Universität Bonn
Institut für Angewandte Mathematik

S2B2 HS PDG: An introduction to geometric measure theory

Winter Semester 2025/26

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Motivation:

Consider sets $V \subset \mathbb{R}^n$ with a fixed volume. What shape should the set V have such that its boundary (or perimeter) is as small as possible? This question is known as the *isoperimetric problem.* It is a fundamental question in analysis and appears in many important mathematical problems and in various forms. One of the main goals of this seminar is to understand and eventually answer this question. To do this, we will

- study Hausdorff measures and the notion of the Hausdorff dimension,
- prove the isodiametric inequality,
- investigate covering theorems,
- discuss the area and co-area formulas,
- familiarize ourselves with functions of bounded perimeter,
- learn about Steiner symmetrisation,
- and prove the isoperimetric inequality.

Outline:

In what follows, we provide a preliminary schedule for the content of the seminar series.

1. **Measure Theory**
    - Introduction
2. **Hausdorff Measure**
    - Definition and main remarks [3, pages 5–7]
    - Important properties and remarks [3, Exercise 2.3, Example 2.4, page 19, and Example 2.11]
    - Hausdorff dimension (example: the 1D Cantor set [4, Section 4.8, with $\gamma = \frac{1}{3}$]) [3, Section 3.1]
    - The notion of length and relation with H^1 [3, Section 3.2]
3. $H^n = \mathcal{L}^n$ [3, Section 3.3]
    - Vitali's covering theorem [2, Section 1.5.1]
    - The isodiametric inequality
    - Steiner symmetrization

- Proof that $H^n = \mathcal{L}^n$

4. **Besicovitch's Covering Theorem** [3, Section 5.1]
   - Auxiliary lemmas
   - Proof of Besicovitch's covering theorem
   - Vitali's property of Radon measures (and [1, Example 2.20])
5. **Differentiation of Radon Measures**
   - Absolute continuity and singularity [3, pp. 51–52]
   - Lebesgue-Besicovitch differentiation theorem and examples [3, Section 5.2]
   - Lebesgue points, proof of Lebesgue's theorem and examples [3, Section 5.3]
6. **Lipschitz Functions** [3, Chapter 7]
   - Definition and McShane's Lemma (see also [2, pp. 101–102])
   - Kirszbraun's theorem
   - Weak gradients
   - Rademacher's theorem (see also [2, pp. 103–106])
7. **Sets of Finite Perimeter** [3, Chapter 12]
   - Definition, properties and examples
   - Lower semicontinuity of perimeter
   - Gauss-Green measures
   - Compactness
8. **The Isoperimetric Problem**
   - The Direct Method for geometric variational problems [3, Section 12.5]
   - Steiner inequality [3, Section 14.1]
   - Proof of the isoperimetric inequality [3, Section 14.2]

9*. **Rectifiable Sets** [3, Chapter 10]
   - Definition and relation to the regular case
   - Decomposition by regular Lipschitz images
   - Approximate tangent space
   - Blow-up and rectifiability criterion

Prerequisites: Analysis I - III

Literature:

[1] L. Ambrosio, N. Fusco, D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford Mathematical Monographs. Oxford Univ. Press, 2000.

[2] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions.* Textbooks in Mathematics. CRC Press, revised edition, 2015.

[3] F. Maggi, *Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory.* Cambridge University Press, 2012.

[4] W. P. Ziemer, *Modern Real Analysis.* Graduate Texts in Mathematics. Springer, 2nd edition, 2017.

Remarks:

A preliminary meeting in which the topics of the seminar will be distributed will take place on 29.07.2025 at 9 am in the meeting room 1.007. If the appointment is inconvenient, topics can, if necessary, still be assigned later, provided there are enough available slots. The seminar takes place on Tuesday at 8:15 in SemR 1.007, and the first meeting is on the 7th of October. The presentations can be given in *German or in English.* Please register via BASIS.

Expectations:

Schedule (Weeks Before the Presentation Date)

≥ 3: Read and try to understand the material.

2–3: Prepare a draft of the oral presentation, including time planning.

≥ 2: Meeting to discuss the presentation.

1–2: Practice your talk with other students.

1: Submit the handout.

0: Give your presentation.

Criteria for a Good Seminar Presentation

The main goals of the seminar are:

- to understand your assigned topic well,
- to explain it clearly to the other students,
- to participate actively as a listener.

To make grading clear and fair, you should focus on the following points, with special emphasis on the math:

- **Understanding:** Do you really understand your topic? Are there any serious mistakes?
- **Structure:** Is your talk well-organized and easy to follow?
- **Blackboard:** Avoid being too chaotic or disorganized at the blackboard.
- **Motivation:** Do you explain why the topic matters (and how it fits with earlier talks)?
- **Free Speaking:** Do you speak freely, without reading too much from notes (except during calculations)?
- **Style:** Do you speak to the audience or to the blackboard?
- **Time:** Do you stick to the time limit and cover the most important points?

Handout

Your handout should be no more than 6 pages, written in LaTeX. Please discuss what to include. Bring printed copies of your handout to the presentation and give them to the other students.