"Combinatorics, Graphs, Matroids", Winter Semester 2026/27

Exercise Sheet 2

To be submitted on eCampus by Tuesday October 27 at noon

Recall that $\mathbb{N} = \{0, 1, 2, \dots\}$ (by convention for this course).

Furthermore, recall that for $n, k \in \mathbb{N}$, we defined $P_{n,k}$ to be the number of partitions of n into k parts, i.e. the number of k-tuples $(\lambda_1, \dots, \lambda_k) \in \mathbb{N}^k$ with $\lambda_1 \geq \cdots \geq \lambda_k \geq 1$ and $\lambda_1 + \cdots + \lambda_k = n$. For $n, k \in \mathbb{N}$, we also defined $S_{n,k}$ to be the number of partitions of a given set of size n into k non-empty parts, i.e. the number of different equivalence relations on a given set of size n with exactly k equivalence classes. These numbers $S_{n,k}$ are called *Stirling numbers of the second kind*.

Problem 1 (5 points)**.** Prove that for $n, k \in \mathbb{N}$ with $n, k \geq 1$, we have

$$P_{n,k} = P_{n-k,k} + P_{n-1,k-1}.$$

Problem 2 (5 points)**.** Let $n, k \in N$ with $n, k \geq 1$. Prove that the number of partitions $(\lambda_1, \dots, \lambda_\ell)$ of n (with $\ell \in \mathbb{N}$ and $\lambda_1 \geq \cdots \geq \lambda_\ell \geq 1$ such that $\lambda_1 + \cdots + \lambda_\ell = n$) satisfying $\lambda_1 = k$ is equal to $P_{n,k}$.

Problem 3 (5 points)**.** Let $n \in \mathbb{N}$ with $n \geq 1$.

(a) Prove that $S_{n,2} = 2^{n-1} - 1$. (1 point)

(b) Prove that $S_{n,n-1} = \binom{n}{2}$. (1 point)

(c) Prove that

$$S_{n,k} = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} S_{n-1-\ell,k-1} = \sum_{j=0}^{n-1} \binom{n-1}{j} S_{j,k-1}$$

for all $k \geq 1$. (3 points)

Problem 4 (5 points)**.** Let $n, k, \ell \in \mathbb{N}$. Prove that

$$\binom{k+\ell}{k} \cdot S_{n,k+\ell} = \sum_{j=0}^{n} \binom{n}{j} \cdot S_{j,k} \cdot S_{n-j,\ell}$$

Bonus Problem 5 (2 bonus points)**.** Let $n \in \mathbb{N}$ with $n > k \geq 1$.

(a) Prove that

$$S_{n,k} = \sum_{j=0}^{n-1} k^{n-1-j} S_{j,k-1}.$$

(1 bonus point)

(b) Prove that

$$S_{n,k} = \sum_{j=0}^{k} j \cdot S_{n-k-1+j,j}.$$

(1 bonus point)