Universität Bonn
Institut für Angewandte Mathematik

S4B2: Graduate Seminar on PDE

Evolution Equations Arising in Continuum Mechanics

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Motivation: Variational models in continuum mechanics have received increased attention over the last decades. In recent years, they have successfully been used for the analysis of materials which are subjected to some (time-dependent) external loading. A possible way to describe the motion of such materials is through the balance of momentum, which, in Lagrangian coordinates, takes the form

$$\rho\,\partial_t^2 y - \operatorname{div}(\mathbf{T}) = f \quad \text{in } [0,T]\times\Omega. \tag{1}$$

Here, $\Omega \subset \mathbb{R}^3$ represents a reference configuration of the material, while $y\colon [0,T]\times\Omega \to \mathbb{R}^3$ corresponds to a deformation mapping. The function $f\colon [0,T]\times\Omega \to \mathbb{R}^3$ indicates a density of body forces and $\rho > 0$ denotes a mass density. The choice of the stress tensor $\mathbf{T} \in \mathbb{R}^{3\times 3}$, which depends on the deformation itself, is strongly influenced by the material to be described. For elastic materials, such as rubber, a possible choice could be $\mathbf{T} = \mathbf{T}(\nabla y)$, whereas for so-called viscoelastic deformations, one would additionally assume that the time derivative $\partial_t \nabla y$ affects the stress tensor $\mathbf{T}$. The goal of the seminar series is the development of (variational) methods which can be applied to problems as in (1).

Outline: In what follows, we provide a preliminary schedule for the content of the seminar series.

In the first part of the seminar series, we focus on the analysis of Bochner spaces [Yosida '80] which are fundamental for the description of weak solution concepts. Possible topics in that direction are the characterization of the dual space of $L^p((0,T);X)$ [Diestel, Uhl '77] or the characterization of compact sets in $L^p((0,T);X)$ [Simon '86].

In the second part, we mainly analyze the quasi-static approximation of (1), where we focus on viscoelastic deformations corresponding to a stress tensor of the form $\mathbf{T} = \mathbf{T}(\nabla y, \partial_t \nabla y)$. This leads to a system of parabolic type. Prominent techniques for their analysis are variational time-discretization schemes. As a first step, we justify the equations and present the result by [Krömer, Roubíček '20]. Then, we will extend the analysis in different directions by including thermal effects [Mielke, Roubíček: '20] and linearization results [Badal, Friedrich, Kružík]. To this end, we also discuss parabolic equations with sources with low integrability [Boccardo, Gallouët '89]. The previously mentioned results follow a rather classical approach based on time-discretization of the weak formulations. The problem can also be tackled by means of gradient flows in a metric space [Machill '24]. Due to the relevance of gradient flows, we also include an excursion into other gradient flow structures such as gradient flows in Wasserstein spaces [JKO '98]. Weaker concepts of weak solutions and other rheologies for viscoelastic materials are discussed in [Chiesa, Kružík, Stefanelli '24].

In the third part, we focus our attention on the system (1) by also including inertial effects, leading to a system of hyperbolic type. First of all, we study the linearized case via [Kružík, Roubíček '19] and justify the equations by relating these with an atomistic system [Friedrich, Seitz, Stefanelli '24]. Then, we discuss a Young measure solution [Demoulini '00] and compare it

with the results in [Benešová, Kampschulte, Schwarzacher '24]. This result is complemented with a regularity result by [Almi, Badal, Friedrich, Schwarzacher '24].

Prerequisites: knowledge of Sobolev spaces, weak solution concepts for PDEs are helpful

Literature: Besides the references provided in the text above, one possible reference for an introduction into continuum mechanics and the analysis of evolutionary problems is the book *"Mathematical Methods in Continuum Mechanics of Solids"* by M. Kružík and T. Roubíček from 2019.

Remarks: A preliminary meeting in which the topics of the seminar will be distributed will take place on 11.02.2025 at 3 pm in the meeting room 0.011. If the appointment is inconvenient, topics can, if necessary, still be assigned later, provided there are enough available slots. The seminar will be held in English.