag{6.75}$$

Then one easily sees that $\lambda \in \mathrm{ba}\,(K)$ and

$$\tilde{T}(f) = \int_K f\, d\lambda \quad \text{if } f = \chi_E, \quad E \in \mathcal{B}_0(K). \tag{6.76}$$

By linearity and continuity of $\tilde{T}$ we obtain

$$\forall f \in C(K) \quad T(f) = \tilde{T}(f) = \int_K f\, d\lambda. \tag{6.77}$$

Then by one shows by a careful construction that there exists $\nu \in \mathrm{rca}\,(K)$ such that

$$\forall f \in C(K) \quad \int_K f\, d\lambda = \int_K f\, d\nu. \tag{6.78}$$

Idea: first one shows that there exist nonnegative measures $\lambda^\pm \in \mathrm{ba}\,(K)$ such that $\lambda = \lambda^+ - \lambda^-$ Then one can assume without loss of generality that $\lambda \ge 0$. For $\lambda \ge 0$ one defines $\nu : \mathcal{B}_0 \to [0, \infty)$ by

$$\nu(E) = \sup_{\substack{A \subset E \\ A \text{ closed}}} \inf_{\substack{U \supset A \\ U \text{ open}}} \lambda(U).$$

Then one can show that $\nu \in \mathrm{rba}\,(K)$ and that μ is countable additive on $\mathcal{B}_0$. Finally one can extend ν to an element of $\mu \in \mathrm{rca}\,(K)$, see Alt's book for the details. □

[13.12. 2017, Lecture 17]

[15.12. 2017, Lecture 18]

[5] see N. Dunford, J.T. Schwartz, Linear Operators, Part I, IV 6.3.

Theorem 6.13. *Let $U \subset \mathbb{R}^n$ be open and bounded. Let $C_0(U)$ denote the closure of $C_c(U)$ in the $\|\cdot\|_{C^0}$ norm. Then the map $J : \mathrm{rca}\,(U) \to C_0(U)'$ given by*

$$J(\lambda)(f) := \int_U f \, d\lambda \tag{6.79}$$

is an isometric isomorphism.

Remark. The same assertion holds for unbounded U, the most important case being $U = \mathbb{R}^n$. One considers U and $\overline{U}$ as subsets of the compact metric space $\mathbb{R}^n \cup \{\infty\}$ (the metric on $\mathbb{R}^n \cup \{\infty\}$ can be defined by stereographic projection to the sphere S^n and its restriction to $\mathbb{R}^n$ is equivalent to the standard metric on $\mathbb{R}^n$). Then one can argue as below, using the Riesz-Radon theorem for $S^n \subset \mathbb{R}^{n-1}$ Note that ∞ belongs to $\overline{U}$ if and only if U is unbounded.

Proof. The main point is again surjectivity. The space $C_c(U)$ can be seen as a subspace of $C(\bar{U})$ (by extending a function in $C_c(U)$ by zero to $\mathbb{R}^n$). Hence $C_0(U)$ is a closed subspace of $C(\bar{U})$ (indeed $C_0(U)$ consists of all functions in $C(\bar{U})$ which vanish on ∂U). Therefore $T \in C_0(U)'$ can be extended to $\tilde{T} \in C(\bar{U})'$. Since $\bar{U}$ is compact there exists a $\mu \in \mathrm{rca}\,(\bar{U})$ such that

$$\tilde{T}(f) := \int_{\overline{U}} f \, d\mu. \tag{6.80}$$

Set $\nu(E) := \mu(E)$ for all $E \in \mathcal{B}(U)$. Then $\nu \in \mathrm{rca}\,(U)$ and for all $f \in C_0(U)$

$$T(f) = \tilde{T}(f) := \int_{\overline{U}} f \, d\mu \underbrace{=}_{f=0 \text{ on } \partial U} \int_U f \, d\mu = \int_U f \, d\nu. \tag{6.81}$$

□

Remark 6.14. *Let $\mathcal{M}_n$ denote the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}^n$. Let $E \in \mathcal{M}_n$ and set $\mathcal{M}_n(E) = \{A \in \mathcal{M}_n : A \subset E\}$. Then the dual space of $L^\infty(E)$ is isometric to the space* $\mathrm{ba}\,(E, \mathcal{L}^n)$ *of finitely additive measures on $\mathcal{M}_n(E)$ which are absolutely continuous with respect to the Lebesgue measure (see homework).*

7 The Baire category theorem and the principle of uniform boundedness

Theorem 7.1 (Baire category theorem)**.** *Let (X, d) be a complete metric space. For $k \in \mathbb{N}$ let $A_k \subset X$ be closed and assume that A_k has empty interior. Then*

$$X \neq \bigcup_{k \in \mathbb{N}} A_k. \tag{7.1}$$

In fact $\bigcup_{k \in \mathbb{N}} A_k$ has empty interior.

Proof. We will construct a decreasing sequence of closed balls $\overline{B}_k = \overline{B}(x_k, r_k)$ with $r_k \to 0$ such that $A_k \cap \overline{B}_k = \emptyset$. Then

$$\bigcap_{k\in\mathbb{N}} \overline{B}_k \subset X \setminus \bigcup_{k\in\mathbb{N}} A_k \tag{7.2}$$

and we will show that the left hand side contains one point.

We construct the balls $\overline{B}_k$ be induction. Since A_0 is closed $X \setminus A_0$ is open. Moreover $X \setminus A_0 \neq \emptyset$ since A_0 is nowhere dense. Thus there exists an open ball $B(x_0, 2r_0) \subset X \setminus A_0$ and we may assume that $r_0 \leq 1$. Set $\overline{B}_0 =: \overline{B}(x_0, r_0)$.

Assume now that closed balls $\overline{B}(x_0, r_0) \supset \ldots \supset \overline{B}_K(x_k, r_k)$ with

$$\overline{B}_k(x_k, r_k) \cap A_k = \emptyset, \quad r_k \leq 2^{-k} \tag{7.3}$$

have been constructed for $k \leq K$. Now

$$B(x_K, r_K) \setminus A_{K+1} \quad \text{is open and not empty} \tag{7.4}$$

since A_{K+1} is closed and nowhere dense. Thus there exist $r_{K+1} \leq 2^{-(K+1)}$ and x_{K+1} such that $B(x_{K+1}, 2r_{K+1}) \subset B(x_K, r_K) \setminus A_{K+1}$. Set $\overline{B}_{K+1} = \overline{B}(x_{K+1}, r_{K+1})$.

It is clear that (7.2) holds. To see that $\bigcap_{k\in\mathbb{N}} \overline{B}_k \neq \emptyset$ note that if $k, l \geq m$ then $x_k, x_l \in B(x_m, r_m)$. Thus $l \mapsto x_l$ is a Cauchy sequence. Since (X, d) is complete $x_l \to x_*$ as $l \to \infty$. Now $x_l \in \overline{B}(x_k, r_k)$ for all $l \geq k$ and all $k \in \mathbb{N}$. Since $\overline{B}(x_k, r_k)$ is closed we get $x_* \in \overline{B}(x_k, r_k)$ for all $k \in \mathbb{N}$. Thus $x_* \in \bigcap \overline{B}_k$.

To show that the set $\bigcup_{k\in\mathbb{N}} A_k$ has empty interior it suffices to show that it contains no closed ball $\overline{B}(x, R)$ with $R > 0$. Now $\overline{B}(x, R)$ is a closed subset of the complete space X and hence a complete space. Moreover the sets $A_k \cap \overline{B}(x, R)$ are closed and have empty interior (as subsets of $\overline{B}(x, R)$). Indeed otherwise there existed a $y \in \overline{B}(x, R)$ and an open ball $B(y, \rho)$ such that $B(y, \rho) \cap \overline{B}(x, R) \subset A_k$. Then $B(y, r_0) \cap B(x, R)$ was not empty and open and contained in A_k. This contradicts the assumption that A_k has empty interior. Now what we have already proved implies that $\bigcup_{k\in\mathbb{N}} (A_k \cap \overline{B}(x, R)) \neq \overline{B}(x, R)$. Thus $\bigcup_{k\in\mathbb{N}} A_k$ does not contain the ball $\overline{B}(x, R)$. □

Remark. Terminology: Recall that a general set $E \subset X$ is nowhere dense if its closure has empty interior. According to Baire one calls a set $E \subset X$ of the *first category* (or *meager*) if it is a countable union of nowhere dense sets. The Baire category theorem states X is not of the first category in itself. The complement of a meager set is called *comeager* or *residual*. One should think of meager sets as small sets and residual sets as large sets.

Remark. By considering the complement of A_k we see that the theorem is equivalent to the following statement. If the sets U_k are open and dense in X then $\bigcap_{k\in\mathbb{N}} U_k$ is dense in X.

Remark. The assumption that X is complete is essential. First counterexample: let $X = \mathbb{Q}$, let $a : \mathbb{N} \to \mathbb{Q}$ be a bijection and set $A_k = \{a_k\}$. Second counterexample: let l_c denote the spaces of sequences for which only finitely many elements are nonzero, i.e.,

$$l_c := \{x : \mathbb{N} \to \mathbb{R} : \#\{j \in \mathbb{N} : x_j \neq 0\} < \infty\}. \tag{7.5}$$

Let $A_k := \{x : \mathbb{N} \to \mathbb{R} : x_j = 0, \ \forall j \geq k+1\}$. Then $l_c = \bigcup_{k=0}^{\infty} A_k$.

Theorem 7.2 (Uniform boundedness principle)**.** *Let X be a complete metric space and let Y be a normed space. Consider a family of functions $\mathcal{F} \subset C(X;Y)$ with the property*

$$\sup_{f\in\mathcal{F}} \|f(x)\| < \infty \quad \forall x \in X. \tag{7.6}$$

Then there exists an $x_0 \in X$ and an $\varepsilon > 0$ such that

$$\sup_{x\in\overline{B}(x_0,\varepsilon)} \sup_{f\in\mathcal{F}} \|f(x)\| < \infty. \tag{7.7}$$

Proof. Let $k \in \mathbb{N}$. The sets $\{x \in X : \|f(x)\| \leq k\}$ are closed since f is continuous. Thus

$$A_k := \bigcap_{f\in\mathcal{F}} \{x \in X : \|f(x)\| \leq k\} \tag{7.8}$$

is closed. If $\sup_{f\in\mathcal{F}} \|f(x)\| \leq k$ then $x \in A_k$. Hence by assumption $\bigcup_{k\in\mathbb{N}} A_k = X$. By the Baire category theorem there exists a k_0 such that A_{k_0} contains an open ball and hence a closed ball $\overline{B}(x_0, \varepsilon)$. By definition of A_{k_0} we have $\|f(x)\| \leq k_0$ for all $x \in \overline{B}(x_0, r)$ and all $f \in \mathcal{F}$. □

If we apply the uniform boundedness principle to linear maps we get the following result.

Theorem 7.3 (Banach-Steinhaus)**.** *Let X be a Banach space and let Y be a normed space. Consider a set $\mathcal{T} \subset \mathcal{L}(X,Y)$ with the property*

$$\sup_{T\in\mathcal{T}} \|Tx\| < \infty \quad \forall x \in X. \tag{7.9}$$

Then $\mathcal{T}$ is a bounded set in $\mathcal{L}(X,Y)$, i.e.,

$$\sup_{T\in\mathcal{T}} \|T\|_{\mathcal{L}(X,Y)} < \infty. \tag{7.10}$$

Proof. By the uniform boundedness principle there exist an $x_0 \in X$ and an $\varepsilon > 0$ such that

$$M_1 := \sup_{x \in \overline{B}(x_0,\varepsilon)} \sup_{T \in \mathcal{T}} \|Tx\| < \infty. \tag{7.11}$$

By assumption $M_2 := \sup_{T \in \mathcal{T}} \|Tx_0\| < \infty$. For $x \in \overline{B}(0,\varepsilon)$ we have $x + x_0 \in \overline{B}(x_0, \varepsilon)$ and $Tx = T(x + x_0) - T(x_0)$. Thus

$$\sup_{x \in \overline{B}(0,\varepsilon)} \sup_{T \in \mathcal{T}} \|Tx\| \leq M_1 + M_2. \tag{7.12}$$

Since the order of the two suprema can be exchanged we get $\|T\| \leq \frac{M_1+M_2}{\varepsilon}$ for all $T \in \mathcal{T}$. □

[15.12. 2017, Lecture 18]

[20.12. 2017, Lecture 19]

Definition 7.4. *Let X and Y be topological spaces. Then a map $f : X \to Y$ is open if*

$$U \text{ open in } X \quad \Longrightarrow \quad f(U) \text{ open in } Y. \tag{7.13}$$

Theorem 7.5 (Open mapping theorem)**.** *Let X and Y be Banach spaces, let $T \in \mathcal{L}(X, Y)$. Then*

$$T \text{ surjective} \quad \Longleftrightarrow \quad T \text{ open}. \tag{7.14}$$

Remark. This example was not discussed in class. It is easy to see that completeness of Y is necessary. Let $X = l_\infty$ and let $Y = \{y \in l_\infty : \sup_{k\in\mathbb{N}}(k+1)|y_k| < \infty\}$, equip both spaces with the l_∞ norm and set $(Tx)_k = \frac{1}{k+1}x_k$. Then $T : X \to Y$ is bijective and continuous, but $T(B(0,1))$ does not contain $B(0,\delta) \cap Y$ for any $\delta > 0$. Indeed $y = \frac{\delta}{2}e_k \in B(0,\delta) \cap Y$ but $T^{-1}(y) = (k+1)\delta e_k$ does not belong to $B(0,1)$ if k is large enough.
For a counterexample with Y complete, but X not complete see Homework sheet 10.

Proof. '$\Longrightarrow$':
The assertion is equivalent to the following statement

$$\exists\, \delta > 0 \quad B(0,\delta) \subset T(B(0,1)). \tag{7.15}$$

Indeed let $U \subset X$ be open and $y \in TU$. Then there exist $x \in U$ and $\varepsilon > 0$ such that $Tx = y$ and $B(x,\varepsilon) \subset U$. If (7.15) holds then $TB(x,\varepsilon) = Tx + TB(0,\varepsilon) \supset y + B(0,\varepsilon\delta)$. Hence TU is open.
Conversely if T is open then $TB(0,1)$ must contain an open ball $B(0,\delta)$ around 0.

Step 1: $B(0,\delta) \subset \overline{TB(0,1)}$.
Since T is surjective we have $\bigcup_{k\in\mathbb{N}} \overline{TB(0,k)} = Y$. By the Baire category theorem there exist a $k_0 \in \mathbb{N}$ and a ball $B(y_0,\varepsilon)$ with

$$B(y_0,\varepsilon) \subset \overline{T(B(0,k_0))}. \tag{7.16}$$

Hence for $y \in B(0,\varepsilon)$ there exist $x_i \in B(0,k_0)$ with $Tx_i \to y_0 + y$ as $i \to \infty$. Since T is surjective there exists $x_0 \in X$ with $Tx_0 = y_0$. Then

$$T\left(\frac{x_i - x_0}{k_0 + \|x_0\|}\right) \to \frac{y}{k_0 + \|x_0\|} \quad \text{and} \quad \left\|\frac{x_i - x_0}{k_0 + \|x_0\|}\right\| < 1. \tag{7.17}$$

Thus for all $y \in B(0,\varepsilon)$

$$\frac{y}{k_0 + \|x_0\|} \in \overline{T(B(0,1))} \tag{7.18}$$

and hence

$$B(0,\delta) \subset \overline{T(B(0,1))} \quad \text{with} \quad \delta := \frac{\varepsilon}{k_0 + \|x_0\|}. \tag{7.19}$$

Step 2: $B(0,\frac{\delta}{2}) \subset T(B(0,1))$.

$$y \in B(0,\delta) \implies \exists x \in B(0,1) \quad y - Tx \in B(0,\frac{\delta}{2}) \tag{7.20}$$

$$\implies \exists x \in B(0,1) \quad 2(y - Tx) \in B(0,\delta). \tag{7.21}$$

For $y \in B(0,\delta)$ choose inductively $y_k \in B(0,\delta)$ and $x_k \in B(0,1)$ such that

$$y_0 = y \quad \text{and} \quad y_{k+1} = 2(y_k - Tx_k). \tag{7.22}$$

Then

$$2^{-k-1}y_{k+1} = 2^{-k}y_k - T(2^{-k}x_k). \tag{7.23}$$

and thus

$$T(\sum_{k=0}^{m} 2^{-k}x_k) = y - 2^{-m-1}y_{m+1} \to y \quad \text{as } m \to \infty. \tag{7.24}$$

Now the partial sums $s_m = \sum_{k=0}^{m} 2^{-k}x_k$ form a Cauchy sequence and thus $s_m \to x$ in X and $\|x\| < 2$ since $\|x_k\| < 1$. Since T is continuous we get $Tx = y$. Therefore $B(0,\delta) \subset T(B(0,2))$ and this implies the assertion.

'$\Longleftarrow$': If $T(B(0,1)) \supset B(0,\delta)$ then $T(B(0,k)) \supset B(0,k\delta)$ and taking the union over $k \in \mathbb{N}$ we get $TX \supset Y$. □

Theorem 7.6 (Inverse operator theorem)**.** *Let X and Y be Banach spaces, let $T \in \mathcal{L}(X,Y)$ and assume that T is bijective. Then $T^{-1} \in \mathcal{L}(Y,X)$.*

Proof. By the open mapping theorem the preimage of an open set in X under T^{-1} is open in Y. Hence T^{-1} is continuous. □

Definition 7.7 (Closed operator)**.** *Let X and Y be normed spaces. A linear map is called closed if its graph* $\operatorname{graph} T := \{(x, Tx) : x \in X\}$ *is closed in* $X \times Y$.

Equivalently, T is closed if

$$x_k \to x_* \quad and \quad Tx_k \to y_* \quad \Longrightarrow \quad y_* = Tx_*. \tag{7.25}$$

Example. Let $X := C^1([0,1])$ equipped with the C^0 norm. Let $Y := C^0([0,1])$ and $Tf = f'$. Then T is closed. Note that $T : X \to Y$ is not continuous and X is not complete.

Theorem 7.8 (Closed graph theorem)**.** *Let X and Y be Banach spaces, let $T : X \to Y$ be linear. Then*

$$T \text{ closed} \quad \Longleftrightarrow \quad T \text{ continuous} \tag{7.26}$$

Remark. The example after Definition 7.7 shows that the assumption that X is complete cannot be dropped. For an example that shows that completeness of Y is necessary, too, see Homework sheet 10.

Proof. The implication $\Longleftarrow$ follows directly from (7.25).

For the converse implication first note that $X \times Y$ is a Banach space with $\|(x,y)\| = \|x\| + \|y\|$. Let $Z = \operatorname{graph} T$. By assumption T is a closed linear subspace of $X \times Y$ and hence a Banach space (see Proposition 1.26). Let $\pi_1 : Z \to X$ and $\pi_2 : Z \to Y$ be the projections to X and Y, i.e., $\pi_1(x,y) = x$, $\pi_2(x,y) = y$. Then π_1 and π_2 are continuous. Moreover π_1 is bijective and $\pi_2 \circ \pi_1^{-1}(x) = \pi_2((x, Tx)) = Tx$. Since Z and X are Banach spaces the Inverse Operator Theorem implies that π_1^{-1} is continuous and thus $T = \pi_2 \circ \pi_1^{-1}$ is continuous. □

The closed graph theorem can often be used to show that an equation has a solution for all right hand sides only if the solution can be bounded by the right hand side. As an example consider a bounded and open set $U \subset \mathbb{R}^n$ and the equation

$$-\Delta u = f \quad \text{in } U. \tag{7.27}$$

We know that for $f \in L^2(U)$ this equation has a unique weak solution $u \in W_0^{1,2}(U)$. Moreover the map $f \mapsto u$ is linear and continuous, i.e.,

$$\|u\|_{W^{1,2}(U)} \le C(U)\|f\|_{L^2(U)} \quad \forall f \in L^2(U).$$

One might expect that the solution u is in fact two orders of differentiability better than f. This can be proved in a variety of function spaces, for example the Hölder spaces $C^{k,\alpha}$ or the Sobolev spaces $W^{k,p}$ for $1 < p < \infty$ (this will be discussed in the course 'Nonlinear PDE' in the MSc programme). The seemingly most natural conclusion $f \in C(U)$ implies $u \in C^2(U)$ is, however, not true. Even if we fix a non-empty open set V with $\bar{V} \subset U$ we can find a function $f \in C_0(U)$ such that u is not in $C^2(V)$ and does not even have a weak second derivative in L^∞.

Proposition 7.9. *Let $n \geq 2$. Let U, V, f and u. Then there exists $f \in C_0(U) \subset L^\infty(U)$ such that the second weak derivative of u in V does not exist or*

$$\nabla^2 u \notin L^\infty(V). \tag{7.28}$$

Remark. One can actually show quite easily that even under the weaker condition $f \in L^2(U)$ the weak solution u always has weak second derivative and $\nabla^2 u \in L^2(V)$. We have formulated the proposition in such a way that this result on existence of the second derivative is not required.

Proposition 7.9 implies in particular that there exists $f \in C_0(U)$ such that there is no classical solution $u \in C^2(U) \cap C^1(\overline{U})$ of the Dirichlet problem

$$-\Delta u = f \quad \text{in } \Omega, \tag{7.29}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{7.30}$$

Indeed, if there was such a u it would also be a weak solution and then the uniqueness of weak solutions yields a contradiction with Proposition 7.9.

Proof of Proposition 7.9. **Step 1.** Reduction to non-existence of bounds on the solution.
Assume that the proposition is false. Then for each $f \in C_0(U)$ the weak solution u has a weak second derivative in V and $\nabla^2 u_{|V} \in L^\infty(V)$. Let T denote the map given by

$$Tf := \nabla^2 u_{|V}. \tag{7.31}$$

Then (by uniqueness of the weak solution) T is a linear map from $C_0(U)$ to $L^\infty(V)$.

The main point is to show that T is closed. Thus a ssume that

$$f_k \to f \quad \text{in } C_0(U), \qquad \nabla^2 u_k \to h \quad \text{in } L^\infty(V), \tag{7.32}$$

where $u_k \in W_0^{1,2}(U)$ is the weak solution of $-\Delta u_k = f_k$. In the second convergence h is a matrix-value map and the convergence means more explicitly

$$\partial_i \partial_j u_k \to h_{ij} \quad \text{in } L^\infty(V) \text{ for all } i, j \in \{1, \ldots, n\}. \tag{7.33}$$

The first convergence in (7.32) implies in particular $f_k \to f$ in $L^2(U)$ and therefore

$$u_k \to u \quad \text{in } W_0^{1,2}(U)$$

since the solution operator $f \mapsto u$ is bounded and continuous as a map from $L^2(U)$ to $W_0^{1,2}(U)$. With the definition of the weak second derivative it follows that for all $\varphi \in C_c^\infty(V)$

$$\int_V \partial_i \partial_j u_k \, \varphi \, dx = \int_V u_k \, \partial_i \partial_j \varphi \, dx \to \int_V u \partial_i \partial_j \varphi \, dx.$$

On the other hand (7.33) implies that

$$\int_V \partial_i \partial_j u_k \, \varphi \, dx \to \int_V h_{ij} \, \varphi \, dx.$$

Thus

$$\int_V h_{ij} \, \varphi \, dx = \int_V u \, \partial_i \partial_j \varphi \, dx$$

for all $\varphi \in C_c^\infty(V)$. It follows that the second weak derivatives of u exist and are given by $\partial_i \partial_j u = h_{ij}$. Therefore $Tu = h$. This shows that T is a closed operator.

The closed graph theorem implies that there exists a constant K such that

$$\|Tf\|_{L^\infty(V)} \le K \|f\|_{L^\infty(U)} \quad \forall f \in C_0(U). \tag{7.34}$$

Step 2. The bound (7.34) does not hold.
We will now show that (7.34) leads to a contradiction. Since V is open and non-empty it contains a ball. After translation we may assume that $B(0,\rho) \subset V$. Let $\eta \in C_0^\infty(B(0,1))$ and assume that $\eta = 1$ on $B(0, \frac{1}{2})$. Set

$$w(x) = x_1^2 - x_2^2. \tag{7.35}$$

Then

$$-\Delta w = 0, \quad \nabla^2 w \equiv S_0 = \begin{pmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & -2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{7.36}$$

and $v = \eta w$ satisfies

$$-\Delta v = \begin{cases} 0 & \text{in } B(0, \frac{1}{2}) \cup (\mathbb{R}^n \setminus B(0,1)), \\ -\Delta \eta \, w + 2 \nabla \eta \cdot \nabla w & \text{in } B(0,1) \setminus B(0, \frac{1}{2}). \end{cases} \tag{7.37}$$

In particular

$$\|\Delta v\|_{L^\infty(U)} \le C_0, \qquad \nabla^2 v(0) = S_0. \tag{7.38}$$

Now we get a contradiction to (7.34) by *scaling* and *summation.* Set

$$v_k(x) = 2^{-2k} v(2^k x). \tag{7.39}$$

Then $\nabla^2 v_k(x) = (\nabla^2 v)(2^k x)$ and

$$\operatorname{supp} \Delta v_k = B(0, 2^{-k}) \setminus B(0, 2^{-k-1}). \tag{7.40}$$

Let k_0 be so large that $2^{-k_0} \le \rho$ and set

$$u := \sum_{k=k_0+1}^{k_0+m} v_k, \qquad f := -\Delta u. \tag{7.41}$$

Then $u \in C_c^\infty(U)$ and thus $Tf = \nabla^2 u_{|V}$. Since the supports of the functions Δv_k are disjoint we get

$$\|f\|_{L^\infty} \le C_0, \qquad Tf(0) := mS_0. \tag{7.42}$$

Since $S_0 \neq 0$ this leads to a contradiction with (7.34) if $m > C_0 K/|S_0|$. □

[20.12. 2017, Lecture 19]

[22.12. 2017, Lecture 20]

8 Weak convergence

8.1 Motivation

Motivation: in functional analysis and the theory of partial differential equations one is often confronted with the following situation. One has a sequence of approximate solutions f_k of a problem. One would like to extract a limit and to show that the limit solves the problem. We have seen this already in the proof of the projection theorems, Theorem 3.2 and Theorem 3.7. In that case we could show that the sequence $k \mapsto f_k$ is a Cauchy sequence and hence converges.

Often, however, one can only show that the sequence is bounded in a suitable Banach space X. This is not enough to extract a convergent subsequence since by Theorem 3.13 the closed unit ball in X is only compact if X is finite dimensional.

In this chapter we will systematically develop weaker topologies in which we can extract a convergent subsequence from a bounded sequence. The following example already contains the heart of the matter.

Example. Let $E \subset \mathbb{R}^n$ be measurable and let $k \mapsto f_k$ be a bounded sequence in $L^2(E)$, i.e., $\sup_{k\in\mathbb{N}} \|f_k\| \le R$. We claim that there exist a subsequence f_{k_j} and an $f \in L^2(E)$ such that

$$\int_E f_{k_j} g \, d\mathcal{L}^n \to \int_E f g \, d\mathcal{L}^n \quad \forall g \in L^2(E). \tag{8.1}$$

Proof: let $h : \mathbb{N} \to L^2(E)$ be a sequence such that $h(\mathbb{N})$ is dense in $L^2(E)$. Then

$$\left| \int_E f_k h_1 \, d\mathcal{L}^n \right| \le \|f_k\|_{L^2} \|h_1\|_{L^2} \le R\|h_1\|. \tag{8.2}$$

Hence there exists a subsequence $f^{(1)}$ such that

$$\int_E f_k^{(1)} h_1 \, d\mathcal{L}^n \to L(h_1) \quad \text{and} \quad \|L(h_1)\| \le R\|h_1\|. \tag{8.3}$$

Now we can take successive subsequences $f^{(2)}, f^{(3)}, \ldots$ such $\int_E f_k^{(l)} h_m \, d\mathcal{L}^n \to L(h_m)$ for all $m \le l$. Finally let $\tilde{f}$ be the diagonal sequence $\tilde{f}_j = f_j^{(j)}$. Then

$$\lim_{j\to\infty} \int_E \tilde{f}_j g \, d\mathcal{L}^n = L(g) \quad \text{and} \quad \|L(g)\| \le R\|g\| \quad \text{if } g \in \{h_1, h_2, \ldots\}. \tag{8.4}$$

It follows easily that convergence holds for all $g \in Y = \operatorname{span}\{h_1, h_2, \ldots\}$ and that $L : Y \to \mathbb{R}$ is linear and $|L(g)| \le R\|g\|$. Since Y is dense in $L^2(E)$ there exist a unique bounded linear map $L : L^2(E) \to \mathbb{R}$ which extends L_Y. Moreover one easily sees that the convergence in (8.4) holds for all $g \in L^2(E)$.

Finally by the Riesz representation theorem there exists $f \in L^2(E)$ such that

$$L(g) = \int_E f g \, d\mathcal{L}^n. \tag{8.5}$$

This proves (8.1).

The same argument works with $L^2(E)$ replaced by $L^p(E)$ if $p \in (1, \infty]$ and $g \in L^{p'}(E)$ if we use the fact that $(L^{p'}(E))'$ is isometrically isomorphic to $L^p(E)$ (see Theorem 6.1). The argument does not work for $L^1(E)$ (since $L^\infty(E)$ is not separable) and the assertion is not true for $L^1(E)$ (hint: consider the sequence of standard mollifiers η_j).

8.2 Weak topology, weak convergence, and weak compactness

The point in (8.1) is that convergence in norm is replaced by the convergence of certain linear functionals applied to the sequence $j \mapsto f_j$. We now consider this in a more general context. This will lead to the definition of the weak and weak* topologies which correspond to the convergence introduced ad hoc in (8.1).

Question 1. Let X be a set and let Y be a topological space. Consider a collection of maps $\varphi_\alpha : X \to Y$ where α runs through some index set A. What is the coarsest topology $\mathcal{T}$ on X such all the maps φ_α are continuous ?

Clearly $\mathcal{T}$ must contain all sets $\varphi_\alpha^{-1}(V)$ where V is an open set in Y and $\alpha \in A$. Conversely if $\mathcal{T}$ is a topology which contains all these sets then all the maps φ_α are continuous (by the definition of continuity). This leads us to

Question 2. Let X be a set and let $\mathcal{S} \subset 2^X$. What is the coarsest topology $\mathcal{T}$ with $\mathcal{T} \supset \mathcal{S}$?

Lemma 8.1. *Let X be a set and let $\mathcal{S} \subset 2^X$ and suppose that $\bigcup_{W\in\mathcal{S}} W = X$. Let $\mathcal{B}$ denote the collection of sets obtained by taking finite intersections of sets in $\mathcal{S}$ and let $\mathcal{T}$ denote the collection of sets formed by (arbitrary) union of sets in $\mathcal{B}$. More formally:*

$$\mathcal{B} :=\{\bigcap_{i=1}^{k} W_i : k \in \mathbb{N}\setminus\{0\}, W_i \in \mathcal{S} \quad \forall\, i \in \{1,\dots,k\}\}, \tag{8.6}$$

$$\mathcal{T} :=\{\bigcup_{\alpha\in A} V_\alpha : A \text{ set}, V_\alpha \in \mathcal{B} \quad \forall\, \alpha \in A\} \cup \emptyset. \tag{8.7}$$

Then $\mathcal{T}$ is the coarsest topology which contains $\mathcal{S}$.

Proof. Homework. Show first that $\mathcal{T}$ is a topology. Then it is easy to see that if $\mathcal{T}'$ is another topology and $\mathcal{T}' \supset \mathcal{S}$ then $\mathcal{T}' \supset \mathcal{T}$. □

Definition 8.2. *Let X be a set and let $\mathcal{T} \subset 2^X$ be a topology.*

(i) We say that $\mathcal{T}' \subset \mathcal{T}$ is a base of the topology $\mathcal{T}$ if every set in $\mathcal{T}$ can be written as a union of sets in $\mathcal{T}'$.

(ii) We say that $\mathcal{T}'' \subset \mathcal{T}$ is a subbase of $\mathcal{T}$ if the the collection of all finite intersection of sets in $\mathcal{T}''$ is a base of $\mathcal{T}$.

(iii) Let $x_0 \in X$. We say that $\mathcal{T}''' \subset \mathcal{T}$ is a neighbourhood base at x_0 if every $U \in \mathcal{T}$ with $x_0 \in U$ contains a non-empty set in $\mathcal{T}'''$ which contains x_0.

Remark. (i) A base $\mathcal{T}'$ can be equivalently characterized as follows: if $U \in \mathcal{T}$ then for every $x \in U$ there exists a $V_x \in \mathcal{T}'$ such that $x \in V_x$ and $V_x \subset U$. Indeed if $\mathcal{T}'$ has this property then $U = \bigcup_{x\in U} V_x$. Conversely if $\mathcal{T}'$ has the property in (i) of the definition and $U \in \mathcal{T}$ then there exist $V_\alpha \in \mathcal{T}'$ such that $U = \bigcup_{\alpha\in A} V_\alpha$. In particular $V_\alpha \subset U$ for all $\alpha \in A$. Now if $x \in U$ there exists an α such that $x \in V_\alpha$.
(ii) The set $\mathcal{B}$ in Lemma 8.1 is a base of $\mathcal{T}$ and $\mathcal{S}$ is a subbase of $\mathcal{T}$.

Proposition 8.3. *Let X be a set, let Y be a topological space. Let A be a set and consider a collection of maps $\varphi_\alpha : X \to Y$, where $\alpha \in A$. Let $\mathcal{T}$ be the coarsest topology such that all the maps φ_α are continuous. Let $x : \mathbb{N} \to X$ be a sequence. Then*

$$x_n \to x_* \quad \text{in } \mathcal{T} \quad \iff \quad \varphi_\alpha(x_n) \to \varphi_\alpha(x_*) \quad \forall\, \alpha \in A. \tag{8.8}$$

Proof. '$\Longrightarrow$': This follows from the fact that continuity of φ_α implies sequential continuity. Details: Let $W \subset Y$ be open with $\varphi_\alpha(x_*) \in W$. Then $U := \varphi_\alpha^{-1}(W)$ is open and $x_* \in U$. Hence the set $\{n : x_n \notin U\}$ is finite. Thus the set $\{n : \varphi_\alpha(x_n) \notin W\}$ is finite. Thus $\varphi_\alpha(x_n) \to \varphi(x_*)$.

'$\Longleftarrow$': Let $U \in \mathcal{T}$ and $x_* \in U$. By Lemma 8.1 there exists a set $V \subset U$ with $x_* \in V$ and $V = \bigcap_{i=1}^k \varphi_{\alpha_i}^{-1}(W_i)$ with W_i open in Y. Thus $\varphi_{\alpha_i}(x_*) \in W_i$ and the sets $\{n : \varphi_{\alpha_i}(x_n) \notin W_i\}$ are finite since $\varphi_{\alpha_i}(x_n) \to \varphi_{\alpha_i}(x_*)$. Hence

$$\{n : x_n \notin U\} \subset \{n : x_n \notin V\} = \bigcup_{i=1}^k \{n : \varphi_{\alpha_i}(x_n) \notin W_i\} \quad \text{is finite.} \tag{8.9}$$

Thus $x_n \to x_*$. □

Example. (i) Let $X = \mathbb{R}^{\mathbb{N}}$ be the space of sequences with values in $\mathbb{R}$, i.e., $X = \{x : x : \mathbb{N} \to \mathbb{R}\}$. For $k \in \mathbb{N}$ let $\varphi_k(x) := x(k)$ be the projection to the k-th factor. Let $\mathcal{T}$ be the coarsest topology such that all the maps $\varphi_k : X \to \mathbb{R}$ are continuous. Then by Proposition 8.3 and we have $x^{(n)} \to x_*$ in $\mathcal{T}$ if and only if $x^{(n)}(k) \to x_*(k)$ for all $k \in \mathbb{N}$. A base of $\mathcal{T}$ is given by the collection of all the Cartesian products $\prod_{k\in\mathbb{N}} U_k$ where all sets $U_k \subset \mathbb{R}$ are open and only finitely many are different from $\mathbb{R}$. Note that the (non empty) sets in this base are all rather large in the sense that they contain an infinite dimensional affine subspace.
(ii) (*not discussed in class*) If we slightly generalize the setting and consider maps $\varphi_\alpha : X \to Y_\alpha$ with possibly different topological spaces Y_α we can in the same way put a topology on an arbitrary product of topological spaces $(X_\alpha, \mathcal{T}_\alpha)$. Let $X = \prod_{\alpha\in A} X_\alpha$ (more precisely X is the space of all maps $x : A \to \cup_{\alpha\in A} X_\alpha$ with $x(\alpha) \in X_\alpha$) and let the projections $\pi_\alpha : X \to X_\alpha$ be given by $\pi_\alpha(x) = x(\alpha)$. The product topology on X is defined as the coarsest topology for which all the projections π_α are continuous. We have $x_n \to x_*$ in $\mathcal{T}$ if and only if $x_n(\alpha) \to x_*(\alpha)$ for all $\alpha \in A$. A base of the product topology is given by the collection of all sets of the form $U = \prod_{\alpha\in A} U_\alpha$ where $U_\alpha \in \mathcal{T}_\alpha$ for all $\alpha \in A$ and where only finitely many of the U_α are different from X_α.

We now apply the above reasoning to the situation where X is a Banach space and the family of maps on X is given by the elements of the dual space X'.

Definition 8.4. *Let X be a Banach space with dual X'. The weak topology $\sigma(X, X')$ on X is the coarsest topology for which all $x' \in X'$ are continuous.*

Remark. (i) Let $\mathcal{T}_{\text{strong}}$ be the topology on X induced by the norm. Then $\sigma(X, X') \subset \mathcal{T}_{\text{strong}}$ since by definition of X' all $x' \in X'$ are continuous with respect to $\mathcal{T}_{\text{strong}}$.
(ii) If X is infinite dimensional then it follows from the remark after Proposition 8.6 that $\sigma(X, X') \neq \mathcal{T}_{\text{strong}}$, i.e., the weak topology is strictly coarser than the norm topology. Indeed every set in $\sigma(X, X')$ contains an (infinite) line, while the open unit ball does not contain a line.
(iii) If X is finite dimensional then it is easy to see that $\sigma(X, X') = \mathcal{T}_{\text{strong}}$.

Notation: If X is a normed space with dual space X' and $x \in X$, $x' \in X'$ we write [6]

$$\langle x, x' \rangle_X := x'(x). \tag{8.10}$$

We often write $\langle x, x' \rangle$ instead of $\langle x, x' \rangle_X$.

Proposition 8.5. *The weak topology $\sigma(X, X')$ is Hausdorff.*

Proof. Let $x, y \in X$ with $x \neq y$. By Corollary 6.6 there exists $x' \in X'$ such that $\varepsilon := \frac{1}{2}\langle x - y, x' \rangle \neq 0$. Set $a = \langle x, x' \rangle$, $b = \langle y, x' \rangle$ and

$$U_x := (x')^{-1}(B(a, \varepsilon)), \quad U_y := (x')^{-1}(B(b, \varepsilon)). \tag{8.11}$$

Then $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \emptyset$. Moreover U_x and U_y belong to the topology $\sigma(X, X')$ since x' is continuous with respect to this topology. □

Proposition 8.6. *Let X be a Banach space. Let $x_0 \in X$ and let $\{x'_1, \ldots, x'_k\} \subset X'$. Define*

$$V(x'_1, \ldots, x'_k; \varepsilon) := \{x \in X : |\langle x - x_0, x'_i \rangle| < \varepsilon \quad \forall\, i = 1, \ldots, k\}. \tag{8.12}$$

Then $V(x'_1, \ldots, x'_k; \varepsilon)$ is a neighbourhood of x_0 in the $\sigma(X, X')$ topology. Moreover the collection of sets $V(x'_1, \ldots, x'_k; \varepsilon)$ with $\varepsilon > 0$, $k \in \mathbb{N} \setminus \{0\}$ and $x'_i \in X'$ for $i = 1, \ldots, k$ forms a neighbourhood base of x_0 in the $\sigma(X, X')$ topology.

Proof. Let $a_i = \langle x_0, x'_i \rangle$. The sets $V_i := (x'_i)^{-1}(B(a_i, \varepsilon))$ belong to $\sigma(X, X')$. Hence $V(x'_1, \ldots, x'_k; \varepsilon) = \bigcap_{i=1}^k V_i$ belongs to $\sigma(X, X')$.

Conversely let $U \in \sigma(X, X')$ and $x_0 \in U$. Then it follows from Lemma 8.1 that U contains a set of the form $\bigcap_{i=1}^k (x'_i)^{-1}(W_i)$ where $W_i \subset \mathbb{K}$ is open

[6] This notation deliberately resembles the notation of the scalar product and is motivated by the Riesz representation theorem. If X is a Hilbert space then every element in X' can be represented by the scalar product and we have $\langle x, R_X(y) \rangle = R_X(y)(x) = (x, y)_X$ and $R_X : X \to X'$ is an isometric isomorphism. The notation above follows H.W. Alt's book. Some authors, e.g., Brezis, write $\langle x', x \rangle$ instead of $\langle x, x' \rangle$.

and $a_i = \langle x_0, x_i' \rangle \in W_i$. Hence there exists an $\varepsilon > 0$ such that $B(a_i, \varepsilon) \subset W_i$ for all $i = 1, \ldots, k$ (here we use the fact that we deal only with finitely many x_i'). Thus

$$V(x_1', \ldots, x_k'; \varepsilon) = \bigcap_{i=1}^{k} x_i'^{-1}(B(a_i, \varepsilon)) \subset \bigcap_{i=1}^{k} x_i'^{-1}(W_i) \subset U. \tag{8.13}$$

□

Remark. If X is infinite dimensional then all the sets $V(x_1', \ldots, x_k'; \varepsilon)$ (with $\varepsilon > 0$) contain a line (in fact an infinite dimensional affine subspace). Indeed set $L := \{y : \langle y, x_i' \rangle = 0 \quad \forall\, i = 1, \ldots, k\}$. Then $x_0 + L \subset V(x_1', \ldots, x_k'; \varepsilon)$. To see that $L \neq \{0\}$ consider the map $T : X \to \mathbb{K}^k$ given by $T(x) = (\langle x, x_1' \rangle, \ldots, \langle x, x_k' \rangle)$. If $L = \{0\}$ then $T : X \to R(T)$ is bijective. Since $R(T)$ is finite dimensional the space X must be finite dimensional. This contradiction finishes the proof. More generally one sees that the quotient space X/L is isomorphic to the finite dimensional space $R(T)$. Thus if X is infinite dimensional, L must be infinite dimensional.

Notation: If a sequence converges in the $\sigma(X, X')$ topology we write

$$x_n \rightharpoonup x_* \quad \text{weakly in } \sigma(X, X') \tag{8.14}$$

or shorter

$$x_n \rightharpoonup x_*, \tag{8.15}$$

i.e., we use the halfarrow $\rightharpoonup$ to denote convergence in the weak topology (or 'weak convergence' for brevity). Note that the weak limit is unique since the $\sigma(X, X')$ topology is Hausdorff. We write

$$x_n \to x_* \quad \text{strongly in } X \tag{8.16}$$

or shorter

$$x_n \to x_* \tag{8.17}$$

if $\|x_n - x_*\| \to 0$ and we call this convergence strong convergence (or 'norm-convergence').

[22.12. 2017, Lecture 20]

[10.1. 2018, Lecture 21]

Proposition 8.7. *Let $x : \mathbb{N} \to X$ be a sequence. Then*

(i) $x_n \rightharpoonup x_* \quad \iff \quad \langle x_n, x' \rangle \to \langle x_*, x' \rangle \quad \forall x' \in X'$.

(ii) If $x_n \to x_*$ then $x_n \rightharpoonup x_*$.

(iii) If $x_n \rightharpoonup x_*$ *then* $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$ *and* $\|x_*\| \leq \liminf_{n\to\infty} \|x_n\|$.

(iv) If $x_n \rightharpoonup x_*$ *(weakly) and* $x'_n \to x'$ *(strongly) then* $\langle x_n, x'_n\rangle \to \langle x_*, x'_*\rangle$.

Proof. (i): This follows from Proposition 8.3 and the definition of the $\sigma(X, X')$ topology.

(ii): This follows from the fact that the weak topology is coarser than the strong topology or from (i) and the estimate $|\langle x_n, x'\rangle - \langle x_*, x'\rangle| \leq \|x'\| \, \|x_n - x_*\|$.

(iii): The first assertion follows from the Banach-Steinhaus theorem. More precisely define a map $J : X \to (X')'$ by $J(x)(x') = \langle x, x'\rangle$. Then $\|J(x)\|_{(X')'} \leq \|x\|$ and by Corollary 6.6 we have $\|J(x)\|_{(X')'} = \|x\|$. Now for each x' we have

$$\sup_{n\in\mathbb{N}} |J(x_n)(x')| = \sup_{n\in\mathbb{N}} |\langle x_n, x'\rangle| < \infty \qquad \text{since } \langle x_n, x'\rangle \to \langle x_*, x'\rangle. \tag{8.18}$$

The Banach-Steinhaus theorem implies that $\sup_{n\in\mathbb{N}} \|J(x_n)\|_{(X')'} < \infty$. Hence $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$.
To prove the second assertion pass to the limit in the inequality

$$|\langle x_n, x'\rangle| \leq \|x'\| \, \|x_n\|. \tag{8.19}$$

This yields

$$|\langle x_*, x'\rangle| \leq \|x'\| \, \liminf_{n\to\infty} \|x_n\|. \tag{8.20}$$

Using again Corollary 6.6 we get $\|x_*\| = \sup_{\|x'\|=1} \langle x_*, x'\rangle \leq \liminf_{n\to\infty} \|x_n\|$.

(iv): This follows from the inequality

$$|\langle x_n, x'_n\rangle - \langle x_*, x'_*\rangle| \leq |\langle x_n, x'_n - x'_*\rangle| + |\langle x_n - x_*, x'_*\rangle| \leq \|x_n\| \, \|x'_n - x'_*\| + |\langle x_n - x_*, x'_*\rangle| \tag{8.21}$$

combined with (i) and (iii). □

Let X be a Banach space with dual X'. So far we have two topologies on X'.

(i) The norm topology (or strong topology) $\mathcal{T}_{\text{strong}}$.

(ii) The weak topology $\sigma(X', (X')')$ defined as coarsest topology such that all the maps $x' \mapsto \langle x', y\rangle_{X'} = y(x')$ are continuous for all y in the bidual space $(X')'$.

We now introduce a third topology on X', the weak* or $\sigma(X', X)$ topology which is the coarsest topology such that the maps $x' \mapsto \langle x', y\rangle_{X'} = y(x')$ are continuous for all y for the form $y \in J(X) \subset (X')'$ where $J : X \mapsto (X')'$ is given by $J(x)(x') = x'(x) = \langle x, x'\rangle_X$.

Definition 8.8. *Let* X *be a Banach space with dual* X'*. The weak* topology* $\sigma(X', X)$ *on* X' *is the coarsest topology such that for all* $x \in X$ *the functionals* $J(x) : X' \to \mathbb{K}$ *given by* $J(x)(x') = \langle x, x'\rangle$ *are continuous.*

Remark. We have already seen that $\sigma(X', (X')') \subset \mathcal{T}_{\text{strong}}$ (and the inclusion is strict if and only if X is infinite dimensional). Note that also $\sigma(X', X) \subset \sigma(X', (X')')$ since the definition of $\sigma(X', X)$ involves only a subset of functionals $J(X) \subset (X')'$. If $J(X) = (X')'$, then, by definition $\sigma(X', X) = \sigma(X', (X')')$, i.e., the weak and weak* topologies agree.

Warning. If X is a Banach space and $Y \subset X$ is a dense subspace then $Y' = X'$ since every bounded linear map on Y has a unique extension to X. Nonetheless the $\sigma(X', Y)$ topology can be strictly coarser than the $\sigma(X', X)$ topology.

Proposition 8.9. *The weak* topology is Hausdorff and a neighbourhood base at x_0' is given by the collection of the sets*

$$V(x_1, \ldots, x_k; \varepsilon) := \{x' \in X' : |\langle x_i, x' - x_0' \rangle| < \varepsilon \ \forall\, i \in \{1, \ldots, k\}\}, \quad (8.22)$$

where $k \in \mathbb{N} \setminus \{0\}$, $\varepsilon > 0$ and $x_1, \ldots, x_k \in X$.

Proof. This is proved like Proposition 8.5 and Proposition 8.6. More precisely to show that the weak* topology is Hausdorff one uses that for $x' \neq y'$ there exists $x \in X$ such that $\langle x, x' \rangle \neq \langle x, y' \rangle$. □

Notation: If a sequence $x' : \mathbb{N} \to X'$ converges in the weak* topology to x_*' we write

$$x_n' \overset{*}{\rightharpoonup} x_*'. \quad (8.23)$$

Proposition 8.10. *Let $x' : \mathbb{N} \to X'$ be a sequence. Then*

(i) $x_n' \overset{*}{\rightharpoonup} x_*' \quad \Longleftrightarrow \quad \langle x, x_n' \rangle \to \langle x, x_*' \rangle \quad \forall x \in X.$

(ii) If $x_n' \to x_*'$ *then* $x_n' \overset{*}{\rightharpoonup} x_*'$.

(iii) If $x_n' \overset{*}{\rightharpoonup} x_*'$ *then* $\sup_{n \in \mathbb{N}} \|x_n'\| < \infty$ *and* $\|x_*'\| \leq \liminf_{n \to \infty} \|x_n'\|$.

(iv) If $x_n' \overset{*}{\rightharpoonup} x_*'$ *(weakly*) and* $x_n \to x$ *(strongly) then* $\langle x_n, x_n' \rangle \to \langle x_*, x_*' \rangle$.

Proof. The proof is analogous to the proof of Proposition 8.7. □

We now come to the first key compactness property.

Theorem 8.11 (Sequential weak* compactness of the closed unit ball)**.** *Let X be a separable Banach space with dual X'. Then the closed unit ball $\overline{B}(0,1) \subset X'$ is sequentially compact in the weak* topology, i.e., every sequence $x' : \mathbb{N} \to \overline{B}(0,1)$ contains a subsequence which converges to a point $x_*' \in \overline{B}(0,1)$ in the weak* topology.*

Remark. (i) This assertion about sequential compactness does in general not hold if X is not separable. Example: for $X = l_\infty$ consider the $f_k \in l'_\infty$ given by $f_k(x) = x_k$. Then $\|f_k\|_{l'_\infty} = 1$, but the sequence $k \mapsto f_k$ has no weak* convergent subsequence. Proof: let $j \to k_j$ be any strictly increasing map from $\mathbb{N}$ to $\mathbb{N}$ and let $g_j = f_{k_j}$. Define $x \in l_\infty$ by $x_m = (-1)^m$ if $m = k_j$ and $x_m = 0$ else. Then $g_j(x) = (-1)^j$ and hence $g_j(x)$ has no limit as $j \to \infty$.

(ii) It is, however, true that for any Banach space X the closed unit ball is compact in the weak* topology (Banach-Alaoglu theorem). The proof is based on Tychonoff's theorem which asserts that an arbitrary product of compact topological spaces is compact in the product topology[7].

(iii) If X is separable than one can show that the restriction of the weak* topology to the closed unit ball is metrizable (see homework sheet 12, problem 3). Hence compactness implies sequential compactness and the above result could be deduced from the Banach-Alaoglu theorem. We prefer to give a short direct proof.

Proof. This is the abstract version of the example given at the beginning of this chapter. Let $y : \mathbb{N} \to X$ be a sequence such that $y(\mathbb{N})$ is dense in X. Let $f : \mathbb{N} \to \overline{B}(0,1) \subset X'$ (for ease of notation we call the sequence f rather than x'). Then

$$|\langle y_0, f_k \rangle| \le \|y_0\| \, \|f_k\| \le \|y_0\|. \tag{8.24}$$

Hence there exists a subsequence $f^{(0)}$ such that

$$\langle y_0, f_k^{(0)} \rangle \to L(y_0) \quad \text{and } |L(y_0)| \le \|y_0\| \tag{8.25}$$

Now we can take successive subsequences $f^{(1)}, f^{(2)}, \ldots$ such that $\langle y_m, f_k^{(l)} \rangle \to L(y_m)$ as $k \to \infty$ for all $m \le l$. Let $\tilde{f}_j := f_j^{(j)}$ be the diagonal sequence. Then

$$\langle x, \tilde{f}_j \rangle \to L(x) \quad \text{and } |L(x)| \le \|x\| \quad \forall x \in y(\mathbb{N}). \tag{8.26}$$

Since $x \mapsto \langle x, \tilde{f}_j \rangle$ is linear, convergence and the estimate for $L(x)$ also hold for $x \in Y := \operatorname{span}\{y(\mathbb{N})\}$ and $L : Y \to \mathbb{K}$ is linear. Since Y is dense in X it follows that there exists a unique $f_* \in X'$ such that

$$\langle y, f_* \rangle = f_*(y) = L(y) \quad \forall\, y \in Y. \tag{8.27}$$

Moreover $|\langle x, f_* \rangle| \le \|x\|$ for all $x \in X$ and hence $\|f_*\| \le 1$. Finally for each y_k we have

$$\limsup_{j \to \infty} |\langle x, \tilde{f}_j - f_* \rangle| \tag{8.28}$$

$$\le \limsup_{j \to \infty} |\langle (x - y_k), \tilde{f}_j - f_* \rangle| + \underbrace{\limsup_{j \to \infty} |\langle y_k, \tilde{f}_j - f_* \rangle|}_{=0 \text{ by (8.26), (8.27)}} \le 2\|x - y_k\|. \tag{8.29}$$

[7] See, e.g., the book by Brezis or W. Rudin, Functional analysis. Rudin's book also contains a proof of Tychonoff's theorem.

Since $y(\mathbb{N})$ is dense in X the right hand side can be made arbitrarily small and hence $\tilde{f}_j \overset{*}{\rightharpoonup} f_*$. □

We will see shortly that a similar compactness result holds for weak convergence if X satisfy an extra condition. First we look at some examples.

Example.

(i) L^p spaces: Let $E \subset \mathbb{R}^n$ be measurable and let $p \in [1, \infty)$. Recall that there exists an isometric isomorphism $\Phi : L^{p'}(E) \to (L^p(E))'$ given by $\Phi(g)(f) = \int_E fg\, dx$. Thus

$$f_k \rightharpoonup f_* \text{ in } L^p(E) \iff \int_E f_j g\, dx \to \int_E f_* g\, dx \quad \forall g \in L^{p'} \quad \text{for } p \in [1, \infty). \tag{8.30}$$

$$f_k \overset{*}{\rightharpoonup} f_* \text{ in } L^\infty(E) \iff \int_E f_j g\, dx \to \int_E f_* g\, dx \quad \forall g \in L^1. \tag{8.31}$$

(ii) $C(K)$ and $\mathrm{rca}\,(K)$: Let $K \subset \mathbb{R}^n$ be compact. By the Riesz-Radon theorem (Theorem 6.12) there exists an isometric isomorphism Φ : $\mathrm{rca}\,(K) \to (C(K))'$ given by $\Phi(\mu)(f) = \int_E f\, d\mu$. Thus

$$\mu_k \overset{*}{\rightharpoonup} \mu_* \text{ in } \mathrm{rca}\,(K) \iff \int_K f\, d\mu_k \to \int_K f\, d\mu_* \quad \forall f \in C(K). \tag{8.32}$$

(iii) $W^{1,p}(U)$: Let $U \subset \mathbb{R}^n$ be open. Even though we have not computed the dual space of $W^{1,p}(U)$ explicitly it is easy to see that (see Homework sheet 11, Problem 1)

$$f_k \rightharpoonup f_* \in W^{1,p}(U) \iff f_k \rightharpoonup f_* \text{ and } \partial_j f_k \rightharpoonup \partial_j f_* \text{ in } L^p(U) \quad \forall\, j = 1, \ldots, n \tag{8.33}$$

if $p \in [1, \infty)$. Similarly

$$f_k \overset{*}{\rightharpoonup} f_* \in W^{1,\infty}(U) \iff f_k \overset{*}{\rightharpoonup} f_* \text{ and } \partial_j f_k \overset{*}{\rightharpoonup} \partial_j f_* \text{ in } L^\infty(U) \quad \forall\, j = 1, \ldots, n. \tag{8.34}$$

The main idea is that the map $j(f) := (f, \partial_1 f, \ldots, \partial_n f)$ is a linear and continuous map (with continuous inverse) from $W^{1,p}(U)$ to a closed subspace of $(L^p(U))^{n+1}$.

[10.1. 2018, Lecture 21]

[12.1. 2018, Lecture 22]

In Theorem 8.11 we have shown that the closed unit ball in X' is sequentially weak* compact if X is separable. We now aim for a similar result for weak convergence. The idea is essentially to consider a situation where X agrees with the bidual space $X'' := (X')'$ and hence can be viewed as a dual space. More specifically we recall the definition of the map $J_X : X \to X''$ given by

$$J_X(x)(x') = x'(x) \qquad \text{or} \quad \langle x', J_X(x)\rangle := \langle x, x'\rangle.$$

We have seen in the proof of Proposition 8.7 (iii) that J_X is an isometric immersion, i.e., $\|J_X(x)\|_{X''} = \|x\|_X$. In particular J_X is injective. The definition of J_X and of weak and weak* convergence imply that

$$x_k \rightharpoonup x_* \quad \text{in } X \quad \Longleftrightarrow \quad J_X(x_k) \overset{*}{\rightharpoonup} J(x_*) \quad \text{in } X'' \tag{8.35}$$

since both statements are equivalent to $\langle x_k, x'\rangle \to \langle x_*, x'\rangle$ for all $x' \in X'$.

Definition 8.12. *A Banach space X is called reflexive if $J_X : X \to X''$ is surjective.*

Equivalently X is reflexive if $J_X : X \to X''$ is an isometry. In particular for a reflexive space X'' is isometrically isomorphic to X.

Proposition 8.13. *Let X be a Banach space. Then the following assertions hold.*

(i) If X is reflexive then the weak and the weak topology on X' are identical.*

(ii) If X is reflexive and $Y \subset X$ is a closed subspace than Y is reflexive.

(iii) If Y is a Banach space and $T : X \to Y$ is an isomorphism then

$$X \text{ reflexive} \quad \Longleftrightarrow \quad Y \text{ reflexive}. \tag{8.36}$$

(iv) X reflexive $\quad \Longleftrightarrow \quad$ X' reflexive.

Remark. One can show we that X is reflexive if and only if the weak and the weak* topology on X' agree.

Proof. (i): The weak topology on X' is by definition the coarsest topology such that all the maps

$$x' \mapsto \langle x', x''\rangle \tag{8.37}$$

are continuous for all $x'' \in X''$. The weak* topology on X' is by definition the coarsest topology such that all the maps

$$x' \mapsto \langle x, x'\rangle \tag{8.38}$$

are continuous for all $x \in X$. If J_X is surjective then

$$\langle x', x''\rangle = \langle J_X^{-1}(x''), x'\rangle, \tag{8.39}$$

Hence the class of maps considered for the weak and weak* topology is the same and thus these two topologies agree.

(ii) Let $y'' \in Y''$. Define $x'' : X' \to \mathbb{K}$ by

$$x''(x') = \langle x'_{|Y}, y'' \rangle. \tag{8.40}$$

Then $x'' \in X''$ and thus there exists $x \in X$ such that $J_X x = x''$. Hence

$$\langle x'_{|Y}, y'' \rangle = x''(x') = \langle x, x' \rangle. \tag{8.41}$$

We now claim that $x \in Y$. Indeed if $x \notin Y$ then by Theorem 6.5 there exists $x' \in X'$ such that $x'_{|Y} = 0$ and $x'(x) = \operatorname{dist}(x, Y)$. Then (8.41) yields the contradiction

$$0 \neq \operatorname{dist}(x, Y) = \langle x, x' \rangle = \langle x'_{|Y}, y'' \rangle = \langle 0, y'' \rangle = 0. \tag{8.42}$$

Thus $x \in Y$ and $J_X(x) = x''$.

Finally we show that $J_Y(x) = y''$. Let $y' \in Y'$. By the Hahn-Banach theorem (Theorem 6.4) there exists an $x' \in X'$ such $x'_{|Y} = y'$. Thus (8.41) yields

$$\langle x, y' \rangle_Y = \langle x, x' \rangle_X = \langle x'_{|Y}, y'' \rangle_{Y'} = \langle y', y'' \rangle_{Y'}, \tag{8.43}$$

i.e. $J_Y(x) = y''$.

(iii): It suffices to show X reflexive $\implies$ Y reflexive, since the roles of X and Y can be interchanged. Let $y'' \in Y''$. Define x'' by

$$x''(x') := \langle x' \circ T^{-1}, y'' \rangle. \tag{8.44}$$

Then $\|x' \circ T^{-1}\|_{X'} \leq \|x'\|_{X'} \|T^{-1}\|_{\mathcal{L}(X)}$ and thus

$$\|x''(x')\| \leq \|y''\| \, \|x' \circ T^{-1}\| \leq \|y''\| \, \|x'\| \, \|T^{-1}\|. \tag{8.45}$$

Thus $x'' \in X''$. Let $x = J_X^{-1} x''$. For $y' \in Y'$ let $x' := y' \circ T$. Then $x' \in X'$ and

$$\langle y', y'' \rangle = \langle x' \circ T^{-1}, y'' \rangle = \langle x', x'' \rangle = x'(x) = y'(Tx). \tag{8.46}$$

Thus $y'' = J_Y(Tx)$.

(iv): We first show X reflexive $\implies$ X' reflexive. Set $X''' := (X'')'$ and let $x''' \in X'''$. Define x' by

$$x'(x) := \langle J_X x, x''' \rangle. \tag{8.47}$$

Then $x' \in X'$ and

$$\langle x', x'' \rangle = \langle J_X^{-1}(x''), x' \rangle = \langle x'', x''' \rangle \tag{8.48}$$

for all $x'' \in X''$. Thus $J_{X'} x' = x'''$. This shows that $J_{X'}$ is surjective and thus X' is reflexive.

Now we show X' reflexive $\implies$ X reflexive. From what we have already shown it follows that X'' is reflexive. Let $Y = J_X(X) \subset X''$. Then Y is a closed subspace because J_X is an isometry. By (ii) the space Y is reflexive. Then by (iii) the space X is reflexive. □

Theorem 8.14. *Let X be a reflexive Banach space. Then the closed unit ball $\overline{B}(0,1)$ is sequentially compact, i.e., every sequence in $\overline{B}(0,1)$ contains a subsequence which converges weakly in $\overline{B}(0,1)$.*

Lemma 8.15. *Let X be a Banach space. Then*

$$X' \text{ separable} \quad \Longrightarrow \quad X \text{ separable} \tag{8.49}$$

Proof. Homework sheet 9, Problem 2. □

Proof of Theorem 8.14. Consider a sequence $x : \mathbb{N} \to \overline{B}(0,1) \subset X$. Let $Y := \overline{\operatorname{span}\{x_k : k \in \mathbb{N}\}}$. Then Y is separable. Moreover Y is a closed subspace of the reflexive space X and hence reflexive. Thus $Y'' = J_Y(Y)$ and therefore Y'' is separable. By Lemma 8.15 the space Y' is also separable.

Let $z_k := J_Y x_k$. Then $z_k \subset \overline{B}(0,1) \subset Y''$ since J_Y is an isometry. Since Y' is separable the closed unit ball in Y'' is weak* sequentially compact by Theorem 8.11. Thus

$$z_{k_j} \overset{*}{\rightharpoonup} z_* \quad \text{in } \overline{B}(0,1) \subset Y'' \tag{8.50}$$

Set $x_* = J_Y^{-1}(z_*) \in Y$. Then for all $y' \in Y$

$$\langle x_k, y' \rangle = \langle y', J_Y(x_{k_j}) \rangle \to \langle y', z_* \rangle = \langle x_*, y' \rangle. \tag{8.51}$$

Finally for $x' \in X'$ we have $x'_{|Y} \in Y'$ and thus

$$\langle x_{k_j}, x' \rangle = \langle x_{k_j}, x'_{|Y} \rangle \to \langle x_*, x'_{|Y} \rangle = \langle x_*, x' \rangle \tag{8.52}$$

for all $x' \in X'$. Thus $x_{k_j} \rightharpoonup x_*$ in X. □

Examples.

(i) Every Hilbert space X is reflexive.
Let $x'' \in X''$. We have to show that there exists $x \in X$ such that

$$x''(y') = y'(x) \quad \forall y' \in X' \tag{8.53}$$

Let $R : X \to X'$ the Riesz isomorphism introduced in Theorem 5.1. We have shown that R is a conjugately linear isometric isomorphism. Define x' by

$$x'(y) := \overline{x''(Ry)}. \tag{8.54}$$

Then x' is linear (not just conjugately linear) and bounded. Thus $x' \in X'$. By the Riesz representation theorem (Theorem 5.1) there exists an $x \in X$ with $Rx = x'$. Therefore

$$x'(y) = (y, x) \quad \forall y \in Y \tag{8.55}$$

and

$$x''(Ry) = \overline{(y, x)} = (x, y) = Ry(x) \quad \forall y \in X. \tag{8.56}$$

Since $R : Y \to Y'$ is surjective this finishes the proof of (8.53).

(ii) Let $p \in (1,\infty)$. Then $L^p(E,\mu)$ is reflexive.
We write L^p instead of $L^p(E,\mu)$. Let $f'' \in (L^p)''$. We have to show that there exist $f \in L^p$ such that

$$f''(g') = g'(f) \quad \forall g' \in (L^p)'. \tag{8.57}$$

Let

$$(J_p f)(g) := \int_E g\overline{f}\, d\mu. \tag{8.58}$$

In Theorem 6.1 we have shown that J_p defines a conjugately linear isometric isomorphism from $L^{p'}$ to $(L^p)'$. Thus we can define

$$f'(g) := \overline{f''(J_p g)} \quad \forall g \in L^{p'}. \tag{8.59}$$

Then f' is linear and bounded on $L^{p'}$. Again by Theorem 6.1 (applied with p' instead of p) $J_{p'}$ is surjective. Taking into account that $(p')' = p$ we see that there exists an $f \in L^p$ such that $f' = J_{p'} f$. Thus

$$f'(g) = \int_E g\overline{f}\, d\mu \tag{8.60}$$

and

$$f''(J_p g) = \overline{f'(g)} = \int_E f\overline{g}\, d\mu = (J_p g)(f) \quad \forall g \in L^{p'}. \tag{8.61}$$

Since $J_p : Lp' \to (L^p)'$ is surjective this proves (8.57).

(iii) Let $p \in (1,\infty)$, let $U \subset \mathbb{R}^n$ be open. Then $W^{1,p}(\mathbb{R}^n)$ is reflexive. (Exercise. Hint: map $W^{1,p}$ bijectively to a closed subspace of $(L^p)^{n+1}$ as homework sheet 11, Problem 2, and use Proposition 8.13).

(iv) The spaces l_1, l_∞, $L^1(E,\mu)$, $L^\infty(E,\mu)$, $C(K)$ and $\mathrm{rca}\,(K)$ are *not* reflexive. Hint: if X is reflexive and separable then X' is separable. Note that for any infinite, compact set K the space $\mathrm{rca}\,(K)$ is not separable since for any $x, y \in K$ we have $\|\delta_x - \delta_y\|_{\mathrm{rca}\,(K)} = 2$ if $x \neq y$.

8.3 Weak convergence in L^p spaces

The results in the subsection were not discussed in class, but on the homework sheets 12 and 13.

The most important examples for weak convergence will be weak convergence in L^p (for $p < \infty$) and weak* convergence in L^∞. We have already shown (see 8.30 and 8.31)

$$f_k \rightharpoonup f_* \text{ in } L^p(E) \quad \Longleftrightarrow \quad \int_E f_j g\, dx \to \int_E f_* g\, dx \quad \forall g \in L^{p'} \quad \text{for } p \in [1,\infty). \tag{8.62}$$

$$f_k \overset{*}{\rightharpoonup} f_* \text{ in } L^\infty(E) \quad \Longleftrightarrow \quad \int_E f_j g\,dx \to \int_E f_* g\,dx \quad \forall g \in L^1. \tag{8.63}$$

Thus for weak convergence we only require convergence of suitable averages rather then pointwise convergence (almost everywhere) or convergence in measures. For bounded domains E there are two prototypical examples of weakly, but not strongly, convergent sequence.

(i) ('oscillation') First let $Q = (0,1)$. Let $h : \mathbb{R} \to \mathbb{R}$ be 1-periodic, i.e., $h(x+1) = h(x)$ for all $x \in \mathbb{R}$ and assume that $\|h\|_{L^p(Q)} < \infty$. Let

$$f_k(x) = h(kx). \tag{8.64}$$

If $p \in [1,\infty)$ then

$$f_k \rightharpoonup \text{const. in } L^p(Q), \quad \text{where} \quad \text{const.} = \int_Q h(z)\,dz. \tag{8.65}$$

For $p = \infty$ one has similarly $f_k \overset{*}{\rightharpoonup}$ const.. Similar results hold in the higher dimensions. Let $Q = (0,1)^n$, assume that $h : \mathbb{R}^n \to \mathbb{R}^n$ is 1-periodic in each coordinate, i.e., $h(x+z) = h(x)$ for all $z \in \mathbb{Z}^n$, and let $f_k(x) = h(kx)$. If $\|h\|_{L^p(Q)} < \infty$ and $p \in [1,\infty)$ then (8.65) holds. The corresponding result with weak* convergence holds if $p = \infty$.
For the proof of these results for $p > 1$, see Homework sheet 12. For $p = 1$ and uses the fact that for each $\varepsilon > 0$ there exists a decomposition $h = h^1 + h^2$ with $h^1 \in L^\infty(Q)$ and $\|h^2\|_{L^1(Q)} \leq \varepsilon$ and the first apply the result for h^1.

Weak convergence and nonlinar functions: For $f_k(x) = \sin 2\pi kx$

$$f_k \overset{*}{\rightharpoonup} 0, \quad f_k^2 \overset{*}{\rightharpoonup} \frac{1}{2} \qquad \text{in } L^\infty((0,1)). \tag{8.66}$$

This shows that weak convergence does not commute with nonlinear functions. This is a weakness of weak convergence but not surprising. Taking averages only commutes with applying a function if f is affine.

(ii) ('concentration') Let $p \in (1,\infty)$, ket $E = B(0,1) \subset \mathbb{R}^n$ and let $h \in L^p(\mathbb{R}^n)$ with $\operatorname{supp} h \subset B(0,R)$ for some $R > 0$. Set

$$f_k(x) = k^{n/p} h(kx). \tag{8.67}$$

Then

$$f_k \rightharpoonup 0 \quad \text{in } L^p(E). \tag{8.68}$$

Note that $\|f_k\|_{L^p(E)} = \|h\|_{L^p(\mathbb{R}^n)}$ if $k > R$. Hence f_k does not converge strongly to 0 (unless $h = 0$ a.e.).

Hint for the proof of (8.68): let $g \in L^{p'}$ and note that $p' < \infty$ since $p > 1$. Note that for $k > R$

$$\left| \int_{B(0,1)} f_k(x) g(x) \, dx \right| = \left| \int_{B(0,\frac{R}{k})} f_k(x) g(x) \, dx \right| \tag{8.69}$$

and use Hölder's inequality.

If E is unbounded, e.g., if $E = \mathbb{R}^n$, then two further prototypical examples can arise.

(i) ('espace to infinity by translation') Let $p \in (1, \infty]$. Let $h \in L^p(\mathbb{R}^n)$. Let $e \in \mathbb{R}^n \setminus 0$ and set

$$f_k(x) := h(x - ke). \tag{8.70}$$

Then

$$h_k \rightharpoonup 0 \quad \text{in } L^p(\mathbb{R}^n) \quad \text{if } p < \infty \tag{8.71}$$

and $h_k \overset{*}{\rightharpoonup} 0$ in $L^\infty(\mathbb{R}^n)$ if $p = \infty$. Moreover $\|f_k\|_{L^p} = \|h\|_{L^p}$.

(ii) ('vanishing'/ 'escape to infinity by dilation') Let $p \in (1, \infty)$ and let $h \in L^p(\mathbb{R}^n)$. For $k \in \mathbb{N} \setminus \{0\}$ let

$$f_k(x) = \frac{1}{k^{n/p}} h(\frac{x}{k}). \tag{8.72}$$

Then

$$f_k \rightharpoonup 0 \quad \text{in } L^p(\mathbb{R}^n) \tag{8.73}$$

and $\|f_k\|_{L^p} = \|h\|_{L^p}$

Hint for the proof of weak convergence in (i) and (ii): use Lemma 8.16 below with $D = C_c^\infty(\mathbb{R}^n)$.

We know that weakly or weakly* convergent sequences are bounded. If we know already that a sequence is bounded, then it suffices to check the conditions for weak or weak* convergence on a dense subset.

Lemma 8.16 (Criterion for weak convergence of bounded sequences)**.** *Let X be a Banach space.*

(i) Let $x : \mathbb{N} \to X$ with $\sup_{k \in \mathbb{N}} \|x_k\| < \infty$. If $D \subset X'$ and $\operatorname{span} D$ is dense in X' then

$$x_k \rightharpoonup x_* \quad \Longleftrightarrow \quad \langle x_k, x' \rangle \to \langle x_*, x' \rangle \quad \forall x' \in D.$$

(ii) Let $x' : \mathbb{N} \to X$ with $\sup_{k \in \mathbb{N}} \|x'_k\| < \infty$. If $D \subset X$ and $\operatorname{span} D$ is dense in X then

$$x'_k \overset{*}{\rightharpoonup} x'_* \quad \Longleftrightarrow \quad \langle x, x'_k \rangle \to \langle x, x'_* \rangle \quad \forall x \in D.$$

Proof. We only show (i) since the proof of (ii) is analogous. The implication $\Longrightarrow$ is obvious. To show the reverse implication let $M = \sup_{k\in\mathbb{N}} \|x_k\|$, let $x' \in X'$ and let $\varepsilon > 0$. By linearity we have

$$\langle x_k, z'\rangle \to \langle x_*, z'\rangle \quad \forall z' \in \operatorname{span}(D).$$

Since $\operatorname{span}(D)$ is dense in X' there exist a $z' \in \operatorname{span}(D)$ such that $\|x'-z'\| \le \varepsilon$. Thus

$$\limsup_{k\to\infty}\langle x_k - x_*, x'\rangle = \limsup_{k\to\infty}\langle x_k - x_*, x' - z'\rangle \le (M + \|x_*\|)\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary this implies the assertion. $\square$

8.4 Convex sets, Mazur's lemma, and existence of minimizers for convex variational problems

We now study in a general setting the relation between weak convergence and convexity.

Theorem 8.17. *Let X be a normed space and let $M \subset X$ be convex and closed. If the sequence $x : \mathbb{N} \to M$ converges weakly to x_* then $x_* \in M$.*

Remark. (i) One can also show that M is closed in the weak topology. Thus strongly closed sets which are convex are also weakly closed.
(ii) Warning: the corresponding assertion for weak* convergence is in general *not* true.

Proof. Suppose that $x_* \notin M$. By the separation theorem (Theorem 6.8) there exists $x' \in X'$ and $\alpha \in \mathbb{R}$ such that

$$\operatorname{Re}\langle x_*, x'\rangle > \alpha, \qquad \operatorname{Re}\langle y, x'\rangle \le \alpha \quad \forall y \in M. \tag{8.74}$$

By assumption $\langle x_k, x'\rangle \to \langle x_*, x'\rangle$. Since $x_k \in M$ this gives $\operatorname{Re}\langle x_*, x'\rangle \le \alpha$, a contradiction. $\square$

Theorem 8.18 (Mazur's lemma)**.** *Let X be a normed space an assume that the sequence $x : \mathbb{N} \to X$ converges weakly to x_*. Then*

$$x_* \in \overline{\operatorname{conv}\{x_k : k \in \mathbb{N}\}} \tag{8.75}$$

(where the closure is taken in the norm topology).

Notation: for a subset E of a vector space X the set $\operatorname{conv} E$ is defined as the smallest convex set which contains E. It is easy to see that $\operatorname{conv} E$ is given by all finite convex combinations of elements of E, that is,

$$\operatorname{conv} E = \{z : z = \sum_{k=1}^{K} \lambda_k x_k,\ K \in \mathbb{N}\setminus\{0\},\ x_k \in E,\ \lambda_k \ge 0,\ \sum_{k=1}^{K}\lambda_k = 1\}.$$

Proof. Apply the previous theorem with $M := \overline{\text{conv}\,\{x_k : k \in \mathbb{N}\}}$. $\square$

Corollary 8.19. *Let X be a normed space and suppose that $x_k \rightharpoonup x_*$ in X. Then a suitable convex combination of the x_k converges strongly to x_*. More precisely there exist $\lambda_{j,k}$ such that*

$$\lambda_{j,k} \geq 0, \quad \sum_{k=j}^{K_j} \lambda_{j,k} = 1, \tag{8.76}$$

$$z_j := \sum_{k=j}^{K_j} \lambda_{j,k} x_k \to x. \tag{8.77}$$

Proof. Consider the sets $M_j := \text{conv}\,\{x_k : k \geq j\}$ and note that by Mazur's lemma $x_* \in \overline{M_j}$. Thus there exist $z_j \in M_j$ such that $\|z_j - x_*\| \leq \frac{1}{j}$. $\square$

Theorem 8.20. *Let X be a reflexive Banach space and let $M \subset X$ be closed, convex and not empty. Let $y \in X$ Then there exists an $x_* \in M$ such that*

$$\|x_* - y\| = \text{dist}\,(y, M) \tag{8.78}$$

Remark. We have shown earlier in Theorem 3.7 that such an x_* exists if X is uniformly convex. The Milman-Pettis theorem states that every uniformly convex Banach space is reflexive (see, e.g., Brezis, Theorem 3.31).

Proof. By definition of the distance there exist x_j such that

$$\|x_j - y\| \leq \text{dist}\,(y, M) + \frac{1}{j+1}. \tag{8.79}$$

In particular

$$\|x_j\| \leq \|y\| + \text{dist}\,(y, M) + 1 \tag{8.80}$$

and thus the sequence $j \mapsto x_j$ is bounded. Since X is reflexive there exists a weakly convergent subsequence

$$x_{j_k} \rightharpoonup x_* \tag{8.81}$$

Since M is closed and convex we have $x_* \in M$. Finally weak lower semicontinuity of the norm (see Proposition 8.7 (iii)) yields that

$$\|x_* - y\| \leq \liminf_{k\to\infty} \|x_{j_k} - y\| \leq \text{dist}\,(y, M). \tag{8.82}$$

Since $y \in M$ we must have equality and the proof is finished. $\square$

We now consider existence results for more general convex variational problems. *These problems were not discussed in class but on homework sheets 12 and 13.*

Theorem 8.21. *Let $U \subset \mathbb{R}^n$ be open and bounded and assume that $f \in L^2(U)$. For $u \in W_0^{1,2}(U)$ let*

$$E(u) := \frac{1}{2}\int_U |\nabla u|^2 \, d\mathcal{L}^n - \int_U fu \, d\mathcal{L}^n. \tag{8.83}$$

Let $M \subset W_0^{1,2}(U)$ be closed, convex and not empty. Then the following assertions hold.

(i) The functional E attains its minimum in M, i.e., there exists $u \in M$ such that

$$E(u) \le E(v) \quad \forall v \in M. \tag{8.84}$$

(ii) An element $u \in M$ is a minimizer of E if and only if u satisfies the variational inequality

$$\int_U \sum_{i=1}^n \partial_i(u-v)\partial_i u - (u-v)f \, d\mathcal{L}^n \le 0 \quad \forall v \in M \tag{8.85}$$

(iii) If M is a closed subspace then (8.85) *is equivalent to the weak form of the Euler-Lagrange equation*

$$\int_U \sum_{i=1}^n \partial_i w \partial_i u - wf \, d\mathcal{L}^n = 0 \quad \forall w \in M \tag{8.86}$$

Remark. The analogous assertions hold if $\int_U |\nabla u|^2$ is replaced by $\int_U \sum_{i,j} a_{ij}\partial_i u \partial_j u$ with $a_{ij} = a_{ji}$, $a_{ij} \in L^\infty(U)$ and the ellipticity condition

$$\exists c > 0 \; \forall \xi \in \mathbb{R}^n \quad \sum_{i,j} a_{ij}\xi_i\xi_j \ge c|\xi|^2 \tag{8.87}$$

holds. In this setting assertion (iii) with $M = W_0^{1,2}(U)$ provides an alternative proof of the existence of a weak solution of the problem

$$-\sum_{i,j} \partial_i(a_{ij}\partial_j u) = f \quad \text{in } U \qquad \text{and} \qquad u = 0 \quad \text{on } \partial U. \tag{8.88}$$

Proof. See homework sheet 13. □

Example. (Obstacle problem) Let $\varphi \in W^{1,2}(U)$ with $\varphi^+ = \max(\varphi, 0) \in W_0^{1,2}$ and let

$$M := \{u \in W_0^{1,2}(U) : u \ge \varphi \text{ a.e.}\}. \tag{8.89}$$

Then M is clearly convex. Moreover M is closed. Indeed, if $u_k \ge \varphi$ a.e. and $u_k \to u$ in $L^2(U)$ then there exists a subsequence such that $u_{k_j} \to u$ a.e.

Hence $u \geq \varphi$ a.e. Moreover M is not empty since $\varphi^+ \in M$. If we assume that the minimizer u is in $W^{2,2}$ then we get

$$\int_U w(-\Delta u - f)\, dx \geq 0 \quad \text{if } u + w \in M \tag{8.90}$$

Since every positive $w \in W_0^{1,2}$ is admissible we get

$$-\Delta u - f \geq 0 \quad \text{a.e. in } U. \tag{8.91}$$

Moreover if u and φ are in addition continuous we get

$$-\Delta u - f = 0 \quad \text{a.e. on the open set } \{x : u(x) > \varphi(x)\}. \tag{8.92}$$

One can show[8] see that under suitable assumptions on f, φ and U the minimizer u is indeed in $W^{2,2}$ and even in $W^{2,\infty}$ (and hence in particular C^1). Even in one dimension and for $f = 0$ the minimizer is is general not in C^2.

8.5 Completely continuous operators

We finally discuss briefly the relation between compactness and weak convergence.

Definition 8.22. *Let X and Y be Banach spaces. Then a linear map $T : X \to Y$ is called completely continuous if*

$$x_n \rightharpoonup x_* \quad \text{in } X \quad \Longrightarrow \quad Tx_n \to Tx_* \quad \text{in } Y. \tag{8.93}$$

Proposition 8.23. *Let X be a reflexive Banach space and let Y be a Banach space. Then*

$$T : X \to Y \quad \text{completely continuous} \quad \Longleftrightarrow \quad T \in \mathcal{L}(X,Y) \text{ and } T \text{ compact.} \tag{8.94}$$

Proof. Homework. Hint: for '$\Longleftarrow$' show first that for $T \in \mathcal{L}(X,Y)$ the convergence $x_n \rightharpoonup x_*$ in X implies $Tx_n \rightharpoonup Tx_*$ in Y. □

[12.1. 2018, Lecture 22]

[17.1. 2018, Lecture 23]

[8] For a general introduction see D. Kinderlehrer, G. Stampacchia, An introduction to variational inequalities and their applications, reprinted by SIAM, 2000.

9 Finite dimensional approximation

Definition 9.1 (Hamel basis)**.** *Let X be a vector space. A set $A \subset X$ is called a Hamel basis of X if every $x \in X$ can be written in a unique way as a finite linear combination of elements of A.*

Every vector space posesses a Hamel basis (this follows from Zorn's lemma: order the set of all linear independent subsets of X by inclusion). From the point of view of analysis a Hamel basis is, however, not very useful. Indeed, if X is an infinite dimensional Banach space then it cannot have a countable Hamel basis (exercise; hint: otherwise X can be written as a countable union of finite dimensional subspace and this contradicts Baire's theorem)

Definition 9.2 (Schauder basis)**.** *Let X be a normed space. A sequence $e : \mathbb{N} \to X$ is called a Schauder basis if for every $x \in X$ there exist uniquely determined $\alpha_k \in \mathbb{K}$ such that*

$$\lim_{n\to\infty} \sum_{k=0}^{n} \alpha_k e_k = x \tag{9.1}$$

Remark. We write $x = \sum_{k=0}^{\infty} \alpha_k e_k$. Note that it is not required that the sum converges absolutely. Thus a reordering of a Schauder basis is not necessarily a Schauder basis.

¿From uniqueness of α_k one easily deduces that the map $e'_k : x \mapsto \alpha_k$ is linear. If X is a Banach space then one can show that e'_k is a continuous map from X to $\mathbb{K}$, i.e., an element of X'. In this case the sequence $e' : \mathbb{N} \to X'$ is called the dual base[9] and it is characterized by the property

$$e'_k(e_l) = \delta_{kl} \quad \forall k, l \in \mathbb{N}. \tag{9.2}$$

To prove that the e'_k are continuous if X is a Banach space one considers the space of sequences

$$Y := \{\alpha : \mathbb{N} \to \mathbb{K} : \lim_{n\to\infty} \sum_{k=0}^{n} \alpha_k e_k \text{ exists}\} \tag{9.3}$$

with the norm $\|\alpha\|_Y := \sup_{n\in\mathbb{N}} \|\sum_{k=0}^{n} \alpha_k e_k\|$. One defines $T : Y \to X$ by $T(\alpha) = \lim_{n\to\infty} \sum_{k=0}^{n} \alpha_k e_k$. Then it is easy to see that $T \in \mathcal{L}(Y, X)$. By definition of a Schauder basis T is bijective. The main point is to show that Y is complete. Then the inverse operator theorem implies that T^{-1} is continuous and this easily yields the continuity of the maps e'_k (see Alt's book for the details).

[9] Note the so-called 'dual basis' is not necessarily a Schauder basis of the dual space X'. Indeed if $X = l_1$ then $X' = l_\infty$ and X' is not separable and hence cannot have a Schauder basis

Let X and Y be Banach spaces with Schauder bases e and f, respectively, and let $T \in \mathcal{L}(X,Y)$. Then there are unique t_{jk} such that $Te_k = \lim_{n\to\infty} \sum_{j=0}^{n} t_{jk} f_j$. Moreover $x = \sum_{k=0}^{\infty} \alpha_k e_k$ implies that $Tx = \sum_{j=1}^{\infty} \beta_j f_j$ with

$$\beta_j = f_j'(Tx) = \sum_{k=0}^{\infty} t_{jk}\alpha_k. \tag{9.4}$$

In this way a bounded operator can be identified with an infinite matrix. One has to be careful, however, in performing calculation with these matrices since the sums involved do not need to converge absolutely.

If X has a Schauder basis then X is necessarily separable (approximate the coefficients by rational coefficients). It was quite a surprise that there exists a separable Banach space which does not have Schauder basis[10]. Many of the standard spaces such as $L^p(U)$ for $p \in [1,\infty)$ do have a Schauder basis.

In separable Hilbert spaces one can construct a particularly nice Schauder basis by using orthogonality. We begin with the definition of an orthonormal system.

Definition 9.3 (Orthonormal system). *Let X be a pre-Hilbert space. Let $N \subset \mathbb{N}$ be finite or infinite. A map $e : N \to X$ is an orthogonal system if*

$$(e_k, e_l) = 0 \quad \text{if } k \neq l \qquad \text{and } e_k \neq 0 \quad \forall k \tag{9.5}$$

and an orthonormal system if

$$(e_k, e_l) = \delta_{kl} \quad \forall k, l. \tag{9.6}$$

Lemma 9.4 (Bessel's inequality). *Let $e_0, \dots, e_n$ be a finite orthonormal system of the pre-Hilbert space X. Then*

$$0 \leq \|x\|^2 - \sum_{k=0}^{n} |(x, e_k)|^2 \tag{9.7}$$

$$= \left\| x - \sum_{k=0}^{n} (x, e_k) e_k \right\|^2 = \operatorname{dist}^2(x, \operatorname{span}\{e_0, \dots, e_n\}). \tag{9.8}$$

Proof. For $\alpha_0, \dots \alpha_n \in \mathbb{K}$ we have

$$\left\| x - \sum_{k=0}^{n} \alpha_k e_k \right\|^2 = \|x\|^2 - \sum_{k=0}^{n} (x, e_k)\overline{\alpha}_k - \sum_{k=0}^{n} (e_k, x)\alpha_k + \sum_{k=0}^{n} |\alpha_k|^2 \tag{9.9}$$

$$= \|x\|^2 - \sum_{k=0}^{n} |(x, e_k)|^2 + \sum_{k=0}^{n} |(x, e_k) - \alpha_k|^2. \tag{9.10}$$

The expression on the right hand side becomes minimal if $\alpha_k = (x, e_k)$ and the minimum is given by $\|x\|^2 - \sum_{k=0}^{n} |(x, e_k)|^2$. □

[10]Per Enflo, A counterexample to the approximation problem in Banach spaces. Acta Math. 130 (1973), 309–317

Definition 9.5 (Orthonormal basis)**.** *Let X be a pre-Hilbert space and let $e : \mathbb{N} \to X$ be an orthonormal system. Then e is called an orthonormal basis if*

$$\operatorname{span}\{e_k : k \in \mathbb{N}\} \quad \textit{is dense in } X. \tag{9.11}$$

Theorem 9.6. *Let X be a pre-Hilbert space and let $e : \mathbb{N} \to X$ be an orthonormal system. Then the following assertions are equivalent*

(i) e is an orthonormal basis.

(ii) e is a Schauder basis of X.

(iii) (representation formula)

$$x = \sum_{k=0}^{\infty} (x, e_k) e_k \quad \forall x \in X. \tag{9.12}$$

(iv) (Parseval identity)

$$(x, y) = \sum_{k=0}^{\infty} (x, e_k)\overline{(y, e_k)} \quad \forall x, y \in X. \tag{9.13}$$

(v) (completeness relation)

$$\|x\|^2 = \sum_{k=0}^{\infty} |(x, e_k)|^2 \quad \forall x \in X \tag{9.14}$$

¿From the completeness relation and the orthonormality relation one easily sees that the sequence $k \mapsto (x, e_k)e_k$ converges absolutely. Hence a reordering of an orthonormal basis is still an orthonormal basis.

Proof. We show that (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v) $\Longrightarrow$ (iii).

(i) $\Longrightarrow$ (iii): Let $x \in X$. By (i) there exist $x_n = \sum_{k=0}^{m_n} \alpha_{n,k} e_k$ with $x_n \to x$. For $m \geq m_n$ Bessel's inequality yields

$$\left\| x - \sum_{k=0}^{m} (x, e_k) e_k \right\|^2 = \operatorname{dist}^2(x, \operatorname{span}\{e_0, \dots, e_m\}) \tag{9.15}$$

$$\leq \operatorname{dist}^2(x, \operatorname{span}\{e_0, \dots, e_{m_n}\}) = \|x - x_n\|^2. \tag{9.16}$$

Thus $\limsup_{m\to\infty} \|x - \sum_{k=0}^{m} (x, e_k) e_k\| \leq \|x - x_n\|$. This holds for all $n \in \mathbb{N}$ and taking $n \to \infty$ we get (iii).

(iii) $\Longrightarrow$ (ii): We have to show uniqueness of the coefficients. Assume that

$$0 = \sum_{k=0}^{\infty} \alpha_k e_k. \tag{9.17}$$

By the continuity of the scalar product we have for all l

$$0 = (\sum_{k=0}^{\infty} \alpha_k e_k, e_l) = \sum_{k=0}^{\infty} \underbrace{(\alpha_k e_k, e_l)}_{\alpha_k \delta_{kl}} = \alpha_l. \tag{9.18}$$

(ii) $\implies$ (i): Let $x \in X$. By definition of the Schauder basis there exist $\alpha_k \in \mathbb{K}$ such that $x_n := \sum_{k=0}^{n} \alpha_k e_k \to x$. Since $x_n \in \operatorname{span}\{e_k : k \in \mathbb{N}\}$ this proves (i).

(iii) $\implies$ (iv): By the continuity of the scalar product we have

$$(x, y) = \lim_{n\to\infty} \left(\sum_{k=0}^{n} (x, e_k) e_k, \sum_{l=0}^{n} (y, e_l) e_l \right) \tag{9.19}$$

$$= \lim_{n\to\infty} \sum_{k,l=0}^{n} (x, e_k)\overline{(y, e_l)}\, \underbrace{(e_k, e_l)}_{\delta_{kl}} = \lim_{n\to\infty} \sum_{k=0}^{n} (x, e_k)\overline{(y, e_k)}. \tag{9.20}$$

(iv) $\implies$ (v): Take $y = x$.

(v) $\implies$ (iii): Bessel's inequality yields

$$\left\| x - \sum_{k=0}^{n} (x, e_k) e_k \right\|^2 = \|x\|^2 - \sum_{k=0}^{n} |(x, e_k)|^2 \to 0 \quad \text{as } n \to \infty. \tag{9.21}$$

□

Theorem 9.7. *Let X be an infinite-dimensional Hilbert space. Then the following statements are equivalent.*

(i) X is separable.

(ii) X has an orthonormal basis.

(iii) X is isometrically isomorphic to l_2.

Proof. (i) $\implies$ (ii) (Schmidt orthogonalization):
This was not discussed in detail in class
Let $E \subset X$ be a countable dense subset and let $y : \mathbb{N} \to E$ be a bijection. Let $z : \mathbb{N} \to E$ be a subsequence such that $\{z_0, \dots z_k\}$ is linear independent for all k and $X_k = \operatorname{span}\{z_0, \dots, z_k\} \supset \operatorname{span}\{y_0, \dots, y_k\}$. Note that $\dim X_k = k+1$. Set $\hat{e}_0 := z_0/\|z_0\|$ and define $\hat{e}_n$ inductively as follows. Let $e_n \in X_n \setminus X_{n-1}$ and set

$$\tilde{e}_n := e_n - \sum_{k=0}^{n-1} (e_n, \hat{e}_k)\hat{e}_k \tag{9.22}$$

$$\hat{e}_n := \frac{\tilde{e}_n}{\|\tilde{e}_n\|} \tag{9.23}$$

Note that $\tilde{e}_n \neq 0$ since $e_n \notin X_{n-1}$ but $\sum_{k=0}^{n-1}(e_n, \hat{e}_k)\hat{e}_k \in X_{n-1}$. Then $n \mapsto \hat{e}_n$ is an orthonormal system. Moreover span $\{\hat{e}_0, \dots, \hat{e}_n\} = X_n$ and $\cup_{n\in\mathbb{N}} X_n \supset E$ is dense in X. Hence $\hat{e}$ is an orthonormal basis.

(ii) $\implies$ (iii): Let e be an orthonormal basis of X. By Theorem 9.6 the map

$$T(\alpha) = \sum_{k=0}^{\infty} \alpha_k e_k \tag{9.24}$$

defines an isometry from l_2 to X.

(iii) $\implies$ (i): Clear since l_2 is separable. $\square$

Example. (Fourier series) Let $X = L^2((-\pi, \pi); \mathbb{C})$. Then the functions

$$e_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikz}, \quad k \in \mathbb{Z} \tag{9.25}$$

form an orthonormal basis. Indeed, $e : \mathbb{Z} \to X$ is clearly an orthonormal system. To see that the span of the e_k is dense let

$$Y := \{f_{|(-\pi,\pi)} : f \in C^2(\mathbb{R}),\ f(x + 2\pi) = f(x)\ \forall x \in \mathbb{R}\}. \tag{9.26}$$

We have shown in Analysis 1 (or see Lemma 9.8 below) that for $f \in Y$

$$\left\| f - \sum_{k\in\mathbb{Z}, |k|\leq n} (f, e_k)e_k \right\|_{L^2((-\pi,\pi))} \to 0 \quad \text{as } n \to \infty. \tag{9.27}$$

Thus $\overline{\operatorname{span}(e_k : k \in \mathbb{Z})} \supset Y$. Now Y is dense in $L^2((-\pi, \pi); \mathbb{C})$. This can be easily seen by considering the extension $g \in L^2((-\pi, \pi); \mathbb{C})$ to a 2π-periodic function on $\mathbb{R}$ and approximating the extended function in the usual way by convolution. Thus $\overline{\operatorname{span}(e_k : k \in \mathbb{Z})} = L^2((-\pi, \pi); \mathbb{C})$.

Lemma 9.8 (Convergence of Fourier series of C^2 functions)**.** *Let Y be given by* (9.26)*, let $f \in Y$ and define*

$$f_n = \sum_{k\in\mathbb{Z}, |k|\leq n} (f, e_k)e_k.$$

Then (9.27) *holds. $f_n \to f$ uniformly and in particular* (9.27) *holds.*

Proof. Not discussed in class.
Integration by parts shows for $k \in \mathbb{N} \setminus \{0\}$

$$(f, e_k) = \frac{1}{ik}(f', e_k) = -\frac{1}{k^2}(f'', e_k) \leq \frac{1}{k^2}\sqrt{2\pi} \sup \|f''\|.$$

Since $\sup |e_k| \leq 1$ it follows that f_n is a Cauchy sequence in $C^0(\mathbb{R})$ and hence convergence uniformly to a 2π periodic function f_*. It only remains to show that $f_*(x) = f(x)$ for all $x \in (-\pi, \pi)$. We have

$$(f, e_k)e_k(x) = \frac{1}{2\pi} \int_{\pi}^{\pi} f(y)e^{-iky}\, dy\; e^{ikx} = \int_{-\pi}^{\pi} \frac{1}{2\pi} e^{ik(x-y)} f(y)\, dy.$$

Thus

$$f_n(x) = \int_{-\pi}^{\pi} K_n(x-y) f(y)\, dy$$

where

$$K_n(z) = \sum_{k=-n}^{n} e^{iz} = e^{-inz} \sum_{k=0}^{2n+1} e^{ikz} = e^{-inz} \frac{1 - e^{i(2n+1)z}}{1 - e^{iz}}.$$

Thus

$$K_n(z) = \frac{\sin(n + \frac{1}{2})z}{\sin \frac{1}{2} z}.$$

Since $\int_{-\pi} \pi e^{ikz} = 0$ for $k \in \mathbb{Z} \setminus \{0\}$ we have

$$\int_{-\pi}^{\pi} K_n(z)\, dz = 1. \tag{9.28}$$

Therefore we get

$$f_n(x) = \int_{-\pi}^{\pi} K_n(x-y) f(y)\, dy = \int_{x-\pi}^{x+\pi} K(z) f(x-z)\, dz.$$

Now for a 2π periodic function h we have $\int_{a-\pi}^{a+\pi} h(z)\, dz = \int_{-\pi}^{\pi} h(z)\, dz$. In view of (9.28) it follows that

$$f_n(x) - f(x) = \int_{-\pi}^{\pi} K(z)[f(x-z) - f(x)] = \int_{-\pi}^{\pi} \sin{(n + \frac{1}{2})z}\, \frac{f(x-z) - f(x)}{\sin \frac{1}{2} z}\, dz.$$

Set

$$g(z) := \frac{f(x-z) - f(x)}{\sin \frac{1}{2} z}.$$

A short calculation shows that $f \in C^2$ implies that $g \in C^1(-\pi, \pi)$ and integration by parts and the fact that $\cos(n + \frac{1}{2})z = 0$ for $z = \pm\pi$ yield

$$f_n(x) - f(x) = \frac{1}{n + \frac{1}{2}} \int_{-\pi} \pi \cos(n + \frac{1}{2}) z\, g'(z)\, dz \to 0 \quad \text{as } n \to \infty.$$

Thus $f_*(x) = f(x)$. □

[17.1. 2018, Lecture 23]

[19.1. 2018, Lecture 24]

10 Compact operators and Sobolev embeddings

10.1 Sobolev embeddings

Theorem 10.1. *Let $U \subset \mathbb{R}^n$ be open, $1 \le p < n$ and let*

$$p^* := \frac{np}{n-p} \quad \text{or, equivalently,} \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}. \tag{10.1}$$

Then there exists a constant $C_{n,p}$ which only depends on n and p such that

$$\|u\|_{L^{p^*}} \le C_{n,p} \|\nabla u\|_{L^p} \quad \forall u \in W_0^{1,p}(U). \tag{10.2}$$

Proof. Step 1: $u \in C_c^1(U)$, $p = 1$
We extend u by zero to a function in $C_c^1(\mathbb{R}^n)$. Since u has compact support the fundamental theorem of calculus yields

$$u(x_1, x_2, \ldots, x_n) = \int_{-\infty}^{x_1} \partial_1 u(y_1, x_2, \ldots, x_n)\, dy_1 \tag{10.3}$$

and thus

$$|u(x)| \le \int_{\mathbb{R}} |\partial_1 u|(y_1, x_2, \ldots, x_n)|\, dy_1 \tag{10.4}$$

and similarly

$$|u(x)| \le \int_{\mathbb{R}} |\partial_i u|(x_1, \ldots, y_i, \ldots, x_n)|\, dy_i. \tag{10.5}$$

We now illustrate the argument by considering the case $n = 3$. We raise (10.5) to the power $\frac{1}{n-1} = \frac{1}{2}$ and multiply over $i = 1, \ldots, n$. Moreover we abbreviate the right hand side of (10.5) by $\int_{\mathbb{R}} |\partial_i u|\, dy_i$. This yields

$$|u(x)|^{\frac{3}{2}} \le \underbrace{\Big(\int_{\mathbb{R}} |\partial_1 u|\, dy_1\Big)^{\frac{1}{2}}}_{h_1(x_2,x_3)} \underbrace{\Big(\int_{\mathbb{R}} |\partial_2 u|\, dy_2\Big)^{\frac{1}{2}}}_{h_2(x_1,x_3)} \underbrace{\Big(\int_{\mathbb{R}} |\partial_3 u|\, dy_3\Big)^{\frac{1}{2}}}_{h_3(x_1,x_2)}. \tag{10.6}$$

Now we integrate over x_1, note that the first term on the right does not depend on x_1 and for the other two terms use Hölder's inequality in the form $\int_{\mathbb{R}} f^{\frac{1}{2}} g^{\frac{1}{2}} \le (\int_{\mathbb{R}} f)^{\frac{1}{2}} (\int_{\mathbb{R}} g)^{\frac{1}{2}}$ (for $f, g \ge 0$). This gives

$$\int_{\mathbb{R}} |u(x)|^{\frac{3}{2}}\, dx_1 \le h_1(x_2, x_3)^{\frac{1}{2}} \left(\int_{\mathbb{R}} h_2(x_1, x_3)\, dx_1\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} h_3(x_1, x_2)\, dx_1\right)^{\frac{1}{2}} \tag{10.7}$$

Now we integrate with respect to x_2 use again Hölder's inequality (in this case for the first and third term, since the second term is independent of x_2. Finally we integrate with respect to x_3 and use once more Hölder's inequality. Thus we get

$$
\begin{aligned}
&\int_{\mathbb{R}^3} |u(x)|^{\frac{3}{2}}\,dx \\
\le &\left(\int_{\mathbb{R}^2} h_1(x_2,x_3)\,dx_2dx_3\right)^{\frac12}\left(\int_{\mathbb{R}^2} h_2(x_1,x_3)\,dx_1dx_3\right)^{\frac12}\left(\int_{\mathbb{R}^2} h_3(x_1,x_2)\,dx_1dx_2\right)^{\frac12} \\
= &\left(\int_{\mathbb{R}^3} |\partial_1 u|\,dx\right)^{\frac12}\left(\int_{\mathbb{R}^3} |\partial_2 u|\,dx\right)^{\frac12}\left(\int_{\mathbb{R}^3} |\partial_3 u|\,dx\right)^{\frac12} \le \left(\int_{\mathbb{R}^3} |\nabla u|\,dx\right)^{\frac32}
\end{aligned}
\tag{10.8}
$$

This proves the estimate for $p = 1$ and $n = 3$. For general n we start from the estimate

$$
|u(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^{n}\left(\int_{\mathbb{R}} |\partial_i u|\,dy_i\right)^{\frac{1}{n-1}} \tag{10.9}
$$

and use the generalized Hölder inequality $\int_{\mathbb{R}} f_1^{\frac{1}{n-1}} \dots f_{n-1}^{\frac{1}{n-1}} \le (\int_{\mathbb{R}} f_1)^{\frac{1}{n-1}} \dots (\int_{\mathbb{R}} f_{n-1})^{\frac{1}{n-1}}$.

Step 2: $u \in C_c^1(U)$, $p \in (1, n)$

Note that for $\gamma > 1$ we have by the chain rule $|u|^\gamma$ in $C_c^1(\mathbb{R}^n)$ and

$$
|\nabla |u|^\gamma| \le \gamma |u|^{\gamma-1}\,|\nabla u|. \tag{10.10}
$$

Thus by the result for $p = 1$ and the Hölder inequality

$$
\begin{aligned}
\frac{1}{\gamma}\|u\|^\gamma_{L^{\gamma\frac{n}{n-1}}} = \frac{1}{\gamma}\||u|^\gamma\|_{L^{\frac{n}{n-1}}} \le \||u|^{\gamma-1}|\nabla u|\|_{L^1} \\
\le \||u|^{\gamma-1}\|_{L^{p'}}\,\|\nabla u\|_{L^p} \le \|u\|^{\gamma-1}_{L^{p'(\gamma-1)}}\,\|\nabla u\|_{L^p}.
\end{aligned}
\tag{10.11}
$$

Now let

$$
\gamma = \frac{(n-1)p}{n-p} = \frac{np-p}{n-p}. \tag{10.12}
$$

Then $\gamma > 1$ and

$$
(\gamma - 1)p' = \frac{np-n}{n-p}\frac{p}{p-1} = \frac{np}{n-p} = p^* = \gamma\frac{n}{n-1}. \tag{10.13}
$$

Thus (10.11) yields the desired estimate for $u \in C_c^1(U)$.

Step 3: $u \in W_0^{1,p}(U)$, $p = 1$

Let $u \in W_0^{1,p}$. By the definition of $W_0^{1,p}$ there exist $u_k \in C_c^1(U)$ such that $u_k \to u$ in $W_0^{1,p}$. Since we have shown that (10.1) holds for u_k it follows that $k \mapsto u_k$ is a Cauchy sequence in $L^{p*}(U)$. Hence there exist $v \in L^{p^*}(U)$ such that $u_k \to v$ in $L^{p*}(U)$ and

$$
\|v\|_{L^{p^*}} = \lim_{k\to\infty}\|u_k\|_{L^{p^*}} \le C_{n,p}\limsup_{k\to\infty}\|\nabla u_k\|_{L^p} \le C_{n,p}\|\nabla u\|_{L^p}. \tag{10.14}
$$

On the other hand $u_k \to u$ in $L^p(u)$. Thus $u = v$ a.e. and this finishes the proof. □

Theorem 10.2. *Let $U \subset \mathbb{R}^n$ be open, let $p \in (n, \infty]$ and let*

$$\alpha := 1 - \frac{n}{p}, \quad \textit{with } \alpha = 1 \textit{ for } p = \infty. \tag{10.15}$$

Then every $u \in W_0^{1,p}(U)$ has a Hölder continuous representative $\overline{u}$ and

$$[\overline{u}]_{C^{0,\alpha}} := \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq C_p \|\nabla u\|_{L^p}. \tag{10.16}$$

Warning. Note that the case $p = n$ is excluded in Theorems 10.1 and in Theorem 10.2. Indeed for $n \geq 2$ the function $u(x) = \ln |\ln |x||$ belongs to $W^{1,n}(B(0, \frac{1}{2})$ but is not in L^∞ (see Homework sheet 13). For $n = 1$ we have shown in Theorem 2.42 that every $W^{1,1}$ function has a continuous representative.

Remark. (i) Assume that U is bounded. Since $\overline{u} = 0$ on ∂U the estimate 10.2 implies that

$$\sup_U |u| \leq C_p \|\nabla u\|_{L^p} (\mathrm{diam} U)^\alpha. \tag{10.17}$$

(ii) We will show that for $p = \infty$ the estimate holds with $C_p = 1$. One call also show a converse statement: if $v \in C^{0,1}(U)$ then $v \in W^{1,\infty}(U)$ and $\|\nabla v\|_{L^\infty} \leq [v]_{C^{0,1}}$. Idea of proof: extend v to a Lipschitz function on $\mathbb{R}^n$ (with the same Lipschitz constant) and then consider the difference quotients

$$g_a^{(k)}(x) := k(v(x + \frac{1}{k}a) - u(x)).$$

Then $|g^{(k)}| \leq [v]_{C^{0,1}} |a|$. Thus for each a there exists a subsequence k_j such that

$$g_a^{(k_j)} \overset{*}{\rightharpoonup} g_a^* \quad \text{in } L^\infty(\mathbb{R}^n) \text{ as } j \to \infty.$$

Now

$$\int_{\mathbb{R}^n} g_a^{(k)} \varphi \, dx = \int_{\mathbb{R}^n} v(x) k(\varphi(x - \frac{1}{k}a) - \varphi(x)) \, dx \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n). \tag{10.18}$$

Taking a subsequence which converges for $a \in \{e_1, \ldots, e_n\}$ and passing to the limit we get

$$\int_{\mathbb{R}^n} g_{e_i}^* \varphi \, dx = -\int_{\mathbb{R}^n} v \partial_i \varphi \, dx \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n).$$

Thus v is weakly differentiable and the weak derivatives $\partial_i v$ are given by $g_{e_i}^* \in L^\infty$. Hence $v \in W^{1,\infty}(\mathbb{R}^n)$. Moreover we the weak* sequential lower semicontinuity of the L^∞ norm (see Proposition 8.10 (iii))) we get

$$\|\partial_i v\|_{L^\infty} = \|g_{e_i}^*\|_{L^\infty} \leq [v]_{C^{0,1}}.$$

To show that even $|\nabla v(x)| = (\sum_{i=1}^n |\nabla v_i|^2)^{1/2} \leq [v]_{C^{0,1}}$ one can argue as follows. Let D be a countable dense subset of $\mathbb{R}^n$. By taking a diagonal sequence we can find a subsequence such that $g_a^{(k_j} \overset{*}{\rightharpoonup} g_a^*$ in $L^\infty(\mathbb{R}^n)$ for all $a \in D$. Passing to the limit in (10.18) we get

$$\int_{\mathbb{R}^n} g_a^* \varphi \, dx = -\int_{\mathbb{R}^n} v \sum_{i=1}^n a_i \partial_i \varphi \, dx = \int_{R^n} \sum_{i=1} a_i \partial_i v_i \, \varphi \, dx \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n).$$

Since $|g_a^{(k)}| \leq [v]_{C^{0,1}}$ weak* lower semicontinuity of the L^∞ norm gives $|g_a^*| \leq [v]_{C^{0,1}}$

$$(a, \nabla v(x)) = \sum_{i=1} a_i \partial_i v_i(x) = g_a(x) \leq [v]_{C^{0,1}} |a| \quad \forall x \in \mathbb{R}^n \setminus N \ \forall a \in D$$

where N is a null set. Since D is dense we get $(a, \nabla v(x)) \leq\leq [v]_{C^{0,1}} |a|$ for all $a \in \mathbb{R}^n$ and all $x \in \mathbb{R}^n \setminus N$ we get the desired assertion by taking $a = \nabla v(x)$ (if $\nabla v(x) \neq 0$).

Lemma 10.3. *Let $u \in C_c^1(\mathbb{R}^n)$ and let $p \in (n, \infty]$. Then for all $x \in \mathbb{R}^n$*

$$\int_{B(x,1)} |u(y) - u(x)| \, dy \leq \frac{1}{n} \int_{B(x,1} \frac{|\nabla u(z)|}{|z - x|^{n-1}} \, dz \tag{10.19}$$

$$\leq C_{n,p} \|\nabla u\|_{L^p(B(x,1))}. \tag{10.20}$$

Proof. The second inequality follows from Hölder's inequality since $p' < n' = \frac{n}{n-1}$. To prove the first inequality we may assume that $x = 0$. Using polar coordinates $y = r\omega$ we get

$$\int_{B(0,1)} |u(y) - u(0)| \, dy = \int_0^1 \int_{S^{n-1}} |u(r\omega) - u(0)| \, d\mathcal{H}^{n-1}(\omega) \, r^{n-1} \, dr. \tag{10.21}$$

By the fundamental theorem of calculus we get

$$|u(r\omega) - u(0)| = \left| \int_0^r \frac{d}{ds} u(s\omega) \, ds \right| \leq \int_0^r |\nabla u(s\omega)| \, ds. \tag{10.22}$$

Hence

$$\int_{S^{n-1}} |u(r\omega) - u(0)| \, d\mathcal{H}^{n-1}(\omega) \leq \int_0^r \int_{S^{n-1}} \frac{|\nabla u(s\omega)|}{s^{n-1}} \, d\mathcal{H}^{n-1}(\omega) \, s^{n-1} \, ds$$
$$= \int_{B(0,r)} \frac{|\nabla u(z)|}{|z|^{n-1}} \, dz \leq \int_{B(0,1} \frac{|\nabla u(z)|}{|z|^{n-1}} \, dz. \tag{10.23}$$

Now the assertion follows by multiplying by r^{n-1}, integrating over r and using (10.21). □

Proof of Theorem 10.2. First assume $p \in (n, \infty)$ and $u \in C_c^1(\mathbb{R}^n)$. Extend u by zero to $\mathbb{R}^n$.

Step 1: For $x, z \in \mathbb{R}^n$ with $|x - z| = 1$ we have

$$|u(x) - u(z)| \le C'_{n,p} \left(\int_{B(x,2)} |\nabla u|^p \, d\mathcal{L}^n \right)^{\frac{1}{p}}. \tag{10.24}$$

We have for all $y \in B(x,1) \cap B(z,1)$

$$|u(x) - u(z)| \le |u(x) - u(y)| + |u(y) - u(z)| \tag{10.25}$$

and integration over $B(x,1) \cap B(z,1)$ yields in connection with the lemma

$$\begin{aligned} &\mathcal{L}^n(B(x,1) \cap B(y,1)) \; |u(x) - u(z)| \\ \le &\int_{B(x,1)\cap B(z,1)} |u(x) - u(y)| + |u(z) - u(y)| \, dy \\ \le &C_{n,p} \left(\int_{B(x,1)} |\nabla u(y)|^p \, dy \right)^{\frac{1}{p}} + C_{n,p} \left(\int_{B(z,1)} |\nabla u(y)|^p \, dy \right)^{\frac{1}{p}} = 2C_{n,p} \left(\int_{B(x,2)} |\nabla u(y)|^p \, dy \right)^{\frac{1}{p}} \end{aligned} \tag{10.26}$$

since $B(y,1) \subset B(x,2)$. The assertion follows since $\mathcal{L}^n(B(x,1) \cap B(z,1)) > 0$.

Step 2: For $x, z \in \mathbb{R}^n$ with $|x - z| = r$ we have

$$|u(x) - u(z)| \le C'_{n,p} r^{1-\frac{n}{p}} \left(\int_{B(x,2r)} |\nabla u|^p \, d\mathcal{L}^n \right)^{\frac{1}{p}}. \tag{10.27}$$

Let $x' = \frac{x}{r}$, $z' = \frac{z}{r}$ and apply the estimate in Step 1 to the function $v(\xi) := u(r\xi)$. and the points x' and z'. Now $\nabla v(\xi) = r(\nabla u)(r\xi)$ and thus

$$\int_{B(x',2)} |\nabla v(\xi)|^p \, d\xi = \int_{B(x',2)} r^p |\nabla u(r\xi)|^p \, d\xi = \int_{B(x,2r)} r^{p-n} |\nabla u(y)| \, dy. \tag{10.28}$$

Taking the p-th root we obtain the desired estimate.

Step 3: The estimate holds for $u \in W_0^{1,p}(U)$ and $p \in (n, \infty)$.
This follows by density as in the proof of Theorem 10.1. Let $u_k \in C_c^1(U)$ such that $u_k \to u$ in $W_0^{1,p}(U)$. Extend u_k by zero to $\mathbb{R}^n$. By Step 2 (and the fact that U is bounded) the sequence $k \mapsto u_k$ is a Cauchy sequence in $C^{0,\alpha}(U)$. Hence $u_k \to \overline{u}$ in $C^{0,\alpha}(U)$ and $[\overline{u}]_{0,\alpha} \le C \|\nabla u\|_p$. On the other hand $u_k \to u$ in $L^p(U)$. Thus $\overline{u} = u$ a.e. This finishes the proof for $p \ne \infty$.

Step 4: $p = \infty$.
For $u \in C^1(\mathbb{R}^n)$ we have

$$|u(z) - u(x)| = \left| \int_0^1 \nabla u((1-t)x + tz) \cdot (z - x) \, dt \right| \le \|\nabla u\|_{L^\infty} |z - x|. \tag{10.29}$$

Let $u \in W^{1,\infty}_0(U)$. Extend u by zero outside U. Then $u \in W^{1,\infty}(\mathbb{R}^n)$ (see the examples of Definition 2.44). Let $V = B_1(U) = \{x : \operatorname{dist}(x, U) < 1\}$. Then V is open and bounded and $\overline{U} \subset V$. Thus exist $u_k \in C^1_c(V)$ such that

$$u_k \to u \quad \text{in } W^{1,p}_0(V) \quad \text{for all } p < \infty \qquad \text{and } \|\nabla u_k\|_{L^\infty} \le \|\nabla u\|_{L^\infty}. \tag{10.30}$$

To see approximate u as usual by convolution. Hence by the result for $p \in (n, \infty)$ we have $u_k \to \overline{u}$ in $C^{0,\alpha}(V)$ for all $\alpha < 1$. Thus for all $x, z \in \mathbb{R}^n$

$$|\overline{u}(x) - \overline{u}(z)| = \lim_{k\to\infty} |u_k(x) - u_k(z)| \le \limsup_{k\to\infty} \|\nabla u_k\|_{L^\infty} \le \|\nabla u\|_{L^\infty}. \tag{10.31}$$

Moreover $u_k \to u$ in $L^p(U)$. Thus $\overline{u} = u$ a.e. in U. □

By induction one can easily obtain corresponding results for the space $W^{m,p}_0$ and we will state them below. Before doing so we show that the exponents in the embedding theorem as $p^* = \frac{np}{n-p}$ and $\alpha = 1 - \frac{n}{p}$ above are determined entirely by scaling. Let

$$u \in C^\infty_c(B(0,1)) \quad u_r(x) := u(rx). \tag{10.32}$$

Then (with the change of variables $y = rx$)

$$\left(\int_{\mathbb{R}^n} |\nabla^m u_r(x)|^p \, dx\right)^{\frac{1}{p}} = r^{m-\frac{n}{p}} \left(\int_{\mathbb{R}^n} |\nabla^m u(y)|^p \, dy\right)^{\frac{1}{p}} \tag{10.33}$$

The exponent $m - \frac{n}{p}$ is sometimes called the Sobolev number of the space $W^{m,p}$. An estimate of the form $\|f\|_{L^q} \le C\|\nabla^m f\|_{L^p}$ can only hold for all $f \in C^\infty_c(\mathbb{R}^n)$ it holds for all the functions u_r. This yields the necessary condition $-\frac{n}{q} = m - \frac{n}{p}$, i.e., if the Sobolev numbers of $W^{m,p}$ and L^q must agree. For $m = 1$ we recover the condition $q = p^*$.

If we want the estimate only for function with the support in a fixed bounded set, e.g., the unit ball, then we need the estimate for all u_r with $r > 1$. This leads to the necessary condition $-\frac{n}{q} \le m - \frac{n}{p}$, i.e., the Sobolev number of L^q has to be less than or equal to the Sobolev number of $W^{m,p}$.

A similar reasoning applies to the Hölder spaces. We have

$$[\nabla^k u_r]_{C^{0,\beta}} = r^{k+\beta}[\nabla^k u]_{C^{0,\beta}}. \tag{10.34}$$

Hence for an estimate $[\nabla^k f]_{C^{0,\beta}} \le C\|\nabla^m f\|_{L^p}$ for all $f \in C^\infty_c(\mathbb{R}^n)$ we need $k + \beta = m - \frac{n}{p}$. For $k = 0$ and $m = 1$ we recover the condition $\beta = 1 - \frac{n}{p}$. On bounded domains we obtain the corresponding inequality as a necessary condition.

We now show that the above conditions on the Sobolev number are also sufficient.

Theorem 10.4 (Sobolev embedding)**.** *Let $U \subset \mathbb{R}^n$ be open and bounded. Assume that*

$$m \in \mathbb{N} \setminus \{0\}, p \in [1, \infty], \tag{10.35}$$

$$l \in \mathbb{N}, q \in [1, \infty), \beta \in (0,1) \tag{10.36}$$

Then

(i) $W_0^{m,p}(U) \subset W_0^{l,q}(U)$ *if* $l \le m$ *and* $l - \frac{n}{q} \le m - \frac{n}{p}$.

(ii) $W_0^{m,p}(U) \subset C^{k,\beta}(U)$ *if* $k + \beta \le m - \frac{n}{p}$.

Moreover the corresponding injections are continuous, i.e., the $W^{l,q}$ norm or the $C^{k,\beta}$ norm can be estimated by the $W^{m,p}$ norm.

Proof. This follows by induction from Theorems 10.1 and 10.2 as well as the relation $\|f\|_{L^q(U)} \le C(U)\|f\|_{L^r}$ for bounded sets and $q < r$. *The details were not discussed in class.*

(i): If U is bounded then $L^s(U) \subset L^r(U)$ if $s \ge r$. Thus Theorem 10.1 shows that (for $p < n$)

$$W_0^{1,p}(U) \subset L^q(U) \quad \text{if } \frac{1}{q} \ge \frac{1}{p} - \frac{1}{n}. \tag{10.37}$$

This condition can be rewritten as $-\frac{n}{q} \le 1 - \frac{n}{p}$ or $m - 1 - \frac{n}{q} \le m - \frac{n}{p}$. If we apply (10.37) to all partial derivative of order $\le m-1$ of a function in $W_0^{m.p}(U)$ we get

$$W_0^{m,p}(U) \subset W^{m-1,q}(U) \quad \text{if } m - 1 - \frac{n}{q} \le m - \frac{n}{p}. \tag{10.38}$$

By density of C_c^∞ we get in fact $W_0^{m,p}(U) \subset W_0^{m-1,q}(U)$ if $m-1-\frac{n}{q} \le m-\frac{n}{p}$. Now assertion (i) follows (for $p < n$) by induction. If $p > n$ assertion (i) follows from assertion (ii) with $k = m-1$. If $p = n$ and $q = n$ assertion (i) is trivial. Finally if $p = n$ we can use that $u \in W_0^{1,\tilde{p}}(U)$ for all $\tilde{p} < n$. Thus $u \in W_0^{m-1,q}(U)$ for all $q < \infty$.

(ii): If $k = 0$ this follows directly from 10.2. Now suppose $u \in W_0^{m,p}(U)$ and the condition is satisfied for some $k > 0$. Then all (weak) derivatives of u up to order k belong to $W_0^{m-k,p}$ and hence have a representative in $C^{0,\alpha}$. From this one easily concludes that u has a representative in $C^{k,\alpha}(U)$ (one possibility is to show that if $u_k \in C_c^\infty(U)$ converges in $W_0^{m,p}$ then u_k is Cauchy sequence in $C^{k,\beta}(U)$). The limit in $C^{k,\beta}(U)$ must agree a.e. with the limit in $W_0^{m,p}(U)$. □

[19.1. 2018, Lecture 24]

[24.1. 2018, Lecture 25]

We now ask when the identity map from $W_0^{m,p}$ to $W_0^{l.q}$ is not only continuous but compact. Recall that for normed spaces X and Y a map $T \in \mathcal{L}(X,Y)$ is compact if and only if $\overline{T(B(0,1))}$ is compact. We note that

$$\begin{aligned}&\overline{T(B(0,1))} \text{ compact} \quad \Longleftrightarrow \\ &\text{every sequence } y:\mathbb{N}\to T(B(0,1)) \text{ has a subsequence which converges in } Y\end{aligned} \tag{10.39}$$

The implication $\Longrightarrow$ is clear since compactness of $\overline{T(B(0,1))}$ implies sequential compactness. To show the implication $\Longleftarrow$ we show that $\overline{T(B(0,1))}$ is sequentially compact. Let $z:\mathbb{N}\to\overline{T(B(0,1))}$. By definition of the closure there exist $y:\mathbb{N}\to T(B(0,1))$ such that $\|z_k - y_k\| \le 2^{-k}$. By assumption there exists a subsequence $y_{k_j}\to y_*$ in X. Then $y_*\in\overline{T(B(0,1))}$ and $z_{k_j}\to y_*$. Thus $\overline{T(B(0,1))}$ is sequentially compact and hence compact.

Theorem 10.5 (Compact Sobolev embedding)**.** *Under the conditions of Theorem 10.4 the identity map from $W_0^{m,p}(U)$ to $W_0^{l,q}(U)$ or $C^{k,\beta}(U)$ are compact if*

$$l < m \quad \textit{and} \quad l - \frac{n}{q} < m - \frac{n}{p} \tag{10.40}$$

and

$$k + \beta < m - \frac{n}{p}, \tag{10.41}$$

respectively.

Remark. The scaling $u_k(x) = k^{\frac{n}{p}-m}u(kx)$ shows that the embedding is *not* compact if $l - \frac{n}{q} = m - \frac{n}{p}$ (or if $k+\beta = m - \frac{n}{p}$). Highly oscillating functions of the form $u_k(x) = k^{-m}\sin kx$ show that the embedding is not compact if $l = m$, even for $n = 1$ and $U = (0,1)$.

Proof. Regarding (10.40) it suffices to consider the case $m = 1$, $l = 0$. The other cases follow from Theorem 10.4 and the fact that the composition of a bounded operator and a compact operator is a compact operator. To show that the identity is a compact operator from $W_0^{1,p}(U)$ to $L^q(U)$ for $q < p*$ we have to show that every sequence in $u:\mathbb{N}\to B(0,1)\subset W^{1,p}(U)$ has a convergent subsequence in $L^q(U)$. We know that

$$u_{k_j}\to u \quad \text{in } L^p(U) \tag{10.42}$$

(see the example after Lemma 3.17). Hence we are done if $q \le p$ (since U is bounded). If $p < q < p*$ we use the interpolation inequality

$$\|f\|_q \le \|f\|_{p_1}^{\theta}\,\|f\|_{p_2}^{1-\theta}, \quad \text{where } \theta \text{ is uniquely defined by } \frac{1}{q} = \theta\frac{1}{p_1} + (1-\theta)\frac{1}{p_2} \tag{10.43}$$

which follows from Hölder's inequality $\int gh \leq \|g\|_s \|h\|_{s'}$ with $g = |f|^{\theta q}$, $h = |f|^{(1-\theta)q}$, $s = \frac{p_1}{q\theta}$ and $s' = \frac{p_2}{q(1-\theta)}$. Since u_{k_j} is bounded in L^{p*} and a Cauchy sequence in L^p this shows that $j \mapsto u_{k_j}$ is a Cauchy sequence in $L^q(U)$ and hence convergent.

The proof of (10.41) is similar. It suffices to show that the identity map from $C^{0,\alpha}(U)$ to $C^{0,\beta}(U)$ is compact if $0 \leq \beta < \alpha \leq 1$. Let $u : \mathbb{N} \to B(0,1) \subset C^{0,\alpha}(U)$. By the Arzela-Ascoli theorem (Theorem 3.15) there exists a subsequence such that $u_{k_j} \to u_*$ uniformly. This finishes the proof for $\beta = 0$. Now let $\theta = \beta/\alpha$

$$\frac{|v(x) - v(y)|}{|x-y|^\beta} = \frac{|v(x) - v(y)|^\theta}{|x-y|^{\theta\alpha}} |v(x) - v(y)|^{1-\theta} \tag{10.44}$$

which implies that

$$[v]_{C^{0,\beta}} \leq [v]^\theta_{C^{0,\alpha}} 2^{1-\theta} \|v\|^{1-\theta}_{C^0}. \tag{10.45}$$

Thus $j \mapsto u_{k_j}$ is a Cauchy sequence in $C^{0,\beta}(U)$ and hence convergent. □

Theorem 10.6. *Let $U \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary. Then the results in Theorems 10.4 and 10.5 also hold with $W^{m,p}_0(U)$ replaced by $W^{m,p}(U)$.*

Idea of proof. . Let $U \subset\subset V \subset\subset \mathbb{R}^n$. Then one can show that there exist a bounded extension operator $E : W^{1,p}(U) \to W^{1,p}_0(V)$ such that $Ef_{|U} = f$. Then the assertion follows for $m = 1$ and $l = 0$ or $k = 0$ by applying the result for $W^{1,p}_0(V)$. The general case follows by induction as in the proof of Theorem 10.4.

To construct E for a function with support near ∂U one uses a suitable local reflection and a cut-off. For general f one first uses a partition of unity. For details see, e.g., the book of H.W Alt. □

11 Spectral theory

11.1 The spectrum and the resolvent

For a linear map $A : \mathbb{C}^n \to \mathbb{C}^n$ we have

$$A \text{ injective} \iff A \text{ surjective} \iff A \text{ invertible.} \tag{11.1}$$

Moreover there exist finitely many values (the eigenvalues of A) $\lambda_1, \ldots \lambda_k \in \mathbb{C}$ such that $A - \lambda_i \mathrm{Id}$ is not invertible. There exists a basis of $\mathbb{C}^n$ such that the matrix of A in this basis has Jordan normal form with the values λ_i on the diagonal. If A is self-adjoint, i.e. if $(Ax, y) = (x, Ay)$ for all $x, y \in \mathbb{C}^n$ then all eigenvalues are real and there exists an orthonormal basis such that A is diagonal in this basis (thus the elements of the basis are eigenvectors).

We aim for an extension of these results to bounded operators $T \in \mathcal{L}(X)$ on a complex Banach space X. In this case the situation is more complicated since the counterpart of (11.1) is not true.

Example. (i) Let $X = l_2$ and let $T : l_2 \to l_2$ be the shift operator, i.e., $(Tx)_{k+1} = x_k$, $(Tx)_0 = 0$. Then T is injective but not surjective. More precisely the range $\mathcal{R}(T)$ is a closed subspace of l_2 and we have $l_2 = \mathcal{R}(T) \oplus \operatorname{span}(e_0)$. Similarly the left shift operator defined by $(Tx)_k = x_{k+1}$ is surjective but not injective.
(ii) Let $X = l_2$ and let $T : l_2 \to l_2$ be given by $(Tx)_k = 2^{-k} x_k$. Then T is injective and $\mathcal{R}(T)$ is dense in l_2 since $\mathcal{R}(T)$ contains all sequences which have only finitely many non-zero entries. We have, however, $\mathcal{R}(T) \neq l_2$, since, e.g., the sequence $y \in l_2$ given by $y_k = 2^{-k/2}$ is not in $\mathcal{R}(T)$, Indeed every sequence $x \in l_2$ is bounded. Hence every sequence $y \in \mathcal{R}(T)$ must satisfies $|y_k| \leq C2^{-k}$ for some $C \in \mathbb{R}$ and all $k \in \mathbb{N}$.

These examples motivate the following definition.

Definition 11.1 (Spectrum of an operator)**.** *Let X be a Banach space of $\mathbb{C}$ and let $T \in \mathcal{L}(X)$. Then we define the spectrum of T as*

$$\sigma(T) := \{\lambda \in \mathbb{C} : T - \lambda \mathrm{Id} \text{ is not invertible} \tag{11.2}$$

and we set

$$\sigma_p(T) := \{\lambda \in \mathbb{C} : \mathcal{N}(T - \lambda \mathrm{Id}\,) \neq \{0\}\}, \tag{11.3}$$

$$\sigma_c(T) := \{\lambda \in \mathbb{C} : \mathcal{N}(T - \lambda \mathrm{Id}\,) = \{0\}, \mathcal{R}(T - \lambda \mathrm{Id}\,) \neq X, \overline{\mathcal{R}(T - \lambda \mathrm{Id}\,)} = X\}, \tag{11.4}$$

$$\sigma_r(T) := \{\lambda \in \mathbb{C} : \mathcal{N}(T - \lambda \mathrm{Id}\,) = \{0\}, \overline{\mathcal{R}(T - \lambda \mathrm{Id}\,)} \neq X\}. \tag{11.5}$$

These sets are referred to as the point spectrum, the continuous spectrum and the residual spectrum. The elements of $\sigma_p(T)$ are called eigenvalues.

By the inverse operator theorem we have $\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$.

Notation For $T \in \mathcal{L}(X)$ and $\lambda \in \mathbb{C}$ we write

$$T - \lambda \quad \text{as an abbreviation of } T - \lambda \,\mathrm{Id} \tag{11.6}$$

Theorem 11.2. *The resolvent set*

$$\rho(T) := \mathbb{C} \setminus \sigma(T) \tag{11.7}$$

is open and the map

$$\lambda \mapsto R(\lambda) := (\lambda - T)^{-1} \tag{11.8}$$

$$\rho(T) \to \mathcal{L}(X) \tag{11.9}$$

is analytic.

Proof. Let $\lambda \in \rho(T)$. Then

$$(\lambda + \mu) - T = \lambda - T + \mu = (\lambda - T)(Id + \mu(\lambda - T)^{-1}. \tag{11.10}$$

Now for $|\mu| < r := 1/\|(\lambda - T)^{-1}$ we can express $(Id + \mu(\lambda - T)^{-1}$ as a convergent Neumann series, see Theorem 4.8. This shows that $\rho(T)$ contains the disc $B(\lambda, r)$ and that the resolvent maps is analytic in $B(\lambda, r)$. Hence $\rho(T)$ is open and the resolvent map is analytic in $\rho(T)$. □

11.2 Fredholm operators, index, Fredholm alternative

We next consider a class of operators which are almost invertible, in the sense that they are invertible up to a finite dimensional correction in the domain and the image.

Definition 11.3 (Fredholm operator)**.** . *Let X and Y be Banach spaces. An operator $F \in \mathcal{L}(X, Y)$ is called a Fredholm operator if*

(i) $\dim \mathcal{N}(F) < \infty$,

(ii) $\mathcal{R}(F)$ *is closed,*

(iii) $\operatorname{codim} \mathcal{R}(F) < \infty$ *i.e., there exists a finite dimensional space Y_0 such that*

$$Y = \mathcal{R}(F) \oplus Y_0, \quad \dim Y_0 < \infty \tag{11.11}$$

Proposition 11.4. *Let Y be a Banach space and Z, Y_0 and Y_1 be closed subspaces. Assume that*

$$Y = Z \oplus Y_0 = Z \oplus Y_1. \tag{11.12}$$

Then Y_0 and Y_1 are isometrically isomorphic. In particular if Y_0 is finite dimensional then Y_1 is finite dimensional and $\dim Y_0 = \dim Y_1$.

Proof. Consider the maps $A_i := Z \times Y_i \to Y$ given by $A(z, y_i) = z + y_i$. Then A is bijective (by the definition of the direct sum) and bounded (since the product norm is given by $\|(z, y_i)\| = \|z\| + \|y_i\|$. Hence by the inverse operator theorem the operators A_i are invertible. Now define $B : Y_1 \to Y_0$ by $By_1 := \pi_2 A_0^{-1} y_1$, where $\pi_2(z, y_0) = y_0$. Then B is bounded. Moreover B is injective since $By_1 = 0$ implies that $A_0^{-1} y_1 = (z, 0)$. Thus $z + 0 = y_1$, which implies $y_1 = 0$. Finally B is surjective. Indeed if $y_0 \in Y_0$ then $\pi_2 A_1^{-1} y_0 \in Y_1$. Thus there exists a $z \in Z$ such that $y_1 + z = y_0$. Hence $A_0^{-1}(y_1) = (-z, y_0)$ and $By_1 = y_0$. By the inverse operator theorem B is invertible and hence Y_0 and Y_1 are isometrically isomorphic. □

[24.1. 2018, Lecture 25]

[26.1. 2018, Lecture 26]

Definition 11.5. *Let $F \in \mathcal{L}(X, Y)$ be a Fredholm operator. Then we define* $\operatorname{codim} \mathcal{R}(F) = \dim Y_0$ *if* $\mathcal{R}(F) \oplus Y_0 = Y$ *and we define the index of* T *as*

$$\operatorname{ind} F := \dim \mathcal{N}(F) - \operatorname{codim} \mathcal{R}(F). \tag{11.13}$$

Theorem 11.6. *Let X be a Banach space and let $T \in \mathcal{L}(X)$ be compact. Then*

$$A := \mathrm{Id} - T \tag{11.14}$$

is a Fredholm operator with index zero. In particular we have

(i) $\dim \mathcal{N}(A) < \infty$*;*

(ii) $\mathcal{R}(A)$ *closed;*

(iii) $\mathcal{N}(A) = \{0\} \implies \mathcal{R}(A) = X$*;*

(iv) $\mathcal{R}(A) = X \implies \mathcal{N}(A) = \{0\}$*;*

(v) $\operatorname{codim} \mathcal{R}(A) \le \dim \mathcal{N}(A)$*:*

(vi) $\dim \mathcal{N}(A) \le \operatorname{codim} \mathcal{R}(A)$*.*

Proof. Many arguments of the proof are modelled on the argument that the closed unit ball in a Banach space is only compact if the space is finite dimensional (Theorem 3.13). It might be helpful to reread this short argument as a preparation.

We use the following facts for a closed subspace $Z \subset X$ with $Z \ne X$.

$$\forall x \in X \;\; \exists z \in Z \quad \|x - z\| \le 2\operatorname{dist}(x, Z), \tag{11.15}$$

$$\exists y \in X \quad \|y\| = 1, \quad \operatorname{dist}(y, Z) \ge \frac{1}{2}. \tag{11.16}$$

Indeed the first assertion clearly holds if $\operatorname{dist}(x, Z) = 0$ because closedness of Z then implies $x \in Z$ and we can take $z = x$. If $\operatorname{dist}(x, Z) > 0$ then the definition of the distance implies that for each $\varepsilon > 0$ there exists $z \in Z$ such that $\|x - z\| < \operatorname{dist}(x, Z) + \varepsilon$. Taking $\varepsilon = \operatorname{dist}(x, Z)$ we get (11.15). The second assertion is contained in Lemma 3.8. It follows from the first be taking $x \in X \setminus Z$ and $y = (x - z)/\|x - z\|$.

(i): Note that

$$Ax = 0 \iff x = Tx. \tag{11.17}$$

Let $\overline{B_1(0)} \cap \mathcal{N}(A)$ be the closed unit ball in $\mathcal{N}(A)$. Now $B_1(0) \cap \mathcal{N}(A) = T(B_1(0))$ therefore

$$\overline{B_1(0)} \cap \mathcal{N}(A) \subset \overline{T(B_1(0))} \tag{11.18}$$

Hence $\overline{B_1(0)} \cap \mathcal{N}(A)$ is compact which implies $\dim \mathcal{N}(A) < \infty$.

(ii): Suppose that $Ax_n \to y$. Since $\mathcal{N}(A)$ is closed there exists an $a_n \in \mathcal{N}(T)$ such that

$$\|x_n - a_n\| \leq 2\text{dist}\,(x_n, \mathcal{N}(A)). \tag{11.19}$$

We may assume that $a_n = 0$. Otherwise we consider the sequence $\tilde{x}_n := x_n - a_n$.

Case 1: $\sup_n \|x_n\| < \infty$.

We have

$$x_n = Ax_n + Tx_n. \tag{11.20}$$

Since T is compact there exists a subsequence such that $Tx_{n_j} \to y_*$ and hence $x_{n_j} \to y + y_*$. Therefore

$$y \leftarrow Ax_{n_j} \to A(y + y_*). \tag{11.21}$$

Thus $y = A(y + y_*)$ and $y \in \mathcal{R}(A)$. Hence $\mathcal{R}(A)$ is closed.

Case 2: $\sup_n \|x_n\| = \infty$.

Then there exists a subsequence (which for simplicity we still denote by x_n) such that $\|x_n\| \to \infty$. Set

$$y_n := \frac{x_n}{\|x_n\|}, \quad \text{so that } \|y_n\| = 1, \quad Ay_n = \frac{Ax_n}{\|x_n\|} \to 0. \tag{11.22}$$

Now the compactness of T implies that for a subsequence

$$y_{n_j} = Ay_{n_j} + Ty_{n_j} \to y_*. \tag{11.23}$$

The continuity of A yields

$$Ay_* = \lim_{j\to\infty} Ay_{n_j} = 0. \tag{11.24}$$

Hence $y_* \in \mathcal{N}(A)$ and thus

$$\|y_{n_j} - y_*\| \geq \text{dist}\,(y_{n_j}, \mathcal{N}(A)) = \frac{\text{dist}\,(x_{n_j}, \mathcal{N}(A))}{\|x_{n_j}\|} \geq \frac{1}{2} \tag{11.25}$$

where we used (11.19) with $a_n = 0$ in the last step. This contradicts the convergence $y_{n_j} \to y_*$.

(iii): We always have $\mathcal{R}(A^{n+1}) = A^n(A(X)) \subset \mathcal{R}(A^n)$. We claim that if $\mathcal{N}(A) = \{0\}$ and $\mathcal{R}(A) \neq X$ then

$$\mathcal{R}(A^n) \neq \mathcal{R}(A^{n+1}) \quad \forall n \in \mathbb{N}.$$

Indeed let $x \in X \setminus \mathcal{R}(A)$. We claim that then

$$A^n x \in \mathcal{R}(A^n) \setminus \mathcal{R}(A^{n+1}) \quad \forall n \geq 0 \tag{11.26}$$

If there existed $y \in X$ such that $A^n x = A^{n+1} y$ then $A^n(x - Ay) = 0$. Since $\mathcal{N}(A) = \{0\}$ this implies that $A^{n-1}(x - Ay) = 0$ and by induction we get $x - Ay$. This contradicts the assumption $x \notin \mathcal{R}(A)$.

Moreover $\mathcal{R}(A^{n+1})$ is closed. Indeed

$$A^{n+1} = (I - T)^{n+1} = I + \underbrace{\sum_{k=1}^{n+1} \binom{n+1}{k} (-T)^k}_{\text{compact operator}} \tag{11.27}$$

and thus $\mathcal{R}(A^{n+1})$ is closed by (ii).

Thus there exists $x_n \in \mathcal{R}(A^n) \setminus \mathcal{R}(A^{n+1})$ such that

$$\|x_n\| = 1, \quad \operatorname{dist}(x_n, \mathcal{R}(A^{n+1}) \geq \frac{1}{2}.$$

For $m > n$ we have $Ax_n + (x_m - Ax_m) \in \mathcal{R}(A^{n+1})$. Thus

$$\|Tx_n - Tx_m\| = \|x_n - (Ax_n + x_m - Ax_m)\| \geq \operatorname{dist}(x_n, \mathcal{R}(A^{n+1})) \geq \frac{1}{2} \quad \forall m > n.$$

Therefore the sequence $n \mapsto Tx_n$ cannot contain a convergent subsequence. This contradicts the compactness of T.

(iv): We follow a similar approach as in the proof of (iii). Let $x_1 \in \mathcal{N}(A)$ with $x_1 \neq 0$. Since $\mathcal{R}(A) = X$ we can define inductively x_k such $Ax_k = x_{k-1}$. Then $x_k \in \mathcal{N}(A^k) \setminus \mathcal{N}(A^{k-1})$. Since A^k is continuous the subspaces $\mathcal{N}(A^k)$ are closed. By Lemma 3.8 there exist $z_k \in \mathcal{N}(A^k) \setminus \mathcal{N}(A^{k-1})$ such that

$$\|z_k\| = 1 \quad \text{and} \operatorname{dist}(z_k, \mathcal{N}(A^{k-1})) \geq \frac{1}{2}. \tag{11.28}$$

Thus for $l > k$ we have $Ax_l + x_k - Ax_k \in \mathcal{N}(A^{l-1})$ and hence

$$\|Tx_l - Tx_k\| = \|x_l - (Ax_l + x_k - Ax_k)\| \geq \frac{1}{2}. \tag{11.29}$$

Hence the sequence $k \mapsto Tx_k$ cannot contain a convergent subsequence. This contradicts the compactness of T.

(v): We reduce this to (iii). By (i) we have $n := \dim \mathcal{N}(A) < \infty$. Let $x_1, \ldots, x_n$ be a basis of $N(A)$. Assume the assertion was false. Then there exist linear independent $y_1, \ldots, y_n \in X$ such that $\mathcal{R}(A) \oplus \operatorname{span}\{y_1, \ldots, y_n\} \neq X$.

Every $x \in N(A)$ has a unique decomposition $x = \sum_{i=1}^n \alpha_i x_i$ and the maps $x \mapsto \alpha_i$ are linear and hence bounded since $N(A)$ is finite dimensional. It follows from the Hahn-Banach theorem that these map can be extended to a maps $x_1', \ldots, x_n' \in X'$. We have

$$\langle x_j, x_k' \rangle = \delta_{jk} \quad \forall j, k = 1, \ldots, n. \tag{11.30}$$

Set

$$\tilde{T}x := Tx - \sum_{j=1}^{n} \langle x, x_j \rangle y_j. \tag{11.31}$$

The operator $\tilde{T} - T$ is compact because its range is contained in the finite dimensional space span $(y_1, \dots, y_n)$. Hence $\tilde{T}$ is compact. Set

$$\tilde{A} := \mathrm{Id} - \tilde{T} \quad \text{so that } \tilde{A}x = Ax + \sum_{j=1}^{n} \langle x, x_j \rangle y_j. \tag{11.32}$$

We claim that $\mathcal{N}(\tilde{A}) = \{0\}$. Indeed if

$$0 = \tilde{A}x = \underbrace{Ax}_{\in \mathcal{R}(A)} + \underbrace{\sum_{j=1}^{n} \langle x, x_j \rangle y_j}_{\in \mathrm{span}\, (y_1, \dots y_n\}} . \tag{11.33}$$

then we get $Ax = 0$ and $\sum_{j=1}^{n} \langle x, x_j \rangle y_j = 0$ since we assumed that the sum $\mathcal{R}(A) \oplus \mathrm{span}\, \{y_1, \dots, y_n\}$ is direct. Thus $x \in \mathcal{N}(A)$ and since the y_j are linearly independent we get $\langle x, x'_j \rangle = 0$. Now $x \in \mathcal{N}(A)$ implies that $x = \sum_{i=1}^{n} \alpha_i x_i$. The condition $\langle x, x'_j \rangle = 0$ and (11.30) then imply that $x = 0$.

Thus assertion (iii) yields that $\mathcal{R}(\tilde{A}) = X$. On the other hand the definition of $\tilde{A}$ implies that $\mathcal{R}(\tilde{A}) \subset \mathcal{R}(A) \oplus \mathrm{span}\, (y_1, \dots, y_n) \neq X$. This contradiction finishes the proof of (iv).

(vi): We reduce this to (iv). By (v) we have $m := \mathrm{codim}\, \mathcal{R}(A) \leq n := \dim \mathcal{N}(A)$. Let $x_1, \dots, x_n$ and $x'_1, \dots, x'_n$ be as in the proof of (iv) and let $y_1, \dots, y_m$ be such that

$$\mathcal{R}(A) \oplus \mathrm{span}\, (y_1, \dots, y_m) = X. \tag{11.34}$$

Consider the map

$$x \mapsto \tilde{T}x := Tx - \sum_{i=1}^{m} \langle x, x'_i \rangle y_i. \tag{11.35}$$

Then $\tilde{T}$ is compact. We claim that $\tilde{A} = \mathrm{Id} - \tilde{T}$ is surjective. Indeed every $y \in Y$ can be written as $y = z + \sum_{i=1}^{m} \beta_i y_i$ with $z \in \mathcal{R}(A)$, i.e. $z = Ax$. Set $x_0 = \sum_{i=1}^{m} \alpha_i x_i$. Then $x_0 \in \mathcal{N}(A)$ and thus $A(x + x_0) = z$ and

$$\tilde{A}(x + x_0) = z + \sum_{i=1}^{m} (\langle x, x'_i \rangle + \alpha_i) y_i. \tag{11.36}$$

The choice $\alpha_i = \beta_i - \langle x, x'_i \rangle$ shows that $\tilde{A}(x + x_0) = y$. Thus $\mathcal{R}(\tilde{A}) = X$ by (iv) we get $\mathcal{N}(\tilde{A}) = \{0\}$. If $n > m$ then $\langle x_n, x'_i \rangle = 0$ for all $i \leq m$ and hence $\tilde{A}x_n = Ax_n = 0$. Thus $n = m$. □

Theorem 11.7 (Fredholm alternative)**.** *Let X be a Banach space, let $T \in \mathcal{L}(X)$ be compact and let $A = \mathrm{Id} - T$. Then either (i) or (ii) holds.*

(i) For each $y \in X$ the equation $Ax = y$ has a unique solution. Moreover $A^{-1} \in \mathcal{L}(X)$.

(ii) The equation $Ax = 0$ has a nontrivial solution and $\mathcal{R}(A) \neq X$.

Proof. By Theorem 11.6 the operator A is a Fredholm operator of index 0.
Case 1: $\mathcal{N}(A) = \{0\}$.
Then $R(A) = Y$ and thus the equation $Ax = y$ has a unique solution for every $y \in X$, i.e., $A : X \to X$ is bijective. By the inverse operator theorem $A^{-1} \in \mathcal{L}(X)$. Thus alternative (i) holds.
Case 2: $\mathcal{N}(A) \neq \{0\}$.
Then $Ax = 0$ has a nontrivial solution. By Theorem 11.6 we have $\mathrm{codim}\,(\mathcal{R}(A)) \geq 1$ and thus $\mathcal{R}(A) \neq X$. Thus alternative (ii) holds. □

[26.1. 2018, Lecture 26]

[31.1. 2018, Lecture 27]

11.3 Further examples and properties of Fredholm operators

The subsection was only discussed very briefly.

Proposition 11.8. *Let X be a Banach space.*

(i) If $Y \subset X$ is finite dimensional then there exists a closed subspace $Z \subset X$ such that $X = Y \oplus Z$. Moreover the injection $J : Z \to X$ is a Fredholm operator.

(ii) If $X = Y \oplus Z$, Z is a closed subspace and Y is a finite dimensional subspace then there exist $R \in \mathcal{L}(X, Z)$ with $R_{|Z} = \mathrm{Id}_Z$, $\mathcal{R}(R) = Z$, $\mathcal{N}(R) = Y$. In particular R is a Fredholm operator.

(iii) Let X, Y, Z and R be as in (ii) and let $J : Z \to X$ be the injection. Then $P := JR$ is a projection (i.e. $P^2 = P$) $\mathcal{N}(P) = Y$, $\mathcal{R}(P) = Z$. Hence P is Fredholm. Moreover $\mathcal{R}(\mathrm{Id} - P) = Y$ and in particular $\mathrm{Id} - P$ is compact.

Proof. (i): Let $y_1, \ldots y_n$ be a basis of Y and let $y_1', \ldots y_n' \in X'$ be the dual basis, i.e., $\langle y_k, y_l' \rangle = \delta_k l$. Set

$$Qx = \sum_{i=1}^{n} \langle x, y_i' \rangle y_i \tag{11.37}$$

Then $Q \in \mathcal{L}(X)$, $\mathcal{R}(Q) = Y$ and $Q^2 = Q$. Set $Z = \mathcal{N}(Q)$. Then Z is closed. Moreover $x - Qx \in Z$ for all $x \in X$, since $Q^2 = Q$. Since every $x \in X$ can be written as $x = Qx + (x - Qx)$ we get $X = Y + Z$. If $x \in Y \cap Z$ then $x = Qx'$ for some $x' \in X$ and $0 = Qx = Q^2x' = Qx' = x$ and therefore $X = Y \oplus Z$. Finally $\mathcal{N}(J) = \{0\}$ and $\mathcal{R}(J) = Z$ is closed and has finite codimension. Hence J is a Fredholm operator.

(ii): The space $Y \times Z$ is a Banach space with norm $\|(y, z)\| = \|y\| + \|z\|$ and the projection $\pi_2 : Y \times Z \to Z$ given by $\pi(y, z) = z$ is continuous. The map $L : Y \times Z \to X$ given by $L(y, z) = y + z$ is continuous and bijective and hence has a continuous inverse. Set $R = \pi_2 L^{-1}$. Then $R \in \mathcal{L}(X, Z)$ and $R_{|Z} = \mathrm{Id}_Z$. Moreover $\mathcal{N}(R) = Y$. Hence R is a Fredholm operator.

(iii) From the properties of R and J we see that $Pz = z$ for $z \in Z$ and $Py = 0$ for $y \in Y$. Hence $P(y + z) = z$. It follows that $\mathcal{R}(P) = Z$, $\mathcal{N}(P) = Y$. In particular P is a Fredholm operator. Moreover $P^2 = P$. Finally it follows that $(\mathrm{Id} - P)z = 0$ and $(\mathrm{Id} - P)y = y$ and hence $\mathcal{R}(\mathrm{Id} - P) = Y$. □

Theorem 11.9. *Let X,Y and Z be Banach spaces. Then the following assertions hold.*

(i) Let $B \in \mathcal{L}(X, Y)$ and $A \in \mathcal{L}(Y, Z)$. If two of the three operators A, B, AB are Fredholm operators, then the third operator is a Fredholm operator and

$$\mathrm{ind}\, AB = \mathrm{ind}\, A + \mathrm{ind}\, B. \tag{11.38}$$

(ii) The set $\mathcal{F}(X, Y)$ of all Fredholm operators from X to Y is an open subset of $\mathcal{L}(X, Y)$ and the index is locally constant in $\mathcal{F}(X, Y)$.

Remark. Note that while the index is locally constant it is in general not true that $\dim \mathcal{N}(A)$ or $\mathrm{codim}\, \mathcal{R}(A)$ are locally constant (not even for $X = Y = \mathbb{R}^n$; look at a neighbourhood of 0).

Proof. The proof of (i) is essentially a nice exercise in linear algebra (the details of the proof were not discussed in the lecture).

Step 1: Linear algebra.

We first discuss only the aspects of linear algebra, i.e., we consider all the spaces only as vector spaces and all the maps only as linear maps, ignoring questions of closedness or continuity. This argument can be expressed very concisely in the language of exact sequences (see below). For the convenience of the reader we first give a proof that does not use that language.

Preliminaries from linear algebra: If V is a vector space and V_1 is a subspace then there exists a subspace V_2 such that $V = V_1 \oplus V_2$ (this follows from Zorn's lemma[11]). Moreover $V_2 \simeq V/V_1$ in the sense that there exists

[11] One approach is to argue as in the proof of the Hahn-Banach theorem and to construct a linear projection $P : V \to V$ with $\mathcal{R}(P) = V_1$ and $Pv = v$ for $v \in V_1$ (see 7.13 in Alt's book). Then one can take $V_2 = \mathcal{N}(P)$. Alternatively one can start from a Hamel basis B_1 of V_1 and show that there exists a Hamel basis B of V which contains B_1. Then one sets $V_2 = \mathrm{span}\,(B \setminus B_1)$.

a bijective linear map from V_2 to the quotient space V/V_1. Indeed the restriction of the canonical projection projection $\pi : V \to V/V_1$ given by $\pi(x) = x + V_1$ to V_2 is bijective. In particular if $V = V_1 \oplus \tilde{V}_2$ then $V_2 \simeq \tilde{V}_2$. Thus we can define $\operatorname{codim} V_1 := \dim V_2 = \dim V/V_1$ (where we allow the value ∞).

We main assertion is that under the assumption of (i) we have

$$\begin{aligned}\dim \mathcal{N}(AB) &= \dim \mathcal{N}(B) + \dim(\mathcal{N}(A) \cap \mathcal{R}(B)) \qquad (11.39)\\ \operatorname{codim} \mathcal{R}(AB) &= \operatorname{codim} \mathcal{R}(A) + \operatorname{codim} (\mathcal{N}(A) + \mathcal{R}(B))\\ &= \operatorname{codim} \mathcal{R}(A) + \operatorname{codim} \mathcal{R}(B) - \dim \mathcal{N}(A) + \dim(\mathcal{N}(A) \cap \mathcal{R}(B)). \qquad (11.40)\end{aligned}$$

and that all the numbers which appear in these formulae are finite.

To prove (11.39) we use there exists a subspace X_0 such that $X = \mathcal{N}(B) \oplus X_0$. Moreover $B_{|X_0}$ is injective and we have

$$\mathcal{R}(B_{|X_0}) = \mathcal{R}(B) \quad \text{and} \quad B^{-1}(E) = \mathcal{N}(B) \oplus B^{-1}_{|X_0}(E)$$

for any subspace $E \subset Y$. Thus

$$\mathcal{N}(AB) = B^{-1}(\mathcal{N}(A) \cap \mathcal{R}(B)) = \mathcal{N}(B) \oplus B^{-1}_{|X_0}(\mathcal{N}(A) \cap \mathcal{R}(B)). \qquad (11.41)$$

This implies (11.39) since B_{X_0} is a bijective map from X_0 to $\mathcal{R}(B)$. Note also that all the spaces which appear in the formula are finite dimensional. Indeed if A and B are Fredholm operators then both space on the right hand side are finite dimensional. If AB is a Fredholm operator then $\mathcal{N}(AB)$ is finite dimensional and hence both space on the right are finite dimensional.

To prove (11.40) we use the decompositions

$$\begin{aligned}Y &= \mathcal{R}(B) \oplus Y_0 \qquad (11.42)\\ Y &= (\mathcal{R}(B) + N(A)) \oplus Y_1\\ \mathcal{N}(A) &= (\mathcal{R}(B) \cap \mathcal{N}(A)) \oplus Y_2 \qquad (11.43)\\ \mathcal{R}(B) &= Y_3 \oplus (\mathcal{R}(B) \cap \mathcal{N}(A)). \qquad (11.44)\end{aligned}$$

Then $\mathcal{R}(B) + \mathcal{N}(A) = \mathcal{R}(B) \oplus Y_2$ and

$$Y = \mathcal{R}(B) \oplus Y_2 \oplus Y_1 = Y_3 \oplus \underbrace{(\mathcal{R}(B) \cap \mathcal{N}(A)) \oplus Y_2}_{=\mathcal{N}(A)} \oplus Y_1. \qquad (11.45)$$

Thus

$$Y_0 \simeq Y_1 \oplus Y_2. \qquad (11.46)$$

It follows from (11.45) that $A_{|Y_3 \oplus Y_1}$ is injective, that A is a bijective map from $Y_3 \oplus Y_1$ to $\mathcal{R}(A)$ and that

$$\mathcal{R}(A) = AY_3 \oplus AY_1 = A(Y_3 \oplus \mathcal{N}(A)) \oplus AY_1 = \mathcal{R}(AB) \oplus AY_1.$$

Thus

$$Z = \mathcal{R}(A) \oplus Z_0 = \mathcal{R}(AB) \oplus AY_1 \oplus Z_0 \quad \text{and} \quad AY_1 \simeq Y_1. \tag{11.47}$$

We now claim that if three of the operators A, B, AB are Fredholm then the spaces Y_1, Y_2, Z_0 and $\mathcal{N}(A)$ are finite dimensional. Since we already know that $\mathcal{R}(B) \cap \mathcal{N}(A)$ is finite dimensional this yields

$$\begin{aligned}
\operatorname{codim} \mathcal{R}(AB) &\underset{(11.47)}{=} \operatorname{codim} \mathcal{R}(A) + \dim Y_1 \\
&\underset{(11.46),(11.42)}{=} \operatorname{codim} \mathcal{R}(A) + \operatorname{codim} \mathcal{R}(B) - \dim Y_2 \\
&\underset{(11.43)}{=} \operatorname{codim} \mathcal{R}(A) + \operatorname{codim} \mathcal{R}(B) - \dim \mathcal{N}(A) + \dim(\mathcal{R}(B) \cap \mathcal{N}(A)).
\end{aligned} \tag{11.48}$$

This gives (11.40). To see that the spaces Y_1, Y_2, Z and $\mathcal{N}(A)$ are indeed finite dimensional assume first that A and B are Fredholm. Then Y_0 is finite dimensional and hence Y_1 and Y_2 are finite dimensional. Moreover $\dim Z_0 = \operatorname{codim} \mathcal{R}(A) < \infty$ and $\dim \mathcal{N}(A) < \infty$.

Now assume that AB and B are Fredholm. Then Y_0 and hence Y_1 and Y_2 are finite dimensional. Moreover $\dim Z = \operatorname{codim} \mathcal{R}(A) \leq \operatorname{codim} \mathcal{R}(AB) < \infty$. Finally $\dim \mathcal{N}(A) = \dim Y_2 + \dim(\mathcal{R}(B) \cap \mathcal{N}(A)) < \infty$.

Finally consider the case that AB and A are Fredholm. Then $AY_1 \oplus Z_0$ is finite dimensonal and hence Z_0 and Y_1 are finite dimensional. Moreover $\mathcal{N}(A)$ is finite dimensional which implies that Y_2 is finite dimensional. Thus $Y_0 \simeq Y_1 \oplus Y_2$ is finite dimensional.

The calculations so far can be summarized in the statement that the following two sequences are exact.

$$0 \longrightarrow \mathcal{N}(B) \longrightarrow \mathcal{N}(AB) \xrightarrow{B} \mathcal{N}(A) \cap \mathcal{R}(B) \longrightarrow 0, \tag{11.49}$$

$$0 \longrightarrow \frac{\mathcal{R}(B) + \mathcal{N}(A)}{\mathcal{R}(B)} \longrightarrow \frac{Y}{\mathcal{R}(B)} \xrightarrow{A} \frac{Z}{\mathcal{R}(AB)} \longrightarrow \frac{Z}{\mathcal{R}(A)} \longrightarrow 0. \tag{11.50}$$

Moreover one can easily check that if two of the three operators A, B and AB are Fredholm then in each sequence at most one space is not finite dimensional. Exactness then implies that all the spaces are finite dimensional.

Step 2: Closedness of the range.
Assume that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are Fredholm. Then $\mathcal{R}(AB) = AY_3$. Moreover A is a bijective map from $Y_1 \oplus Y_3$ to $\mathcal{R}(A)$ and $\mathcal{R}(A)$ is closed by assumption. Proposition 11.8 (i) implies that Y_3 can be chosen as a closed subspace since $\mathcal{N}(A) \cap \mathcal{R}(B) \subset \mathcal{N}(A)$ is finite dimensional. Since $\dim Y_1 \leq \dim Y_0 < \infty$ the space $Y_1 \oplus Y_3$ is also closed and hence a Banach space. Thus A is an

invertible operator from $Y_1 \oplus Y_3$ to $\mathcal{R}(A)$ and hence the image of the closed subspace Y_3 is closed.
Alternative proof: It suffices to show that $\mathcal{R}(AB)$ is closed in $\mathcal{R}(A)$. The map A is bijective as a map from the quotient space[12] $Y/\mathcal{N}(A) \to \mathcal{R}(A)$ and hence invertible. Now $\mathcal{R}(B)+\mathcal{N}(A)$ is closed since $\mathcal{R}(B)$ is closed and $\mathcal{N}(A)$ is finite dimensional. Hence $(\mathcal{R}(B)+\mathcal{N}(A))/\mathcal{N}(A)$ is closed in $Y/\mathcal{N}(A)$ and thus $\mathcal{R}(AB) = A(\mathcal{R}(B)+\mathcal{N}(A))/\mathcal{N}(A))$ is closed in $\mathcal{R}(A)$ as A is invertible.

Now assume that AB and B are Fredholm operators. Then $\dim Y_1 \le \dim Y_0 < \infty$ and $\mathcal{R}(A) = \mathcal{R}(AB) \oplus AY_1$. Since $\mathcal{R}(AB)$ is closed and AY_1 is finite dimensional it follows that $\mathcal{R}(A)$ is closed.

Finally assume that AB and A are Fredholm operators. By Proposition 11.8 there exists a closed space $X_1 \subset X$ such that $X = \mathcal{N}(AB) \oplus X_1$. Let $J : X_1 \to X$ be the injection and let $R : Z \to \mathcal{R}(AB)$ the restriction as in Proposition 11.8. We first show that $\mathcal{R}(BJ)$ is closed. Then $C := RABJ : X_1 \to \mathcal{R}(AB)$ is bijective and hence invertible. This implies that $\mathcal{R}(BJ)$ is closed. Indeed let $y_n = BJx_n$ and assume that $y_n \to y_*$. Then $RAy_n \to RAy_*$ and thus $x_n = C^{-1}RAy_n \to C^{-1}RAy_* =: x_*$. Thus $BJx_n \to BJx_*$ and hence $y_* \in \mathcal{R}(BJ)$. To show that $\mathcal{R}(AB)$ we used that $\mathcal{N}(B) \supset \mathcal{N}(AB)$ is finite dimensional. Hence there exists a finite dimensional space X_2 such that $\mathcal{N}(AB) = \mathcal{N}(B) \oplus X_2$. Hence $X = \mathcal{N}(B) \oplus X_2 \oplus X_1$ and thus $\mathcal{R}(B) = BX_2 \oplus BX_1 = BX_2 \oplus \mathcal{R}(BJ)$. Since BX_2 is finite dimensional it follows that $\mathcal{R}(AB)$ is closed.

(ii): The main point is to reduce the problem from Fredholm operators to invertible operators. Let $A \in \mathcal{L}(X,Y)$ be a Fredholm operator. By Proposition 11.8 there exists a closed subspace X_0 such that $X = \mathcal{N}(A) \oplus X_0$. Let the injection $J : X_0 \to X$ and the restriction $R : Y \to \mathcal{R}(A)$ be as in Proposition 11.8. Then J and P are Fredholm operators. Define

$$A_0 : X_0 \to \mathcal{R}(A) \quad \text{by } A_0 := PAJ. \tag{11.51}$$

Then A_0 is bijective and continuous and hence invertible. In particular A_0 is a Fredholm operator with index 0. Now assume that $B \in \mathcal{L}(X,Y)$ and $\|B - A\| < \varepsilon$. Set

$$B_0 := PBJ. \tag{11.52}$$

Then $\|B_0 - A_0\| < \|P\|\,\|J\|\,\varepsilon$. Since the set of invertible operators is open (see Corollary 4.9) the operator B_0 is invertible if $\varepsilon > 0$ is sufficiently small.

[12]If X is a normed space and Y is a subspace then the quotient space X/Y consists of equivalence classes $[x] := x + Y$, for $x \in X$. If Y is closed then $\|[x]\| := \inf_{y\in Y} \|x + y\|$ defines a norm on X/Y and if X is a Banach space so is X/Y (see homework sheet 10, problem 3). The canonical projection $\pi : X \to X/Y$ given by $\pi(x) = x + Y$ a Lipschitz continuous map. We claim that $\pi(E)$ is closed in X/Y if and only if $E + Y$ is closed in X. Indeed if $\pi(E)$ is closed then $E + Y = \pi^{-1}(\pi(E))$ is closed since π is continuous. Conversely assume that $E+Y$ is closed and that $\pi(x_n) \to z_*$ for $x_n \in E$. Let $x_* \in \pi^{-1}(z_*)$. Then there exist $y_n \in Y$ such that $x_n + y_n$ converges to x_*. Since $x_n + y_n \in E + Y$ and since this set is closed we have $x_* \in E + Y$. Thus $z_* \in \pi(E + Y)$.

In particular B_0 is a Fredholm operator of index 0. Since P and J are Fredholm operators it follows from (i) that B is a Fredholm operator and $\operatorname{ind} B = \operatorname{ind} A$ since $\operatorname{ind} B_0 = \operatorname{ind} A_0 = 0$. $\square$

Here is another interesting result which gives a precise meaning to the intuition that Fredholm operators are almost invertible or 'invertible modulo compact operators'.

Theorem 11.10 (Atkinson's theorem)**.** *Let X and Y be Banach spaces and let $A \in \mathcal{L}(X, Y)$. Then A is a Fredholm operator if and only if there exists $B \in \mathcal{L}(Y, X)$ such that*

$$AB - \mathrm{Id} \quad \textit{and} \quad BA - \mathrm{Id} \quad \textit{are compact operators.} \tag{11.53}$$

The operator B is often called a parametrix of A; it is determined up to the addition of a compact operator.

11.4 The spectral theorem for compact self-adjoint operators

Definition 11.11. *Let X be a Hilbert space and let $T \in \mathcal{L}(X)$. Then the adjoint operator T^* is defined by*

$$(T^*x, y) = (x, Ty) \quad \forall x, y \in X. \tag{11.54}$$

The operator T is called self-adjoint if $T^* = T$, i.e., if

$$(Tx, y) = (x, Ty) \quad \forall x, y \in X. \tag{11.55}$$

If $A : \mathbb{C}^n \to \mathbb{C}^n$ is selfadjoint then there exist and orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$ such that A is diagonal, i.e., $Ax = \sum_{k=1}^n \lambda_k (x, e_k) e_k$. We now prove a counterpart of this for compact operators on a complex Hilbert space.

Notation. If X is a Hilbert space and Y and Z are subspace we write

$$Y \perp Z \qquad \Longleftrightarrow \qquad (y, z) = 0 \quad \forall y \in Y,\, z \in Z, \tag{11.56}$$

i.e., if the spaces Y and Z are orthogonal. Note that in this case in particular $Y \cap Z = \{0\}$

Theorem 11.12 (Spectral theorem for compact self-adjoint operators)**.** *Let X be a Hilbert space over $\mathbb{C}$ and let $T \in \mathcal{L}(X)$ be self-adjoint and compact, with $T \neq 0$. Then the following assertions hold.*

(i) $\sigma(T) \setminus \{0\}$ consists only of eigenvalues and for each $\lambda \in \sigma(T) \setminus \{0\}$ the eigenspace $\mathcal{N}(T - \lambda)$ is finite dimensional. Moreover $\sigma(T) \setminus \{0\}$ is finite or countable and the only possibly accumulation point is 0.

(ii) We have $\sigma(T) \subset \mathbb{R}$ and there exists an orthonormal system $e : N \to X$ and a sequence $\lambda : N \to \mathbb{R} \setminus \{0\}$ mit $N \subset \mathbb{N}$ such that

$$Te_k = \lambda_k e_k \quad \forall k \in N, \quad \sigma(T) \setminus \{0\} = \{\lambda_k : k \in N\} \tag{11.57}$$

and

$$N(T - \lambda) \subset \operatorname{span}(e_k : k \in N) \quad \forall \lambda \in \sigma(T) \setminus \{0\}. \tag{11.58}$$

If N is infinite then $\lambda_k \to 0$ as $k \to \infty$.

(iii) $\mathcal{N}(T) \perp \overline{\operatorname{span}(e_k : k \in N)}$ and $X = \mathcal{N}(T) \oplus \overline{\operatorname{span}(e_k : k \in N)}$.

(iv) $x = P_{\mathcal{N}(T)}x + \sum_{k \in N}(x, e_k)e_k, \quad Tx = \sum_{k \in N} \lambda_k(x, e_k)e_k \quad \forall x \in X.$

Remark. (*not discussed in class*) (i) By the usual complexification argument one can show that the same assertion holds Y is a Hilbert space over $\mathbb{R}$ and $S \in \mathcal{L}(Y)$ is compact and self-adjoint. Indeed we can extend S to a compact self-adjoint operator T on the complexified space $X = Y + iY$ by $T(a + ib) = S(a) + iS(b)$. If e_k is an eigenfunction of T for the eigenvalue λ_k then the complex conjugate $\overline{e_k}$ is an eigenfunction for the same eigenvalue (since $\lambda_k \in \mathbb{R}$). Thus $\operatorname{Re} e_k$ or $\operatorname{Im} e_k$ is an eigenfunction for S.
(ii) If T is a normal operator, i.e., if $TT^* = T^*T$ then the same conclusions holds with the exception that the spectrum may lie in $\mathbb{C}$ rather than in $\mathbb{R}$ and thus $\lambda : N \to \mathbb{C} \setminus \{0\}$.
(iii) Operator calculus. It follows from assertion (iii) that

$$A^k x = \sum_{k \in N} \lambda^k (x, e_k)e_k. \tag{11.59}$$

Thus for $f : \mathbb{R} \to \mathbb{R}$ we define $f(A)$ by

$$f(A)x := f(0)P_{\mathcal{N}(T)}x + \sum_{k \in N} f(\lambda)(x, e_k, e_k). \tag{11.60}$$

For analytic f this agrees with the definition of $f(A)$ via a power series.
(iv) There is a counterpart of assertion (iii) for merely bounded self-adjoint operators. First note that the assertion for compact T can also be written as

$$x = \sum_{\lambda \in \sigma(T)} Q_\lambda x, \quad Tx = \sum_{\lambda \in \sigma(T)} \lambda Q_\lambda x, \tag{11.61}$$

where Q_λ denotes the orthogonal projection to the eigenspace $\mathcal{N}(T - \lambda)$. Note we have shown that different eigenspaces are orthogonal, i.e., $Q_\lambda Q_\mu = 0$ if $\lambda \neq \mu$. In the case of bounded operators the sum is replaced by a suitable Lebesgue-Stieltjes integral, i.e.,

$$(Tx, y) = \int_{\mathbb{R}} \lambda \, dP_\lambda(x, y) \tag{11.62}$$

where the P_λ are orthogonal projections that are mutually orthogonal in the sense that for $\lambda > \mu$ we have $(P_\lambda - P_{\mu^+})P_\mu = 0$ where $P_{\mu^+}(x,y) = \lim_{t\downarrow 0} P_{\mu+t}$. For compact T the P_λ can be defined by $P_\lambda = \sum_{\mu\in\sigma(T):\mu<\lambda} Q_\mu$. Then the map $\lambda \mapsto (P_\lambda x, y)$ is piecewise constant and has jumps at all values $\mu \in \sigma(T)$ with $\lim_{t\downarrow 0}(P_{\mu+t}x, y) - (P_{\mu-t}x, y) = (Q_\lambda x, y)$.
There is also an extension to unbounded self-adjoint operators $T : D(A) \to X$ where $D(A)$ is a dense subset of X. In this case care has to be taken with the definition of self-adjointness.

Proof. (i): *Step 1: $\sigma(T) \setminus \{0\}$ consists only of eigenvalues and for each $\lambda \in \sigma(T) \setminus \{0\}$ the eigenspace $\mathcal{N}(T - \lambda)$ is finite dimensional.*
Suppose that $\lambda \neq 0$ and that $\lambda\mathrm{Id} - T$ is not invertible. Then $A := \mathrm{Id} - \frac{1}{\lambda}T$ is not invertible. By the Fredholm alternative we must have $\mathcal{N}(A) \neq \{0\}$. Thus λ is an eigenvalue of T and the corresponding eigenspace is finite dimensional by Theorem 11.6.

Step 2: $\sigma(T) \setminus \{0\}$ is countable and the only possible accumulation point of this set is zero.
It suffices to show that the set $S_k := \sigma(T) \setminus B(0, \frac{1}{k})$ is finite for all $k \geq 1$. First note that by definition of $\|T\|$ we have $|\lambda| \leq \|T\|$ for every eigenvalue λ. Thus, if S_k is infinite then there exist a sequence $\lambda_j \to \lambda$ with $\lambda_j \in S_k$ and $\lambda \neq 0$. We may assume that the λ_j are all different. Since the eigenspaces $Z_j := \mathcal{N}(T - \lambda_j)$ are closed and finite dimensional there exist $z_j \in Z_j$ such that $\|z_j\| = 1$ and $\mathrm{dist}\,(z_j, \bigcup_{l=1}^{j-1} Z_l) \geq \frac{1}{2}$. Thus

$$\|z_j - z_k\| \geq \frac{1}{2} \quad \text{if } k < j \tag{11.63}$$

Now

$$z_j = \frac{1}{\lambda_j} T z_j \tag{11.64}$$

and since T is compact and since $\lambda_j \to \lambda \neq 0$ the sequence $j \mapsto z_j$ contains a convergent subsequence. This contradicts (11.63).

(ii): Let $\lambda \in \sigma(T)\setminus\{0\}$ be an eigenvalue and let $x \neq 0$ be an eigenvector.

$$\lambda\|x\|^2 = (\lambda x, x) = (Tx, x) = (x, Tx) = \overline{\lambda}\|x\|^2. \tag{11.65}$$

Hence $\lambda \in \mathbb{R}$.

Now let $\lambda, \mu \in \sigma(T)\setminus\{0\}$ with $\lambda \neq \mu$. Let $x \in \mathcal{N}(T-\lambda)$ and $y \in \mathcal{N}(T-\mu)$ we have

$$\lambda(x, y) = (Tx, y) = (x, Ty) = \mu(x, y). \tag{11.66}$$

Thus $(x, y) = 0$ and the eigenspaces $\mathcal{N}(T - \lambda)$ and $\mathcal{N}(T - \mu)$ are orthogonal. Therefore we can choose a finite orthonormal basis in each eigenspace. Since the number of eigenspaces is at most countable we obtain an at most countable orthonormal system of eigenvectors that contains a basis of each eigenspace $\mathcal{N}(T - \lambda)$ with $\lambda \neq 0$.

(iii): Let $Y := \overline{\operatorname{span}(e_k : k \in N)}$. Then $Y^\perp = \{z \in X : (z, y) = 0\ \forall y \in Y\}$ is a Hilbert space and by the projection theorem we have $X = Y \oplus Y^\perp$. We claim that $Y^\perp = \mathcal{N}(T)$.

Step 1: $\mathcal{N}(T) \subset Y^\perp$
For $x \in \mathcal{N}(T)$ we have

$$\lambda_k (e_k, x) = (Te_k, x) = (e_k, Tx) = 0. \tag{11.67}$$

Since $\lambda_k \neq 0$ it follows that $(e_k, x) = 0$ for all $k \in \mathbb{N}$. By continuity of the scalar product we get $x \in Y^\perp$.

Step 2: $Y^\perp \subset \mathcal{N}(T)$.
We first show that T maps $Y^\perp$ to itself. Indeed for $z \in Y^\perp$ we have for all $k \in N$

$$(Tz, e_k) = (z, Te_k) = \lambda_k (z, e_k) = 0. \tag{11.68}$$

Thus by continuity of the scalar product $Tz \in Y^\perp$.

Let $S = T_{|Y^\perp}$. Then S is compact, self-adjoint operator on $Y^\perp$. If $S = 0$ then we are done. If $S \neq 0$ then by Theorem 11.13 below S has a non-zero eigenvalue. Thus there exist $\mu \neq 0$ and $x \in Y^\perp \setminus \{0\}$ with

$$Tx = Sx = \mu x. \tag{11.69}$$

Thus $\mu \in \sigma(T) \setminus \{0\}$ and by (ii) we must have $x \in \operatorname{span}(e_k : k \in \mathbb{N}) \subset Y$. This contradicts the definition of $Y^\perp$.

(iv) By (iii) we have $y := x - P_{\mathcal{N}(T)} x \in Y = \overline{\operatorname{span}(e_k : k \in N)}$. Since the $e : \mathbb{N} \to X$ is an orthonormal system in X it is actually on orthonormal basis of Y. Thus by Theorem 9.6 we have $y := \sum_{k \in \mathbb{N}} (y, e_k) e_k = \sum_{k \in \mathbb{N}} (x, e_k) e_k$. Finally the formula for Tx follows from the absolute convergence of the series $\sum_{k \in \mathbb{N}} (x, e_k) e_k$ and the boundedness of T. □

Note that for a self-adjoint operator T the expression (Tx, x) is real since $\overline{(x, Tx)} = (x, Tx) = (Tx, x)$. The following result extends the characterization of the maximal and minimal eigenvalue of a symmetric or hermitean matrix.

Theorem 11.13 (Rayleigh quotient)**.** *Let X be a Hilbert space and let $T \in \mathcal{L}(X)$ be a self-adjoint and compact. Set*

$$\lambda_{\min} := \inf_{\|x\|=1} (Tx, x) = \inf_{x \neq 0} \frac{(Tx, x)}{(x, x)}, \quad \lambda_{\max} := \sup_{\|x\|=1} (Tx, x) = \sup_{x \neq 0} \frac{(Tx, x)}{(x, x)}. \tag{11.70}$$

Then the following assertions hold.

(i) If $\lambda_{\max} > 0$ *then there exists* $x_* \in X$ *with* $\|x_*\| = 1$ *and*

$$(Tx_*, x_*) = \lambda_{\max}, \quad Tx_* = \lambda_{\max} x_*. \tag{11.71}$$

(ii) If $\lambda_{\min} < 0$ *then there exists* $x_* \in X$ *with* $\|x_*\| = 1$ *and*

$$(Tx_*, x_*) = \lambda_{\min}, \quad Tx_* = \lambda_{\min} x_*. \tag{11.72}$$

(iii) If $\lambda_{\max} = \lambda_{\min} = 0$ *then* $T = 0$*.*

Remark. If X is infinite dimensional then $\lambda_{\max} \geq 0\lambda_{\min}$. Proof: let $e : \mathbb{N} \to X$ be an infinite orthonormal system (such a system exists by the Schmidt orthonormalization procecure). Since $\sum_{k\in N}(x, e_k)^2 \leq \|x\|^2$ we have $\lim_{k\to\infty}(x, e_k) = 0$ for all x. Thus $e_k \rightharpoonup 0$. Since T is compact this implies $Te_k \to 0$ and thus $(Te_k, e_k) \to 0$.

Proof. (i) Let x_k be a maximizing sequence, i.e., $\|x_k\| = 1$ and $(Tx_k, x_k) \to \lambda_{\max}$. Since every Hilbert space is reflexive there exists a subsequence such that $x_{k_j} \rightharpoonup x_*$. Since compact operators are completely continuous (see Proposition 8.23) we get $Tx_{k_j} \to Tx_*$ (strong convergence !). Hence

$$(Tx_*, x_*) = \lim_{j\to\infty}(Tx_{k_j}, x_{k_j}) = \lambda_{\max}. \tag{11.73}$$

By the weak lower semicontinuity of the norm we have $\|x_*\| \leq 1$. Indeed we must have equality. First $x_* \neq 0$ since $\lambda_{\max} > 0$. Now if $0 < \|x_*\| < 1$ we have $\|x_*/\|x_*\|\,\| = 1$ and

$$(T\frac{x_*}{\|x_*\|}, \frac{x_*}{\|x_*\|}) = \frac{(Tx_*, x_*)}{\|x_*\|^2} > \lambda_{\max} \tag{11.74}$$

and this contradicts the definition of $\lambda_{\max}$. Hence x_* realizes the infimum.

Then the function

$$h(t) := \frac{T(x_* + ty, x_* + ty)}{(x_* + ty, x_* + ty)} \tag{11.75}$$

has a maximum at $t = 0$. Differentiation gives (taking into account that $\|x_*\| = 1$)

$$0 = h'(0) = (Ty, x_*)+(Tx_*, y)-(Tx_*, x_*)((y, x_*)+(x*, y)) = 2\operatorname{Re}(Tx_*-\lambda_{\max}x_*, y). \tag{11.76}$$

Taking $y = Tx_* - \lambda_{\max}x_*$ we obtain the assertion.

(ii): Apply (i) to $-T$.

(iii): In this case $(Tx, x) = 0$ for all $x \in X$. This yields

$$4\operatorname{Re}(Tx, y) = (T(x + y), x + y) - (T(x - y), x - y) = 0. \tag{11.77}$$

Taking $y = Tx$ we get the assertion. □

11.5 An orthonormal system of eigenfunctions for second order elliptic PDE

Theorem 11.14. *Let $U \subset \mathbb{R}^n$ be open and bounded. Then there exists and orthonormal basis $e : \mathbb{N} \to L^2(U)$ which consists of eigenfunctions of $-\Delta$, i.e.,*

$$-\Delta e_k = \lambda_k e_k \quad \text{in } U, \qquad e_k = 0 \quad \text{on } \partial U. \tag{11.78}$$

Moreover the eigenvalues λ_k satisfy $\lambda_k > 0$ and $\lim_{k\to\infty} \lambda_k = \infty$.

Remark. (i) The eigenvalue equation is understood in the sense of weak solutions, i.e., $e_k \in W_0^{1,2}(U)$ and

$$\int_U \nabla\varphi \cdot \nabla e_k \, d\mathcal{L}^n = \int_U \varphi \, \lambda_k e_k \, d\mathcal{L}^n \quad \forall \varphi \in W_0^{1,2}(U). \tag{11.79}$$

One can show that $e_k \in C^\infty(U)$ and that (11.78) actually holds in the classical sense.
(ii) The analogous result (without the regularity statement and with the weaker inequality $\inf_k \lambda_k > -\infty$) holds for the operator L with $Lu = -\sum_{i,j} \partial_i(a_{ij}(x)\partial_j u) + c(x)u$, if $a_{ij} = a_{ji}$ and $a_{ij}, c \in L^\infty(U)$.
(iii) More generally many interesting orthonormal bases of L^2 and other Hilbert space arise as eigenfunctions of suitable differential operators.

Proof. Set $T := (-\Delta)^{-1}$. More precisely we have shown in Theorem 5.7 that for $f \in L^2(U)$ there exists a unique $u \in W^{1,2}(U)$ such that

$$(\nabla\varphi, \nabla u)_{L^2} = (\varphi, f)_{L^2} \quad \forall \varphi \in W_0^{1,2}(U) \tag{11.80}$$

and we set $u = Tf$. Then u is the weak solution of $-\Delta u = f$ with zero boundary conditions.

Step 1: T is a compact operator from $L^2(U)$ to $L^2(U)$.
By Theorem 5.7 T is a bounded operator from $L^2(U) \to W_0^{1,2}(U)$. This implies the assertion since the embedding from $W_0^{1,2}(U)$ to $L^2(U)$ is compact (see the example after Lemma 3.17 or Theorem 10.5).

Step2: T is self-adjoint on $L^2(U)$.
Let $Tf = u$ and $Tg = v$. Then the definition of u yields

$$(Tg, f)_{L^2} = (v, f)_{L^2} = (\nabla v, \nabla u) \tag{11.81}$$

while the definition of v gives

$$(Tf, g)_{L^2} = (u, g)_{L^2} = (\nabla u, \nabla v). \tag{11.82}$$

Thus $(Tg, f) = (Tf, g) = (g, Tf)$ since we work in a real Hilbert space.

Step 3: $\mathcal{N}(T) = \{0\}$.
If $Tf = 0$ then $(\nabla\varphi, 0)_{L^2} = (f, \varphi)_{L^2}$ for all $\varphi \in W_0^{1,2}(U)$. Since $C_c^\infty(U) \subset W_0^{1,2}(U)$ is dense in $L^2(U)$ we get $f = 0$.

Step 4: Conclusion.

It follows from the spectral theorem, Theorem 11.12, that there exist an orthonormal basis $e : \mathbb{N} \to L^2(U)$ and a sequence $\mu : \mathbb{N} \to \mathbb{R} \setminus \{0\}$ such that

$$Te_k = \mu_k e_k \tag{11.83}$$

(note that the set N in Theorem 11.12 cannot be finite since $L^2(U)$ is not finite dimensional. Hence we may take $N = \mathbb{N}$). By the definition of T we have $\mu_k e_k$ in $W_0^{1,2}(U)$ and

$$(\nabla\varphi, \nabla(\mu_k e_k)) = (\varphi, e_k) \quad \forall \varphi \in W_0^{1,2}(U). \tag{11.84}$$

Set $\lambda_k = \frac{1}{\mu_k}$ then (11.84) is equivalent to (11.79).

The choice $\varphi = e_k$ shows that $\mu_k > 0$ and hence $\lambda_k > 0$. The spectral theorem states that the only accumulation point of μ is zero. Hence the sequence λ must converge to ∞. □

The following applications of where only discussed briefly in class.

Theorem 11.14 has many interesting applications. As an example we consider the initial-boundary value problem for the heat equation

$$\partial_t u - \Delta u = 0 \quad \text{in } U \times (0,\infty), \tag{11.85}$$
$$u = 0 \quad \text{on } \partial U \times (0,\infty), \tag{11.86}$$
$$u(x,0) = u_0(x,0) \tag{11.87}$$

for a given function $u_0 \in L^2(U)$. We look for solutions of the form

$$u(x,t) = \sum_{k\in\mathbb{N}} a_k(t) e_k(x) \tag{11.88}$$

and $a_k(0) = (u_0, e_k)$. The heat equation then reduces to a family of decoupled ordinary differential equations

$$a_k'(t) - \lambda_k a_k(t) = 0. \tag{11.89}$$

This yields

$$u(x,t) = \sum_{k\in\mathbb{N}} e^{-t\lambda_k}(u_0, e_k) e_k. \tag{11.90}$$

One can easily check that u is indeed a solution of the heat equation and that $t \mapsto u(\cdot,t)$ is a continuous map from $[0,\infty)$ to $L^2(U)$. Symbolically one can write u as

$$u(\cdot,t) = e^{t\Delta} u_0. \tag{11.91}$$

Similarly the choice

$$u(\cdot,t) = \sin(t(-\Delta)^{\frac{1}{2}})(-\Delta)^{-\frac{1}{2}} u_1 + \cos(t(-\Delta)^{\frac{1}{2}}) u_0 \tag{11.92}$$

provides a solution to the wave equation

$$\partial_t^2 u - \Delta u = 0 \tag{11.93}$$

with initial values

$$u(\cdot,0) = u_0 \in W_0^{1,2}(U), \quad \partial_t u(\cdot,0) = u_1 \in L^2(U). \tag{11.94}$$

More precisely the symbolic notation in (11.92) means

$$u(x,t) = \sum_{k\in\mathbb{N}} \sin(t\lambda_k^{\frac{1}{2}})\lambda_k^{-\frac{1}{2}}(u_1,e_k)e_k + \cos(t\lambda_k^{\frac{1}{2}})(u_0,e_k)e_k. \tag{11.95}$$

The formula can be written even more concisely if one the introduces the complex variable $z = (-\Delta)^{\frac{1}{2}}u + i\partial_t u$. Then

$$z(\cdot,t) = \exp(-it(-\Delta)^{\frac{1}{2}})z_0 \tag{11.96}$$

and one sees immediately that the map $z_0 \mapsto z(\cdot,t)$ is an L^2 isometry.

11.6 The Fredholm altenative for second order elliptic operators

This was not discussed in class.

Lemma 11.15 (Fredholm alternative in Hilbert spaces)**.** *Let X be a Hilbert space, let $T \in \mathcal{L}(X)$ be compact and let $A = \mathrm{Id} - T$. Then*

(i) $\mathcal{R}(A) = \mathcal{N}(A^*)^\perp$,

(ii) $\dim \mathcal{N}(A) = \dim \mathcal{N}(A^*)$.

Proof. (i): We have

$$\begin{aligned} y \in \mathcal{N}(A^*) \quad &\iff \quad \forall x \in X\ (x, A^*y) = 0 \quad \iff \quad \forall x \in X\ (Ax, y) = 0 \\ &\iff \quad y \in \mathcal{R}(A)^\perp. \end{aligned} \tag{11.97}$$

Thus $\mathcal{R}(A)^\perp = \mathcal{N}(A^*)$. Since $\mathcal{R}(A)$ is closed we have $(\mathcal{R}(A)^\perp)^\perp = \mathcal{R}(A)$ which implies the assertion.

(ii) : By Theorem 11.6 the operator A has index zero. Hence

$$\dim \mathcal{N}(A) = \operatorname{codim} \mathcal{R}(A) \underset{(i)}{=} \dim \mathcal{N}(A^*). \tag{11.98}$$

□

We now apply the Fredholm alternative to the existence of weak solutions of the equation $Lu = f$ where

$$Lu := -\sum_{i,j=1}^{n} \partial_i(a_{ij}\partial_j u) - \sum_{i=1}^{n} \partial_i(b_i u) + \sum_{i=1}^{n} c_i \partial_i u + d. \tag{11.99}$$

We assume that

$$a_{ij}, b_i, c_i, d \in L^\infty(U) \tag{11.100}$$

and that the coefficients a_{ij} are elliptic, i.e.,

$$\exists\, c > 0 \quad \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq c|\xi|^2 \quad \text{for a.e. } x. \tag{11.101}$$

Associated to L is the bilinear form

$$B(v,u) = \int_U \sum_{i,j=1}^{n} \partial_i v\, a_{ij}\partial_j u + \sum_{i=1}^{n} \partial_i v\, b_i u + \sum_{i=1}^{n} v c_i \partial_i u + dvu\, d\mathcal{L}^n. \tag{11.102}$$

We recall that u is a weak solution of the boundary value problem

$$Lu = f \quad \text{in } U, \qquad u = 0 \quad \text{on } \partial U \tag{11.103}$$

if $u \in W_0^{1,2}(U)$ and

$$B(v,u) = \int_U vf\, d\mathcal{L}^n \quad \forall v \in W_0^{1,2}(U). \tag{11.104}$$

Let

$$\mathcal{N}(L) := \{u \in W_0^{1,2} : B(v,u) = 0\ \forall v \in W_0^{1,2}(U)\}. \tag{11.105}$$

We define the formal adjoint L^* by

$$L^* u = -\sum_{i,j=1}^{n} (\partial_i a_{ji}\partial_j u) - \sum_{i=1}^{n} \partial_i(c_i u) + \sum_{i=1}^{n} b_i \partial_i u + d. \tag{11.106}$$

The corresponding bilinear form is given by

$$B^*(v,u) = B(u,v). \tag{11.107}$$

Theorem 11.16 (Fredholm alternative for second order elliptic operators)**.** *Assume* (11.100) *and* (11.101)*. Then either (i) or (ii) holds.*

(i) For all $f \in L^2(U)$ there exists a unique weak solution of the boundary value problem (11.103)*.*

(ii) $\mathcal{N}(L) \neq \{0\}$ *and the boundary value problem* (11.103) *has a weak solution if and only if*

$$(v, f)_{L^2} = 0 \quad \forall v \in \mathcal{N}(L^*). \tag{11.108}$$

Moreover

$$\dim \mathcal{N}(L^*) = \dim \mathcal{N}(L) < \infty. \tag{11.109}$$

Proof. Step 1: Reduction to compact operators
Using the Cauchy Schwarz inequality and Young's inequality we get

$$\begin{aligned} B(u,u) &\geq c\|\nabla u\|_{L^2}^2 - C_1\|u\|_{L^2}\,\|\nabla u\|_{L^2} - C_2\|u\|_{L^2}^2 \\ &\geq \frac{c}{2}\|\nabla u\|_{L^2}^2 - C\|u\|_{L^2}^2. \end{aligned} \tag{11.110}$$

For $\gamma \in \mathbb{R}$ set

$$L_\gamma u = Lu + \gamma u, \quad B_\gamma(v,u) = B(v,u) + \gamma(v,u). \tag{11.111}$$

If $\gamma \geq C$ we have by the Poincaré inequality

$$B_\gamma(u,u) \geq \frac{c}{2}\|\nabla u\|_{L^2}^2 \geq \tilde{c}\|u\|_{W_0^{1,2}}^2. \tag{11.112}$$

Thus by the Lax-Milgram theorem for each $f \in L^2$ the boundary value problem for $L_\gamma u$ has a unique weak solution in $W_0^{1,2}(U)$. Denote the solution by $u = L_\gamma^{-1} f$. Then

$$\begin{aligned} Lu = f \quad &\Longleftrightarrow \quad Lu + \gamma u = \gamma u + f \\ \Longleftrightarrow \quad u = \gamma L_\gamma^{-1} u + L_\gamma^{-1} f \quad &\Longleftrightarrow \quad u - Ku = h, \end{aligned} \tag{11.113}$$

where

$$K := \gamma L_\gamma^{-1} \quad \text{and } h := L_\gamma^{-1} f. \tag{11.114}$$

Note that K is bounded as an operator from $L^2(U)$ to $W_0^{1,2}(U)$ and hence compact as an operator from $L^2(U)$ to $L^2(U)$. Note also that (11.113) implies that

$$u - Ku = 0 \quad \Longleftrightarrow \quad u \in \mathcal{N}(L). \tag{11.115}$$

Similarly one can show that (see Step 3 below for the details)

$$u - K^*u = 0 \quad \Longleftrightarrow \quad u \in \mathcal{N}(L^*). \tag{11.116}$$

Step 2: Application of the Fredholm alternative.

First case: $\mathcal{N}(L) = \{0\}$.
Then $\mathcal{N}(\mathrm{Id} - K) = \{0\}$ and thus by the Fredholm alternative, Theorem 11.7, we have $\mathcal{R}(\mathrm{Id} - K) = L^2(U)$. Thus $u - Ku = h$ has a solution for every

$h \in L^2$ and hence the boundary value problem (11.103) has a weak solution for each $f \in L^2(U)$. Moreover the solution is unique since $\mathcal{N}(L) = \{0\}$.

Second case $\mathcal{N}(L) \neq \{0\}$.
Then $\mathcal{N}(\mathrm{Id} - K) \neq \{0\}$. By Lemma 11.15 we have $\mathcal{N}(\mathrm{Id} - K^*) \neq \{0\}$. Moreover the equation $u - Ku = h$ has a solution if and only if

$$(v, h) = 0 \quad \forall v \in \mathcal{N}(\mathrm{Id} - K^*). \tag{11.117}$$

Now for $v \in \mathcal{N}(\mathrm{Id} - K^*)$ we get from (11.114)

$$(v, h) = \frac{1}{\gamma}(v, Kf) = \frac{1}{\gamma}(K^* v, f) = \frac{1}{\gamma}(v, f). \tag{11.118}$$

In view of (11.113) the boundary value problem (11.103) has a weak solution if and only if $(v, f) = 0$ for all $v \in \mathcal{N}(\mathrm{Id} - K^*)$. By (11.116) we have $\mathcal{N}(\mathrm{Id} - K^*) = \mathcal{N}(L^*)$.

Finally the assertion $\dim \mathcal{N}(L) = \dim \mathcal{N}(L^*)$ follows from (11.115), (11.116) and Lemma 11.15 (ii).

Step 3: Computation of K^*.
(*This argument was not discussed in class*). For completeness we give a detailed proof of (11.116) which is similar to the proof the $(-\Delta)^{-1}$ is self-adjoint. If suffices to show that $K^* = \gamma(L^*_\gamma)^{-1}$. Then the assertion follows from (11.113) with L replaced by L^*. To compute K^* let $f, g \in L^2(U)$ and set $u = \gamma L_\gamma^{-1} f = Kf$, $v = \gamma(L^*_\gamma)^{-1} g$. Then the definition of L_γ^{-1} gives

$$B_\gamma(v, u) = \gamma(v, f)_{L^2} \tag{11.119}$$

Similarly the definition of $(L^*_\gamma)^{-1}$ gives

$$B^*_\gamma(u, v) = \gamma(u, g)_{L^2} = (Kf, g)_{L^2} \tag{11.120}$$

Since $B^*_\gamma(u, v) = B(v, u)$ we get $(f, v) = (v, f) = (Kf, g)$ and thus $v = K^* g$, as desired. $\square$

12 Overview

12.1 Abstract results

12.1.1 Hilbert spaces

(i) Cauchy-Schwarz inequality

(ii) Strict convexity, parallelogram identity

(iii) Projection on closed convex sets and closed subspaces

(iv) Riesz representation theorem

(v) Lax-Milgram theorem and its consequences

(vi) Orthonormal basis, Parseval's identity

12.1.2 Metric spaces and Banach spaces

(i) Compactness = sequential compactness = precompact and complete; compactness and existence of minimizers

(ii) Projection theorem in uniformly convex spaces

(iii) The Hahn-Banach theorem and its consequences (separation of subspaces and convex sets)

(iv) The Baire category theorem and its consequences

- Uniform boundedness principle/ Banach-Steinhaus theorem
- Open mapping thm./ inverse operator thm./ closed graph thm.

12.1.3 Weak convergence

(i) Definition of the weak and weak* topology and convergence

(ii) X separable $\Longrightarrow \overline{B}(0,1) \subset X'$ weak* sequentially compact

(iii) If X is reflexive then $\overline{B}(0,1) \subset X$ is weakly sequentially compact.

(iv) Mazur's lemma and convex minimization problems

12.1.4 Spectral theory

(i) Definition of the spectrum and its subsets, analyticity of the resolvent

(ii) Fredholm operator, Fredholm alternative, Continuity of the index

(iii) Spectral theorem for compact self-adjoint operators

12.2 Applications

12.2.1 Function spaces

(i) Definition and properties of $C(X;Y)$, C^k, $C^{k,\alpha}$, completeness, separability

(ii) L^p spaces and Sobolev spaces (Definition, completeness, approximation, product rule, chain rule, boundary values)

(iii) Criteria for compactness in C^0 (Arzela-Ascoli) and L^p (Frechet-Kolmogorov-M. Riesz)

(iv) Duality: $(L^p)' = L^{p'}$ for $1 \leq p < \infty$, $C(K)' = \mathrm{rca}(K)$ for K compact

(v) Weak convergence in L^p and $W^{k,p}$

(vi) Examples of reflexive and non reflexive spaces

(vii) Fourier series in L^2

(viii) (Compact) Sobolev embeddings

12.2.2 Partial differential equations

(i) Weakly harmonic functions by the projection theorem in $W_0^{1,2}(U)$

(ii) Lax-Milgram and weak solutions of second order elliptic pde

(iii) Non-solvability results by the closed graph theorem

(iv) Minimizers of convex variational problems (weak convergence and Mazur's lemma), obstacle problem

(v) Existence of an L^2 orthonormal basis of eigenfunctions for second order elliptic operators

References

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[DS] N. Dunford and J.T. Schwartz, Linear operators, Part I, Interscience Publishers, 1966.

[Evans] L.C. Evans, Partial differential equations, Amer. Math. Society, 1998.

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Notes

The course and the lecture notes follow very closely the book [Alt] by H.W. Alt, usually with identical proofs.

The material in Section 1 is standard is taken from various sources.

Section 2.1 is taken from [Alt] 1.1–1.7, for the separability statements see Alt 2.17. Section 2.2 is classical and is based on the lecture notes from Analysis 3. For a nice very short summary of the properties of the Lebesuge integral and L^p spaces, see also [Brezis]. Section 2.3 on Sobolev spaces follows [GT]. In [Alt] Sobolev spaces are defined by completion and it is then shown that this agrees with the definition by weak derivatives. The one dimensional results are discussed in [Alt] U1.6–1.8. The assertion that Sobolev functions have representatives which are absolutely continuous on a.e. line can, e.g., be found in [EG], Section 4.9.2, Theorem 2. Many further interesting results on pointwise properties of Sobolev functions on lower dimensional sets can be found in in that book, too.

Section 3 is taken from Alt 2.1–2.4, except for Example 3.4 and the discussion of uniformly convex spaces. For uniform convexity see, e.g. [Brezis] Section 3.7 and Theorem 4.10. Section 3.2 is taken from Alt 2.5–2.9. The presentation of the Arzela-Ascoli theorem and the Frechet-Kolmogorov-Riesz theorem is taken from Alt 2.11–2.15, with minor modifications. The Arzela-Ascoli theorem is directly proven for functions with values in a Banach space and the discussion of the approximation by convolution is adapted to the material discussed in the section on L^p spaces. The proof of the compact embedding from $W_0^{1,p}(U)$ to $L^p(U)$ is standard.

Section 4 on linear operators is directly taken from Alt 3.1–3.10 and 3.12, while Sections 5.1 and 5.2 are taken from [Alt] 4.1–4.3 and 4.4–4.8, respectively.

The proof of the duality relation $(L^p)' \simeq L^{p'}$ by uniform convexity is standard and the uniform convexity follows from Clarkson's inequalities; see, e.g., [Brezis] Theorem 4.10 and 4.11, or [Adams] Theorem 2.28, Corollary 2.29 and Theorem 2.33. The rest of the material in Section 6 is taken from [Alt] 4.14–4.23. For Remark 6.14 and a much more comprehensive study of the spaces ba and rca as well as related spaces and their duality relations see [DS].

The material in Section 7 on Baire's category theorem and its consequences is taken from [Alt] 5.1–5.9, with the exception of Proposition 7.9. The result is classical, I know no particular reference for the argument given.

The motivation and definition of the weak topology in Section 8 is taken from [Brezis] Chapter 3.1–3.4. The results starting with the weak* sequential compactness of the closed unit ball in X' are taken from [Alt] 6.4, 6.5, 6.8–6.17. The short discussion of completely continuous operators is based on [Alt] 8.1 and 8.2(i).

The discussion of finite dimensional approximation in Section 9 is taken from [Alt] 7.3–7.9. The proof of the Sobolev embeddings in Section 10 follows [GT] Theorem 7.10 and Corollary 7.11 and [Alt] 8.7–8.13. Many further results, also for the borderline case $W^{1,n}$ can be found in [GT], see also [Adams]. More details on extension operators can be found [Alt] A6.12 and, for higher order Sobolev spaces, in [Stein] Chapter VI.2 and VI.3.

Section 11.1 is taken from [Alt] 9.1–9.3. The results in Section 11.2 up to the Fredholm alternative are taken from [Alt] 9.6, 9.8, and 9.11, The results on the continuity of the index and Atkinson's theorem are adapted from [Notes]. The spectral theorem in Section 11.4 is a special case of [Alt] 10.12, where normal operators are covered. The variational characterization of the largest and smallest eigenvalue is taken from [Alt] 10.14. The result on the eigenfunctions of the Laplace operator and its application to evolution equations follow [Alt] 10.16. The Fredholm alternative in Hilbert spaces follows from [Alt] 10.8 if one takes into account that the dual operator and the adjoint operator are related by the Riesz isomorphism: $T^* = R_X^{-1} T' R_X$. The proof of the Fredholm alternative for second order elliptic PDE is taken from [Evans], Section 6.2.3, Theorem 4.

Further reading

Many interesting topics covered in [Alt] and [Brezis] could not be covered in the course: Distributions, functions of bounded variations, definition of boundary values for Sobolev functions by trace operators, L^p estimates for elliptic operators (Calderon-Zygmund theory), $C^{k,\alpha}$ estimates for elliptic operators (Schauder theory), maximal monotone operators and evolution equations, more general spectral theorems, ...

There are of course many further classical and modern books on functional analysis and on its applications to partial differential equations. The following is a deliberately short list, meant as a starting point to inspire further reading beyond the books by H.W. Alt and H. Brezis.

- For partial differential equations: [GT] and [Evans]
- For further properties of Sobolev spaces: [GT], [Adams] and [EG]
- For a more general set-up (with locally convex topological spaces instead of normed spaces), distributions and their Fourier transform, spectral theory on Banach algebras, and various applications of functional analytic methods: [Rudin]