Variations on the Wave Equation[1]

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To Professor Claus Müller on completion of the eightieth year of his life on February 20th, 2000

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Summary: A lot of vibration processes in mathematical physics are described by the wave equation or by related equations and systems, and plenty of research has been done on this subject. The results and methods obtained thereby have been very important in other fields of application, and they still are. They also had and still have an immense influence on the development of mathematics as a whole.

The following survey tries to convey an impression of how exciting the research in this field of partial differential equations has been in the twentieth century, and it wants to present some of the interesting results achieved. The selection, of course, reflects personal tastes and interests. It starts reporting on the state of the art at the end of the last century. It then describes important solution methods typical for this century, as there are integral equation methods, Hilbert space methods, or spectral representation. Generalized solutions to initial boundary-value problems are mentioned, and some applications are indicated, e.g. on the asymptotic behaviour of solutions, in scattering theory, and in non-linear analysis.

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The wave equation

$$(\partial_t^2 - \Delta)u = o$$

is one of the simplest equations which belong to the class of the so-called hyperbolic partial differential equations. These equations describe a lot of vibration processes in mathematical physics, e.g. the motion of the linearized vibrating string or membrane. With respect to time one has to solve initial value problems, and in contrast to potential theory one expects non-continuous solutions with finite propagation speed. Signals may set in or out sharply, a phenomenon which is described by *Huygens' principle* (Christian Huygens, 1629–1695).

Working on partial differential equations leads to many interesting mathematical questions. The results obtained had an immense influence on the development of mathematics as a whole, and they still have. To set an example: Looking at the problem of the clamped vibrating string of length l, one can follow the development of the concept of a function. The question is to find a solution $u(t, x)$ for $t \geq 0$ and $0 \leq x \leq l$ which solves the wave equation

$$u_{tt} = u_{xx}$$

in $(0, l) \in \mathbb{R}^1$ together with the boundary condition $u(t, 0) = u(t, l) = 0$. Transforming

$$\xi = x + t \text{ and } \eta = x - t,$$

$u_{\xi\eta} = o$ follows. From this Jean le Rond d'Alembert, 1717–1783, obtained his general solution in 1747, namely [2]

$$u(t, x) = f(x + t) + F(x - t).$$

Because of the boundary condition, $f(s) + F(-s) = 0$ and $f(l + s) + F(l - s) = 0$ hold for all $s \geq 0$, and thus $f(s) = f(2l + s)$. According to d'Alembert,

$$u(t, x) = f(t + x) - f(t - x)$$

therefore is the general solution; and this with $2l$-periodic smooth functions f, analytic functions as we would say.

Little later, in 1748, this statement was in two respects decisively improved by Leonhard Euler (1707–1783) [16]. He first noticed that by the very nature of the physical problem u is determined by the initial conditions

$$u(0, x) = g(x) \text{ and } u_t(0, x) = h(x)$$

with prescribed functions g, h. Therefore

$$g(x) = f(x) - f(-x), \quad \int_0^x h(s)ds = f(x) + f(-x) - 2f(0)$$

or

$$f(x) = f(0) + \frac{1}{2}\{g(x) + \int_0^x h(s)\,ds\},$$
$$f(-x) = f(0) + \frac{1}{2}\{-g(x) + \int_0^x h(s)\,ds\}$$

holds. Up to a constant, f is everywhere defined as a $2l$-periodic function herewith, and a representation of the solution which is usually named after d'Alembert follows, namely

$$u(t, x) = \frac{1}{2}\{g(x + t) + g(x - t) + \int_{x-t}^{x+t} h(s)\,ds\}. \tag{1}$$

In order to be able to describe the plucked string Euler allowed piecewise smooth functions g, h in Eq. (1) also. This is the second essential point. The concept of a solution was distinctly generalized by this. A differential equation could have non-differentiable solutions!

This extension of the concept of a solution – and by this the extension of the concept of a function – has become extremely important in the period following. There are classical, strong, weak, or distributional solutions today. Let us remember that the concept of a continuous function, we are using, was introduced only 1817 by Bernhard Bolzano, 1781–1848 [7].

A second development shall be indicated also: Before d'Alembert it was already known that for all $n \in \mathbb{N}$ the

$$u_n(t,x) := \cos\frac{n\pi t}{l}\sin\frac{n\pi x}{l}$$

are solutions of the boundary-value problem for the vibrating string. One speaks of *Taylor's overtones*, after Brook Taylor, 1685–1731. Euler obtained additional solutions from them by superposition. This led Daniel Bernoulli, 1701–1784, in 1753 to the remark that an infinite combination of such u_n,

$$u(t,x) = \sum_{n=1}^{\infty}\{\alpha_n\cos\frac{n\pi t}{l} + \beta_n\sin\frac{n\pi t}{l}\}\sin\frac{n\pi x}{l},$$

solves the boundary-value problem and that this is the general solution [6].

It took quite a while – after many discussions – until this could be mathematically founded. One had to prove that arbitrary functions g, h could be represented by trigonometric series, for instance

$$g(x) = u(0,x) = \sum_{n=1}^{\infty}\alpha_n\sin\frac{n\pi x}{l},$$

and to show how the infinity of unknowns α_n, β_n could be obtained from g, h. Here, above all, Jean-Baptiste-Joseph Fourier, 1766–1830, achieved the breakthrough. Beginning in 1807 he developed functions into the series named after him, functions not necessarily analytic or differentiable [18]. More details can be found in Bernhard Riemann's (1826–1866) paper on this subject written in 1854 [80].

This second development has led to the treatment of more general partial differential equations, for instance of the form

$$u_{tt} + Au = o.$$

A represents a suitable differential operator with respect to the space variables, in case of the string $A = -\partial^2/\partial x^2$ together with a boundary condition. A *spectral theory* of such operators was developed saying that A perhaps has a countably infinite number of *eigenvalues* λ_n and *eigenfunctions* v_n belonging to them,

$$Av_n = \lambda_n v_n.$$

The v_n form a complete orthonormal system, and one obtains the general solution by

$$u(t,x) = \sum_{n=1}^{\infty}\{\alpha_n\cos\sqrt{\lambda_n}t + \beta_n\sin\sqrt{\lambda_n}t\}v_n(x).$$

In case of the string we have $\lambda_n = (n\pi/l)^2$ and $v_n(x) = \sin\sqrt{\lambda_n}x$. In this connection the word *spectrum* was first used 1897 by Wilhelm Wirtinger, 1865–1945 [112], and afterwards by David Hilbert, 1862–1943, and his school.

A lot of fine representations of solutions of partial differential equations were obtained this way, and simultaneously the theory of Fourier series was generalized. Admitting unbounded domains, for instance $x \in \mathbb{R}$, one obtains Fourier integrals correspondingly.

In the following, therefore, one wanted to solve boundary and eigenvalue problems for the Laplace and more general elliptic equations, and this with respect to a most general possible class of boundaries. These initial boundary-value problems in $\mathbb{R}^n$, in general, cannot be solved so easily and directly as our example in $\mathbb{R}^1$. The treatment of the underlying stationary equation $\Delta u = o$, the *potential* or *Laplace equation*, with prescribed inhomogeneous Dirichlet boundary-values $u|\partial G = f$ is well known to present great difficulties. Such problems arise in electrostatics for instance. In $\mathbb{R}^2$ the problem is equivalent to Riemann's mapping theorem. The first chapter reports on some of the difficulties which were encountered in connection with the Dirichlet problem in potential theory at the end of the last century.

Many proposals have been made by famous mathematicians to solve the Dirichlet problem. In this survey we only mention two, both of them have been developed or completed by Hilbert and his students at the turn of the century. First the *integral equation method* will be presented in the second chapter. This method is well suited for smooth boundaries and homogeneous media. It is based on the observation that in electrostatic problems the charge is located at the surface of the conductor. This led to the introduction of a number of special surface layer assumptions which enabled the differential equation to be transformed into an integral equation.

The second method, the *direct* or *Hilbert space method*, is more closely connected with integral principles of mechanics. Quadratic energy functionals are introduced and minimized. In the third chapter we shall report on this.

Both methods are also suited to solve the corresponding eigenvalue problem. In general, of course, one cannot directly calculate the eigenvalues as was done for the string. Asymptotic expansions like

$$N(\lambda) := \sum_{\lambda_n \le \lambda} 1 = \frac{\mathrm{Vol}(G)}{6\pi^2}\lambda^{3/2} + o(\lambda^{3/2}) \text{ in } \mathbb{R}^3, \tag{2}$$

however, hold. They were first obtained by Hermann Weyl, 1885–1955 in 1912 [105], and they originally served for proving Jeans' radiation formula (black body radiation, Planck's cavity radiation).

Further developments are characterized by greater abstraction and stronger use of functional analytic methods. Equations were solved in Banach spaces, and the concept of an operator was made precise. Self-adjoint operators were seen to play an especially important rôle in mathematical physics because the physical observables correspond to them. Since, in general, the various operators are unbounded, there arose the need to clarify the process of extending a symmetric to a self-adjoint operator and of obtaining a normal form (*spectral representation*) for self-adjoint operators. A culmination point of this development was the proof of the spectral theorem for self-adjoint operators by John von Neumann, 1903–1957, between 1929–1932 ([66]–[69]). In the fourth chapter we shall report on this development, and in Chapter 5 we shall briefly discuss generalized solutions of the wave equation obtained that way.

Let us return to initial boundary-value problems however. They can be found very often in mathematical physics. Typical examples are met in acoustics, in the theory of Maxwell's equations, in elasticity, and in the theory of the Schrödinger equation. By means of the spectral theorem one can solve initial boundary-value problems associated with such equations. The case of unbounded domains is of special interest. Thus initial boundary-value problems in *exterior domains* or other unbounded domains have been studied by many different methods (an exterior domain is a domain with bounded complement; one may think of scattering by bounded

obstacles). The spectrum of the operators concerned does not consist of discrete eigenvalues, as with the string; it also or only contains a continuous part. Thus the treatment of the stationary boundary-value problem already requires greater effort. The description of the asymptotic behaviour of solutions for large times and their comparison with corresponding solutions in the free space ($G = \mathbb{R}^n$ with homogeneous medium) are of special importance now. Depending on whether classical equations (like Maxwell's equations) or Schrödinger equations are treated, one speaks of classical or quantum scattering theory. The behaviour of solutions to such equations at high frequencies is also of great interest, since it is the limit towards geometrical optics. In addition, problems of inverse scattering theory, where one wants to recover the shape of a body or the coefficients in the equation from measured data of the solution, are becoming more and more important. From the asymptotic expansion (2), for example, we can immediately obtain the volume of G, and the question is whether we can obtain its shape by improving the expansion. *Can one hear the shape of a drum?* is the title of a paper of Mark Kac, 1914–1984, [39]. Such questions are dealt with in Chapters 6 and 7.

Chapter 8, finally, is devoted to two non-linear problems. First we look at simple conservation laws and explain concepts like *shock waves* or *rarefaction waves*. Afterwards we report on the existence of global classical solutions to non-linear wave equations for small data.

1 Dirichlet's Principle

Solving boundary-value problems was a matter of central concern in mathematical physics during the last century. To have an example in one's mind's eye, let G be a homogeneous medium represented by an open connected set in $\mathbb{R}^n$, $\mathbb{R}^n \backslash G$ the obstacle, $u(t, x)$ an acoustic potential, $v = \nabla u$ the velocity, and $p = u_t$ the pressure. The linearized Euler equations of gas dynamics then show that u solves the wave equation $u_{tt} = \Delta u$ assuming all coefficients to be one. For a stiff boundary u has to comply with the *Neumann condition* $n\nabla u|\partial G = o$, n being the (exterior) normal of ∂G, and u has to comply with the *Dirichlet condition* $u|\partial G = o$ provided the pressure vanishes at the boundary. Mixed boundary-value conditions are also possible.

Let us start with the stationary case and the inhomogeneous Dirichlet condition. That means we are looking for a classical solution $u \in \mathcal{K} := C_2(G) \cap C(\bar{G})$ with

$$\Delta u = o \text{ and } u|\partial G = f. \tag{1.1}$$

$f \in C(\partial G)$ is prescribed, and we say that u solves the *Dirichlet problem*. A solution u of $\Delta u = o$ is called a *potential function* or *harmonic* in G.

Solving the Dirichlet problem for a ball can easily be done using *Poisson's integral formula* (Siméon-Denis Poisson, 1781–1840). The general case, however, turned out to be much more difficult. There are many methods to proceed. Dirichlet's principle played an important rôle in this connection. It goes back to Carl Friedrich Gauss, 1777–1855, and William Kelvin, 1824–1907. Riemann who had heard of it in Dirichlet's lectures, probably named it after him (Peter Lejeune Dirichlet, 1805–1859). He used it 1851 to prove his fundamental existence theorems in geometrical function theory [79].

Dirichlet's principle is deeply connected with classical integral principles in physics which follow from Newton's laws. Think of *Maupertius' law of least action* or *Hamilton's principle*. It says the following: Let

$$D(u) := \int_G (\nabla u)^2$$

be the *Dirichlet integral* with respect to Δ, and $\mathcal{D} := \{u \in \mathcal{K} \mid D(u) < \infty \text{ and } u|\partial G = f\}$ the

underlying function space. Let $D(v)$ be the minimum of $D(u)$, so $v \in \mathcal{D}$ and $D(v) = \min D(u)$ for $u \in \mathcal{D}$. v then solves the Dirichlet problem.

It is easy to see that this is true. From $D(v + \varepsilon\varphi) \geq D(v)$ for all test functions $\varphi \in \mathring{C}_\infty(G)$ and all $\varepsilon \in \mathbb{R}$ we obtain

$$\forall\, \varphi \in \mathring{C}_\infty(G) \qquad \int_G (\nabla v)(\nabla \varphi) = 0,$$

and after partial integration the fundamental lemma of the calculus of variations gives $\Delta v = o$ ($\mathring{C}_\infty(G)$ is dense in $\mathcal{L}^2(G)$).

Because the Dirichlet integral is bounded from below, it was thought to be evident for a long time that the minimum problem has a solution $v \in \mathcal{D}$. One says Dirichlet's principle holds for $D(u)$ in $\mathcal{D}$ provided such a minimum exists. Riemann used it in this way. Criticism was brought in by Karl Weierstrass, 1815–1897, in 1870 [101], who pointed out that an infimum must not necessarily be taken on. It is clear that a minimizing sequence $u_n \in \mathcal{D}$ with $D(u_n) \to \inf D(u)$ for $u \in \mathcal{D}$ exists. But it is not clear whether this sequence converges at all, or in which sense, and whether a limit belongs to $\mathcal{D}$ if it exists.

Thus Riemann's theorems were unfounded, and an intensive effort of many mathematicians started to solve the boundary-value problem and to save Dirichlet's principle. The latter, especially, because Dirichlet's principle was so elegant and physically plausible. Both problems were solved satisfactorily only around 1900. We cannot describe all the different solution methods here. Let us only mention Hermann Amandus Schwarz's (1843–1921) *alternating procedure*, the method of Carl Neumann, 1832–1925, which led to the integral equation method, the method *de balayage* of Henri Poincaré, 1854–1912, and later Oskar Perron's (1880–1975) method of *subharmonic functions* (1923 [72]).

As was already said in the Introduction, in the next two chapters we shall report on the integral equation method and the Hilbert space method. The integral equation method supplies a classical solution u of problem (1.1). The Hilbert space method proves the Dirichlet principle in a suitably chosen function space and thus provides with generalized solutions. For instance, let $G \subset \{\mathbb{R}^n \mid 0 < x_n < d\}$. By partial integration we then obtain a Poincaré's estimate (1894 [73])

$$\forall\, \varphi \in \mathring{C}_\infty(G) \qquad \|\varphi\| \leq d\, \|\nabla \varphi\|. \tag{1.2}$$

Here $\|\varphi\|$ denotes the $\mathcal{L}^2(G)$-norm of φ. Est. (1.2) indicates that a minimizing sequence may be a $\mathcal{L}^2(G)$-Cauchy sequence. The set $\mathcal{D}$, however, is not complete with respect to this norm. This means that a $\mathcal{L}^2(G)$-limit, if it exists, generally does not belong to $\mathcal{D}$. Nevertheless one would consider it to be a generalized solution. We shall make this more precise in ğ3.

This generalization of the solution concept proved to be extremely fruitful, and it has led to many beautiful results. More details will be given in Chapters 3 and 5. Furthermore it turned out that such solutions are especially well suited for physical applications. One should not forget, however, that the formulation of the problem was changed by that. Even if it is true for smooth data, it generally requires great analytical effort to prove that a generalized solution is a classical solution also (the same holds vice versa). Thus one cannot say in the beginning that perhaps the Hilbert space method is superior to the integral equation method. Boundary-value problems exist which are classically solvable but not in the generalized sense and vice versa. Closing this chapter, we present two examples from $\mathbb{R}^2$ to illustrate that.

Friedrich Emil Prym, 1841–1915, first presented an example of a boundary-value problem, classically solvable but for which the corresponding Dirichlet principle fails, in 1871 [74]. Somewhat simpler is the following example which goes back to Jacques Salomon Hadamard, 1865–1963, in 1906 [26]. Let G be the unit disk and $f(\varphi) := \sum n^{-2} \sin(n!\,\varphi)$. f is a continuous

function, and

$$u(r,\varphi) := \sum_{n=1}^{\infty} r^{n!}\,\frac{\sin(n!\,\varphi)}{n^2} \in \mathcal{K}$$

solves the Dirichlet problem. One can take this from the theory of Fourier series or from Poisson's integral formula. For $r < 1$, however,

$$\int_{|x|<r} (\nabla u)^2 = \pi \sum_{n=1}^{\infty} r^{2n!}\,\frac{n!}{n^4}$$

holds, and the limit $r \to 1$ does not exist.

Conversely let $G := \{x \in \mathbb{R}^2 \mid 0 < |x| < 1\}$, and let us look for a solution of the Dirichlet problem with boundary values $f(0) = 1$ and $f(x) = 0$ for $|x| = 1$. In this case the generalized solution $u = o$ exists. But the problem cannot be solved classically. To see that, let u be a classical solution. In analogy to function theory, we then obtain from the maximum principle and the knowledge of the fundamental solution that the origin is a removable singularity. Thus u can be continued to a potential function in the interior of the unit disk, and therefore vanishes because $u(x) = 0$ for $|x| = 1$. This is inconsistent with $u(0) = 1$.

2 Integral Equations

As was already mentioned in the Introduction, physical considerations suggest to try to obtain the solution of the Dirichlet problem (1.1) in $\mathbb{R}^3$ by making the ansatz

$$v(x) := \frac{1}{2\pi} \int_{\partial G} \mu(y)\,\frac{1}{|x-y|}\,dy.$$

v is called a *potential with the simple surface layer* μ. In doing so one assumes the boundary ∂G to be sufficiently smooth and μ to be continuous. Let G be bounded for the present and, simplifying, $\mathbb{R}^3\backslash G$ also be connected. v is formed by convolution of the fundamental solution with the layer μ. Therefore v is a potential function both in G and in $\mathbb{R}^3\backslash\bar{G}$. It is easy to see that v is defined and continuous in all of $\mathbb{R}^3$. $v \in C_\infty(G)$ and $v \in C_\infty(\mathbb{R}^3\backslash\bar{G})$ hold, and some calculation shows that the normal derivative jumps at the boundary whilst the tangential derivatives remain continuous.

To solve the boundary-value problem one has to determine μ such that

$$\forall\, x \in \partial G \qquad f(x) = \frac{1}{2\pi} \int_{\partial G} \mu(y)\,\frac{1}{|x-y|}\,dy$$

holds. In Hilbert's terminology this is an integral equation of first kind. It is difficult of access by analytical treatment.

August Beer, 1825–1863, was the first who obtained the break-through here in 1856 [4], reprinted in C. Neumann [65, p. 220f]. Beer proceeded from Green's formula (1828) applied to the solution u and the fundamental solution, namely from the representation (George Green, 1793–1841)

$$u(x) = \frac{1}{4\pi} \int_{\partial G} \Big\{ \frac{\partial u}{\partial n}(y)\,\frac{1}{|x-y|} - f(y)\,\frac{\partial}{\partial n_y}\,\frac{1}{|x-y|} \Big\}\,dy,$$

and iterated to obtain $\partial u/\partial n$. In today's terminology he did not use a single layer but a *double surface layer ansatz* namely the *dipole potential*

$$w(x) := \frac{1}{2\pi} \int_{\partial G} \nu(y)\,\frac{\partial}{\partial n_y}\,\frac{1}{|x-y|}\,dy.$$

Again ν is supposed to be a continuous function, and n is the exterior normal. The limits of $w(x)$ as $x \to \partial G$, $x \in G$ rsp. $x \in \mathbb{R}^3 \backslash \bar{G}$, exist but differ by 2ν. Looking at the Dirichlet problem in G one obtaines the integral equation of second kind

$$(-id + K)\nu = f \tag{2.1}$$

with $K : C(\partial G) \to C(\partial G)$. Beer solved it by iteration using what we now call *Neumann's series*, $\nu = -\sum K^n f$, $n \in \mathbb{N}_0$. He did not bother about convergence at all, he only writes that it is evident that Kf behaves more regular than f.

Neumann then took up Beer's method again, and he intensively tried to prove convergence in case of convex domains. The difficulty is due to the fact that $\|K\| \leq 1$ holds, but not $\|K\| < 1$. In 1900 Ivar Fredholm, 1866–1927, achieved the break-through, and was able to solve Eq. (2.1) for arbitrary domains using the compactness of K and thereby solving the Dirichlet problem [19]. In 1903 he proved the alternative for integral equations of second kind which was named after him [20]. Afterwards Hilbert, his school, and Frighes Riesz, 1880–1956, worked on such problems. They chose a more abstract and general approach. The so-called *Riesz-Schauder theory* was developed (Juliusz Schauder, 1896–1943). Around 1930 the theory was completely known ([81]–[84]).

The proof of Fredholm's alternative is done by iteration again. Let $\mathcal{X}$ be a Banach space. One then shows the existence of a *Riesz-number* $r \in \mathbb{N}_0$ such that

$$\begin{aligned} \{o\} = \mathcal{N}_0 \subsetneq \cdots \subsetneq \mathcal{N}_r = \mathcal{N}_{r+1} = \cdots \\ \mathcal{X} = \mathcal{R}_0 \supsetneq \cdots \supsetneq \mathcal{R}_r = \mathcal{R}_{r+1} = \cdots . \end{aligned}$$

Here $\mathcal{N}_n := \mathcal{N}(A^n)$, $\mathcal{R}_n := \mathcal{R}(A^n)$, and A is a compact perturbation of the identity. One also considers the adjoint equation.

Applying these results one has to determine the Riesz-number. We have $r(-id + K) = 0$. Because of $r(id + K) = 1$, however, the exterior Dirichlet problem can only be handled after making a modification. E.g. changing the kernel in the double layer ansatz to

$$\Big\{\frac{\partial}{\partial n_y} + \mathrm{i}\Big\}\frac{1}{|x - y|}$$

yields an equation with Riesz-number 0. This problem of making an appropriate ansatz becomes more important when dealing with the Helmholz Equation (vibration problems)

$$(\Delta + k^2)u = o \text{ and } u|\partial G = f. \tag{2.2}$$

In case of exterior boundary-value problems, the usual double layer ansatz leads to an integral equation with Riesz number 0 for all $k \in \mathbb{R}\backslash\{k_n\}$. The $\{k_n\}$ form an infinite exception set, their squares are the eigenvalues of the corresponding interior Neumann boundary-value problem. One wants to obtain an integral equation, however, solvable for all k. In doing so a considerable progress is made. Looking at $u(k)$ for a fixed x or as a mapping

$$u \;:\; \mathbb{C}^+ \longrightarrow \mathcal{L}^2(G \cap B(0, r)),$$

it is analytic in the upper half-plane $\mathbb{C}^+$, and from the foregoing it follows that it can be continued analytically in $\overline{\mathbb{C}^+}$ and meromorphically in $\mathbb{C}$ (Werner 1962 [103], Steinberg 1968 [94]). This continuation played an important rôle in the following. I shall mention it again in ğ6 and at the end of the paper. (We have used $B(0, r) := \{x \mid |x| < r\}$.)

Let me finally once more stress that in many situations (non-smooth boundaries, systems of equations) one usually does not obtain such nice integral equations. There is a broad field of singular integral equations, and much research has been done in it.

3 Hilbert Space Methods

As soon as Hilbert had heard of Fredholm's results, he started working on integral equations himself. Between 1904 and 1906 he published six papers on this subject which were brought together in his book from 1912 [30]. In doing so he deliberately abandoned the view of integral equations subsuming them as a special case of the concept of infinite systems of linear equations. He clearly pointed out in 1905 that *any regular problem in the calculus of variations has a solution, as soon as restrictive assumptions are made with regard to the nature of the boundary condition and, if necessary, the concept of the solution is generalized conveying the general sense* [28, 29].

Everything was said with that. Hilbert's method has been extended by his students, in particular by Richard Courant, 1888–1972, and again by his school, to one of the most efficient methods in the theory of partial differential equations, and especially for solving initial boundary-value problems and scattering problems in mathematical physics. It had an enormous influence on the further development of mathematics. The book by Courant and Hilbert from 1937 [13, p. 471f], reprinted in 1993 [14], contains a detailed presentation of the results.

The generalization of the classical concept of differentiation, of course, is essential in this connection. Let $u \in \mathcal{L}^2(G)$. One then defines $\partial_i u \in \mathcal{L}^2(G)$ to be a weak i-th derivative of u, if for all test functions $\varphi \in \mathring{C}_\infty(G)$

$$(u, \partial_i \varphi) = -(\partial_i u, \varphi)$$

holds. Here (u, v) is the scalar product of $\mathcal{L}^2(G)$. The weak derivatives are uniquely defined, and one denotes $\mathcal{W}_1(G) \subset \mathcal{L}^2(G)$ to be the subset of once weakly differentiable functions of $\mathcal{L}^2(G)$. $\mathcal{W}_1(G)$ itself is a Hilbert space with the scalar product

$$(u, v)_1 := (u, v) + (\nabla u, \nabla v) \tag{3.1}$$

and the norm $\|u\|_1 := \sqrt{(u, u)_1}$. The space $\mathcal{W}_k(G)$ is analogously defined.

In addition to weak derivatives one also needs strong derivatives which are not defined by testing but by approximation. Let $\mathcal{H}_1(G)$ be the completion of $C_1(G) \cap \mathcal{W}_1(G)$ with respect to the $\|\cdot\|_1$-norm. $\mathcal{H}_1(G)$ is also a Hilbert space with the scalar product (3.1), and obviously $\mathcal{H}_1(G) \subset \mathcal{W}_1(G)$ holds. $\mathcal{H}_k(G)$ is also analogously defined.

The spaces $\mathcal{W}_k(G)$ are named after Sergei Lvovich Sobolev, 1908–1989 (1936 [90]). Sobolev also noticed that functions $f \in \mathcal{W}_k(G)$ become more and more regular with increasing k (1936 [91]). Strong derivatives were used by Kurt Otto Friedrichs, 1901–1982, in 1939 [21].

The statement

$$\forall\, k \in \mathbb{N} \qquad \mathcal{H}_k(G) = \mathcal{W}_k(G), \tag{3.2}$$

strong equals weak, is extremely important for further handling with such generalized derivatives. For arbitrary G it was noticed only relatively late; for $k = 1$ it follows easily from Weyl's lemma and the projection theorem, cf. ğ5. Further results were obtained by Friedrichs in 1944 [22]; the general case was done by Meyers and Serrin in 1964 [58] (for $\mathcal{L}^p$-spaces also).

To indicate how the Hilbert space method works, let us look at a simple example. The Dirichlet problem

$$(-\Delta + 1)\, u = o \text{ with } u|\partial G = f$$

in a domain G shall be solved for a given f. To do so, we want to minimize the Dirichlet integral

$$J(u) := \int_G \{(\nabla u)^2 + u^2\}$$

associated with it. The space $\mathcal{H}_1(G)$ seems to be a suitable Hilbert space for doing so.

But the formulation of the behaviour of the desired solution at the boundary is still missing. One therefore defines subspaces $\mathring{\mathcal{H}}_1(G)$ and $\mathring{\mathcal{W}}_1(G)$ of $\mathcal{H}_1(G)$ and $\mathcal{W}_1(G)$ respectively (both are equal again) to describe the vanishing of a function at the boundary, namely

$$\mathring{\mathcal{H}}_1(G) := \{u \in \mathring{\mathcal{C}}_\infty(G)\,;\ \|\cdot\|_1\}^{\sim},$$

$$\mathring{\mathcal{W}}_1(G) := \{u \in \mathcal{W}_1(G)\ \mid\ \forall\,\psi \in \mathcal{W}_1(G) \quad \forall\, i \in \mathbb{N}(n) \quad (u, \partial_i\psi) = -(\partial_i u, \psi)\}\,.$$

In case of smooth data one easily convinces oneself by partial integration that the statement $u \in \mathring{\mathcal{H}}_1(G)$ or $u \in \mathring{\mathcal{W}}_1(G)$ generalizes the statement $u|\partial G = o$.

Finally a suitable class of boundary-values has to be chosen. This may be a restriction as Hilbert had already pointed out, think of Prym's example. We assume $f \in \mathcal{H}_1(G)$, and want to obtain a solution $u \in \mathcal{H}_1(G)$ with $u - f \in \mathring{\mathcal{H}}_1(G)$. This means we want to minimize $J(u)$ in $\mathcal{J} := \{u \in \mathcal{H}_1(G) \mid u - f \in \mathring{\mathcal{H}}_1(G)\}$.

That this can be done follows from the *approximation theorem*. It says the following: Let $\mathcal{H}$ be a Hilbert space and $\mathcal{K} \subset \mathcal{H}$ be a closed and convex subset.

$$\forall\, x \in \mathcal{H} \quad \overset{1}{\exists}\, k \in \mathcal{K} \qquad \|x - k\| = \inf_{y \in \mathcal{K}} \|x - y\|$$

then holds. The proof of the approximation theorem is very simple. One chooses a minimizing sequence $\{y_n\} \in \mathcal{K}$. The parallelogram equation immediately yields its convergence. We then apply the theorem with $\mathcal{H} := \mathcal{H}_1(G)$, $\mathcal{K} := \mathcal{J}$, and $x := o$ to obtain our solution.

4 Spectral Representation of Self-adjoint Operators

We start reformulating the Dirichlet initial boundary-value problem for the wave equation. Let

$$\begin{aligned} A \ :\ \mathcal{D}(A) \subset \mathcal{L}^2(G) &\longrightarrow \mathcal{L}^2(G), \\ u &\longmapsto -\Delta\, u \end{aligned}$$

with

$$\mathcal{D}(A) := \{u \in \mathring{\mathcal{H}}_1(G)\ \mid\ \Delta u \in \mathcal{L}^2(G)\}.$$

Furthermore let functions $u^0 \in \mathcal{D}(A)$ and $u^1 \in \mathcal{D}(A^{1/2}) = \mathring{\mathcal{H}}_1(G)$ be given. Then we are looking for a $u \in C(\mathbb{R}_0^+, \mathcal{D}(A)) \cap C_1(\mathbb{R}_0^+, \mathcal{D}(A^{1/2})) \cap C_2(\mathbb{R}_0^+, \mathcal{L}^2(G))$ with

$$u_{tt} + Au = o$$

as well as (4.1)

$$u(0, x) = u^0(x) \text{ and } u_t(0, x) = u^1(x).$$

The concept of solution we are using here (*strict solution*) can be weakened further, and instead of Δ one can take $\Sigma a_{ik}\partial_i\partial_k$ with positive definite coefficients a_{ik} or, even more general, assume A to be a self-adjoint operator. We shall touch such questions in ğ6. A positive mass density factor usually stands next to u_{tt} in the applications. This factor can be taken as a weight-factor in the underlying Hilbert space, so again we are led to problem (4.1).

Formally our initial-value problem (4.1) is solved by

$$u(t,\,\cdot\,) = (\cos\,\sqrt{A}\,t)u^0 + \frac{\sin\,\sqrt{A}\,t}{\sqrt{A}}\,u^1, \tag{4.2}$$

but for the present we cannot interpret this formal solution since so far we only studied boundary-value problems for A, and can only form the resolvent of A if it exists. What is missing is a calculus with which one can handle Eq. (4.2). One needs a comfortable representation of A for doing so.

An important step in this direction is the *spectral theorem for self-adjoint operators* which was proved 1929 by von Neumann [66]–[69]. In 1932 Marshall Stone's book (1903–1989) on this subject was published [95]. A *spectral family* of orthogonal projectors $P(\lambda)$, $\lambda \in \mathbb{R}$, in a Hilbert space $\mathcal{H}$ is assigned to a self-adjoint operator

$$A \;:\; \mathcal{D}(A) \subset \mathcal{H} \longrightarrow \mathcal{H}$$

with

1. $P(\lambda)\,P(\mu) = P(\min(\lambda,\mu))$.
2. $P(\lambda+0) = P(\lambda)$ (i.e. $P(\lambda)$ is strongly continuous from the right).
3. $P(-\infty) = O$ and $P(+\infty) = id$ (both in the strong sense).

The spectral theorem then says, that

$$\forall\, u \in \mathcal{D}(A) \quad \forall\, v \in \mathcal{H} \qquad (Au, v) = \int_{-\infty}^{\infty} \lambda\, d(P(\lambda)u,\, v)$$

holds together with Stone's formula

$$([P(b) - P(a)]\,f,\, g) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} \frac{1}{2\pi\mathrm{i}} \int_{a+\delta}^{b+\delta} (\{[A - (s+\mathrm{i}\varepsilon)]^{-1} - [A - (s-\mathrm{i}\varepsilon)]^{-1}\}f,\, g)\, ds$$

for all $a, b \in \mathbb{R},\ a < b$, and all $f, g \in \mathcal{H}$.

By means of the spectral theorem, one can represent powers of A and even more general functions of A. Therefore Eq. (4.2) can be read, and the unique solvability of initial boundary-value problem (4.1) follows.

Many proofs have been given of the spectral theorem. There are completely elementary proofs (Leinfelder 1979 [55]), and also very abstract ones using Banach-algebras. A middle course starts with the representation of the resolvent of A in the upper half-plane with help of the residue calculus. One then uses the Riesz representation theorem and has to perform the limit of vanishing imaginary part to obtain the spectral family. To do so, one needs a selection theorem, namely that in a separable normed space $\mathcal{X}$ the closed ball

$$\bar{B}'(o,1) := \{x' \in \mathcal{X}' \;\mid\; \|x'\| \le 1\} \subset \mathcal{X}'$$

is weak-star sequentially compact. Here $\mathcal{X}' := \mathcal{CL}(\mathcal{X}, \mathbb{K})$ denotes the adjoint space of $\mathcal{X}$. Selection theorems play a decisive rôle in the eigenvalue theory also as will be seen in ğ6.

An excellent representation of self-adjoint operators has been given by this. A large class of important operators in mathematical physics belongs to them. But as they are usually unbounded, it is not trivial to see whether a given symmetric operator is self-adjoint, or whether a self-adjoint extension exists or not. This problem can be well understood making use of the Cayley transformation, the analog of the Möbius transformation in function theory. Its definition for linear operators also goes back to von Neumann.

The operators, we are thinking of, are usually densely defined, symmetric, and they are closed or at least closable. The latter means that the graph $G(A) := \{(u, Au) \in \mathcal{H} \times \mathcal{H} \mid u \in \mathcal{D}(A)\}$ of A is closed, or that a closed operator $\bar{A}$ exists such that $G(\bar{A}) = \overline{G(A)}$. In $G(A)$ one can choose the norm $\|u\|_A := \sqrt{\|u\|^2 + \|Au\|^2}$, and one obtains $\mathcal{D}(\bar{A}) = \{\mathcal{D}(A); \|\cdot\|_A\}^{\sim}$ for closable A. Let

$$S \quad : \quad \mathcal{D}(S) \subset \mathcal{H} \longrightarrow \mathcal{H}$$

be a symmetric operator in a Hilbert space $\mathcal{H}$. Then $S \subset S^*$ by definition, and since symmetric operators are always closable, the trivial extension $\bar{S} = S^{**}$ exists. Therefore S always possesses two closed extensions namely S^{**} and S^*. S^{**} is symmetric again, but S^* generally is not. S is called essentially self-adjoint if $S^{**} = S^*$.

So let S be a closed symmetric operator. Then

$$\begin{aligned} U \quad : \quad \mathcal{D}(U) = \mathcal{R}(S + \mathrm{i}) \subset \mathcal{H} &\longrightarrow \mathcal{R}(U) = \mathcal{R}(S - \mathrm{i}) \subset \mathcal{H}, \\ u &\longmapsto (S - \mathrm{i})(S + \mathrm{i})^{-1} u \end{aligned}$$

is the *Cayley-transform* of S. U is closed, surjective and isometric. $\mathcal{D}(U)$ and $\mathcal{R}(U)$ are closed and $S = \mathrm{i}(id + U)(id - U)^{-1}$.

S is self-adjoint if and only if U is unitary (that is when $\mathcal{D}(U) = \mathcal{R}(U) = \mathcal{H}$). The question of extending a symmetric mapping to a self-adjoint is thus reduced to the question of extending an isometric mapping U unitarily; quite often it is easier to answer the latter. To do so, one defines *defect indices* by

$$\dim \mathcal{R}(S + \mathrm{i})^{\perp} = \dim \mathcal{D}(U)^{\perp} \text{ and } \dim \mathcal{R}(S - \mathrm{i})^{\perp} = \dim \mathcal{R}(U)^{\perp}.$$

S is self-adjoint in case both vanish, S can be extended self-adjointly if and only if the defect indices are equal. The formula

$$\mathcal{D}(S^*) = \mathcal{D}(S) \oplus \mathcal{R}(S + \mathrm{i})^{\perp} \oplus \mathcal{R}(S - \mathrm{i})^{\perp}$$

holds. S, therefore, is self-adjoint if one can solve the equations

$$(S \pm \mathrm{i})u = f$$

for all $f \in \mathcal{H}$.

5 Generalized Solutions

Let us take up the problem of generalizing the concept of a function again. We have already seen d'Alembert's smooth solution and strict solutions. We want to indicate some more generalizations. An important step for further understanding is the paper of Weyl from 1940 [106]. It contains both the projection method in potential theory and regularity statements of generalized solutions, even up to the boundary.

Weyl shows that from $u \in \mathcal{L}^1_{loc}(G)$ and

$$\forall\, \varphi \in \mathring{C}_2(G) \qquad \int_G u\, \Delta\varphi = 0$$

$u \in C_\infty(G)$ follows. To prove this, he gives an explicit representation of u by convolution with the fundamental solution of the Laplace equation. This regularity statement can be generalized, see e.g. Hellwig's book from 1960 [27]. Today it reads as follows

Weyl's Lemma: *Let $a_{ik} = a_{ki} \in C_{2+\alpha}(G)$, $a_i \in C_{1+\alpha}(G)$, $a \in C_1(G)$, $f \in C_\alpha(G)$ and $u \in \mathcal{L}^1_{loc}(G)$ be a weak solution of*

$$\forall\, \varphi \in \mathring{C}_\infty(G) \qquad M(u,\varphi) = (f,\varphi).$$

Then $u \in C_{2+\alpha}(G)$ holds a.e.

Here we are using

$$M(u,v) := \sum_{i,k}(a_{ik}\partial_k u, \partial_i v) + \sum_i (a_i \partial_i u, v) + (au, v),$$

and assume a_{ik} to be positive definite. To prove Weyl's lemma, one constructs a singularity function for the leading term $a_{ik}\partial_i\partial_k$ and gives a representation of the solution u by convolution with it. One can also show regularity up to the boundary corresponding to the regularity of the boundary.

Weyl's lemma immediately yields $\mathcal{W}_1(G) = \mathcal{H}_1(G)$: Since the subspace $\mathcal{H}_1(G)$ is closed in $\mathcal{W}_1(G)$, the projection theorem, an easy consequence of the approximation theorem, gives

$$\mathcal{W}_1(G) = \mathcal{H}_1(G) \oplus \mathcal{H}_1(G)^\perp.$$

So let $u \in \mathcal{H}_1(G)^\perp$.

$$\forall \varphi \in \mathring{C}_\infty(G) \qquad (u,\varphi)_1 = 0$$

then follows. u, therefore, is a weak solution of $(-\Delta + 1)u = o$, and Weyl's lemma shows $u \in C_2(G)$. But then $u \in \mathcal{H}_1(G)$ and, therefore, $u = o$.

Let us also look at Weyl's projection method for solving the Dirichlet problem in potential theory. Let

$$\mathcal{D}_0(G) := \{U \in (\mathcal{L}^2(G))^n \mid \forall\, \varphi \in \mathring{C}_\infty(G) \quad (\nabla\varphi, U) = 0\}$$

be the set of divergence-free vector fields $U \in (\mathcal{L}^2(G))^n$. The projection theorem then shows

$$(\mathcal{L}^2(G))^n = \overline{\nabla\mathring{\mathcal{H}}_1(G)} \oplus \mathcal{D}_0(G).$$

The point is that $\nabla\mathring{\mathcal{H}}_1(G)$ is already closed: This follows for domains $G \subset \mathbb{R}^n$ which are bounded in at least one direction, e.g.

$$G \subset \{x \in \mathbb{R}^n \mid 0 < x_n < d\},$$

from *Poincaré's first inequality* (1894 [73])

$$\forall\, u \in \mathring{\mathcal{H}}_1(G) \qquad \|u\|_{\mathcal{L}^2(G)} \le d\, \|\nabla u\|_{(\mathcal{L}^2(G))^n}.$$

Namely, let $\{\varphi_k\}$ be a sequence from $\mathring{\mathcal{H}}_1(G)$ with $\|\nabla\varphi_k - \Phi\|_{(\mathcal{L}^2(G))^n} \to 0$. The inequality then shows the existence of a $\varphi \in \mathcal{L}^2(G)$ with $\|\varphi_k - \varphi\|_{\mathcal{L}^2(G)} \to 0$. From

$$\forall\, \psi \in \mathcal{H}_1(G) \quad \forall\, i \in \mathbb{N}(n) \qquad (\partial_i\psi, \varphi_k) = -(\psi, \partial_i\varphi_k)$$

one therefore gets

$$\forall\, \psi \in \mathcal{H}_1(G) \quad \forall\, i \in \mathbb{N}(n) \qquad (\partial_i\psi, \varphi) = -(\psi, \Phi).$$

Thus $\varphi \in \mathring{\mathcal{W}}_1(G)$, and $\varphi \in \mathring{\mathcal{H}}_1(G)$ follows from $\|\varphi_k - \varphi\|_1 \to 0$. This means

$$(\mathcal{L}^2(G))^n = \nabla\mathring{\mathcal{H}}_1(G) \oplus \mathcal{D}_0(G).$$

Let $f \in \mathcal{H}_1(G)$ be given now. Then we are looking for a $u \in \mathcal{H}_1(G)$ with $u - f \in \mathring{\mathcal{H}}_1(G)$ and $\nabla u \in \mathcal{D}_0(G)$. To get it, we project

$$\begin{aligned}(\mathcal{L}^2(G))^n &= \nabla\mathring{\mathcal{H}}_1(G) \oplus \mathcal{D}_0(G)\\ \nabla f \quad &= \quad \nabla w \quad + \quad U,\end{aligned}$$

and choose $u := f - w \in \mathcal{H}_1(G)$. $u - f \in \mathring{\mathcal{H}}_1(G)$ and $\nabla u = U \in \mathcal{D}_0(G)$ follow. The mapping u therefore solves the Dirichlet boundary-value problem in potential theory. In case of unbounded domains one can prove the analog using a weighted Hilbert space.

But let us come back to the wave equation. Let $\mathring{C}(\mathbb{R})$ be the set of continuous functions with compact support in $\mathbb{R}$; $u^0,\ u^1 \in \mathcal{L}^2(G)$, $u \in C(\mathbb{R}_0^+, \mathcal{L}^2(G))$ and

$$\forall\, v \in \mathcal{V} \qquad \int_{\mathbb{R}^+ \times G} u\,\overline{Wv} \;+ (u^0, v_t(0,\cdot)) - (u^1, v(0,\cdot)) = 0$$

with $W := \partial_t^2 + A$ and

$$\mathcal{V} := \mathring{C}(\mathbb{R}, \mathcal{D}(A)) \cap C_2(\mathbb{R}, \mathcal{L}^2(G)).$$

u is then called a *weak solution* of the wave equation $Wu = o$ for $t \geq 0$ and with $u(0) = u^0$, $u_t(0) = u^1$. This definition goes back to Vishik and Ladyzhenskaya 1956 [99].

In the fields of application one often needs *solutions with finite energy*

$$E(t) := \|u_t(t)\|^2 + \|A^{1/2}u(t)\|^2 < \infty.$$

They are weak solutions in $C(\mathbb{R}_0^+, \mathcal{D}(A^{1/2})) \cap C_1(\mathbb{R}_0^+, \mathcal{L}^2(G))$ and exist provided $u^0 \in \mathcal{D}(A^{1/2})$ and $u^1 \in \mathcal{L}^2(G)$.

Distributional solutions introduced by Schwartz 1945, cf. [86], are important also. To define them we abbreviate $\Box := \partial^2/\partial t^2 - \Delta$, $\mathcal{D} := \mathring{\mathcal{C}}_\infty(\mathbb{R}^n)$, and assume $u_0,\ u_1 \in \mathcal{D}'(\mathbb{R}^n)$ and $f \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^n)$ with $\operatorname{supp} f \subset \mathbb{R}_0^+ \times \mathbb{R}^n$. Let $u \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^n)$ with $\operatorname{supp} u \subset \mathbb{R}_0^+ \times \mathbb{R}^n$. u is then called a *distributional solution* of

$$\Box u = f \;\text{ with } u(0) = u^0 \text{ and } u_t(0) = u^1$$

for $t \geq 0$, if

$$\forall\, \varphi \in \mathring{\mathcal{C}}_\infty(\mathbb{R} \times \mathbb{R}^n) \qquad (u, \Box\varphi)_{\mathbb{R}\times\mathbb{R}^n} = (f, \varphi)_{\mathbb{R}\times\mathbb{R}^n} - (u^0, \varphi_t(0))_{\mathbb{R}^n} + (u^1, \varphi(0))_{\mathbb{R}^n}$$

holds. One symbolically writes $\Box u = g$ with $g(t,x) := f(t,x) + \delta'(t)u^0(x) + \delta(t)u^1(x)$.

Important also are representations of a solution by convolution of the data with a fundamental solution. One may think of Poisson's or Cauchy's integral formula. A distribution $S \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^n)$, with $\operatorname{supp} S \subset \mathbb{R}_0^+ \times \mathbb{R}^n$, is called *fundamental solution* of $\Box$ for $t \geq 0$ if

$$(\Box S)(t,x) = \delta(t,x) := \delta(t)\,\delta(x).$$

In case of arbitrary domains one defines Green's functions analogously.

It is easy to calculate fundamental solutions for the classical wave equation in $\mathbb{R}^n$. We then obtain the solution of our initial-value problem in the form

$$u(t) = \frac{\partial}{\partial t} I(t)u^0 + I(t)u^1.$$

Here $I(t)$ is the *Huygens operator*,

$$I(t)f := \int_{\mathbb{R}^n} S(t,y)f(x-y))\,dy = \frac{\sin(-\Delta)^{1/2}t}{(-\Delta)^{1/2}}\,f,$$

and $S(t,x) = (I(t)\delta)(x)$.

In $\mathbb{R}^1$ one obtains

$$S^{(1)}(t,x) = \frac{1}{2}\,H(t-|x|) \text{ and } (I^{(1)}(t)v)(x) = \frac{1}{2}\int_{x-t}^{x+t} v(z)\,dz$$

leading to the *d'Alembert formula*. In $\mathbb{R}^2$ one gets

$$S^{(2)}(t,x) = \frac{1}{2\pi}\frac{H(t-|x|)}{\sqrt{t^2-|x|^2}} \text{ and } (I^{(2)}(t)v)(x) = \frac{1}{2\pi}\int_{|y|<t}\frac{v(x-y)}{\sqrt{t^2-|y|^2}}\,dy$$

leading to the *Poisson formula*. In $\mathbb{R}^3$

$$S^{(3)}(t,x) = \frac{1}{4\pi}\frac{\delta(t-|x|)}{|x|} \text{ and } (I^{(3)}(t)v)(x) = \frac{t}{4\pi}\int_{S^2} v(x-tz)\,dz$$

follows. This leads to the *Kirchhoff formula*, named after Gustav Robert Kirchhoff, 1824–87.

Remarkable here is first that because of (we are using $r := |x|$)

$$S^{(n+2)}(t;r) = \frac{-1}{2\pi r}\frac{\partial}{\partial r}S^{(n)}(t;r)$$

all fundamental solutions can be calculated from $S^{(1)}$ and $S^{(2)}$, and second that for uneven $n \geq 3$

$$\operatorname{supp} S^{(n)} = \partial C^+(o)$$

holds. For n even and $n = 1$, however, one obtains

$$\operatorname{supp} S^{(n)} = \overline{C^+(o)}.$$

Here

$$C^+(x_0) := \{(t,x) \mid t > 0 \text{ and } |x-x_0| < t\}$$

is the *future-cone*.

One also says that for uneven $n \geq 3$ *Huygens' principle* holds. This behaviour of the fundamental solutions has large influence on the spreading of waves. Signals set in and out sharply if the support of S is contained in the boundary of the future-cone; otherwise they only set in sharply, and there is afterglow. The propagation speed of a signal originating in x_0 is $|x-x_0|/t = 1$.

To understand wave propagation, the study of singularities is extremely important. In more general cases, however, this is no longer so simple as just described. For instance when dealing with the linear system in elasticity and a homogeneous isotropic medium, we encounter two different directions of propagation with different speeds. The propagation of singularities of solutions is being intensively studied in microlocal analysis. One of the first papers in this theory was written by Lax in 1957 [52]. Deep results were also obtained by Hörmander, see [33]. We also draw the attention to Garding's lectures in 1987 [24].

6 Initial Boundary- and Eigenvalue Problems

Let G be a domain in $\mathbb{R}^n$, and let $a_{ik} = a_{ki}$ $(i,k = 1,\cdots,n)$ be real-valued bounded $C_1(G)$-functions with

$$\exists p_0 > 0 \quad \forall x \in G \quad \forall \xi \in \mathbb{R}^3 \qquad \xi_i a_{ik}(x)\xi_k \geq p_0 |\xi|^2.$$

Since we want to restrict ourselves to the Dirichlet problem, let $A : \mathcal{D}(A) \subset \mathcal{L}^2(G) \longrightarrow \mathcal{L}^2(G)$ with

$$\mathcal{D}(A) := \{u \in \mathring{\mathcal{H}}_1(G) \mid \partial_i a_{ik} \partial_k u \in \mathcal{L}^2(G)\} \text{ and } Au := -\partial_i a_{ik} \partial_k u.$$

We are then looking for a $u : \mathbb{R}_0^+ \times G \longrightarrow \mathbb{R}$ with $u_{tt} + A\,u = 0,\ u(0) = u^0$ and $u_t(0) = u^1$.

As we have already seen, several different solution concepts exist for this initial boundary-value problem. In the following we choose solutions with finite energy. Since A is selfadjoint the problem then is uniquely solved by

$$\begin{aligned} u(t) :&= \cos(A^{1/2}t)\,u^0 \ + \ A^{-1/2}\,\sin(A^{1/2}t)\,u^1 \\ &= \int_0^\infty \{\cos(\sqrt{\lambda}\,t) dP(\lambda)\,u^0 \ + \ \frac{\sin(\sqrt{\lambda}\,t)}{\sqrt{\lambda}} dP(\lambda)\,u^1\}. \end{aligned} \tag{6.1}$$

If one wants to deduce further properties of the solution from this representation, one needs better knowledge of the spectrum of A. For bounded G the representation of A by eigenfunctions and -values and the expansion of u with respect to standing waves follow using spectral theory for compact operators. Such results were obtained by Hilbert and especially by Courant. To prove them one needs a selection or a compact imbedding theorem

$$\mathring{\mathcal{H}}_1(G) \hookrightarrow\hookrightarrow \mathcal{L}^2(G)$$

or in general (to solve the Neumann problem)

$$\mathcal{H}_1(G) \hookrightarrow\hookrightarrow \mathcal{L}^2(G),$$

and shows e.g. that $A^{-1} : \mathcal{L}^2(G) \longrightarrow \mathcal{L}^2(G)$ is a compact mapping.

Such selection theorems originate from the Arzelà-Ascoli-theorem (1900) and go back to Franz Rellich, 1906–1955, in 1930 [77]. In case of the Dirichlet problem they hold for bounded domains without further assumptions on the boundary; generally one needs little regularity of the boundary, namely the segment property.

In 1920 Courant also improved Weyl's asymptotic remainder estimate in Exp. (2) replacing the $o(\lambda^{3/2})$-term by $O(\lambda \log \lambda)$ [11]. The proof is included in Courant–Hilbert [14, p. 385f]. In 1978 Seeley was able to even eliminate the $\log \lambda$ factor thereby obtaining a sharp estimate [87]. It is interesting to notice that in Seeleys proof he makes use of results on the asymptotic theory of solutions to the initial boundary-value problem of the wave equation. So he uses the wave equation in order to obtain properties of the spectrum of Δ. Similarly using the heat equation is much easier, but does not lead to sharp estimates.

In case of unbounded domains we choose *exterior domains*, and assume the medium be homogeneous and isotropic sufficiently far outside. It goes without saying that scattering problems for other types of unbounded boundaries are extremely interesting also, e.g. for the half plane, for periodic structures, or for wave guides. In survey articles Meister 1987 reports on diffraction problems for a wedge [57], and Werner 1987 on resonance phenomena in wave guides [104]. In his books Wilcox 1984 develops a scattering theory for diffraction gratings and a theory of sound propagation in stratified fluids [110, 111].

In case of exterior domains one also needs selection theorems (locally). Dealing with the wave equation, one shows that the point spectrum is empty. This result was obtained by Rellich 1943 [78]. For variable coefficients it is a consequence of Müller's principle of unique continuation of solutions of elliptic equations from 1954 [63]. One then solves exterior boundary-value problems. This was first done using the integral equation method. Sommerfeld's radiation condition from 1912 came in (Arnold Sommerfeld, 1868–1951), uniqueness was shown by Kupradze 1934, and existence proofs were given by Vekua in 1943, Weyl in 1951–52, and by Müller in 1952 and also in 1957 for systems in electromagnetic theory [92, 49, 98, 107, 108, 62, 64]. Afterwards Hilbert space methods were used.

The standard way of dealing with exterior boundary-value problems now is Eidus' *principle of limiting absorption* from 1962 [15]. The underlying idea is to use Stone's formula for obtaining the spectral family of A, and to show that the limits

$$\lim_{\varepsilon\downarrow 0}[A-(s\pm \mathrm{i}\varepsilon)]^{-1}f$$

exist and yield the outgoing and incoming solution respectively.

In doing so one uses weighted Hilbert spaces

$$\mathcal{L}^2_s(G) := \{u\in\mathcal{L}^2_{loc}(G)\ \mid\ (1+|\cdot|)^s u\in\mathcal{L}^2(G)\}$$

getting for $s>0$

$$\mathcal{L}^2_s(G)\hookrightarrow\mathcal{L}^2(G)\hookrightarrow\mathcal{L}^2_{-s}(G),$$

and proves a resolvent estimate in $\mathbb{R}^n$: For $s>1/2$ let

$$\mathcal{L}^{2+}_s(G) := \{u\in\mathcal{L}^2_s(G)\ \mid\ r^{s-1}(\partial_r u-\mathrm{i}\,ku)\in\mathcal{L}^2(G)\},$$

$$Q := (\lambda_1,\lambda_2)\times(0,\tau)\in\mathbb{C}^+\backslash\{0\},$$

$f\in\mathcal{L}^2_s(G)$, and $z\in Q$ with $\operatorname{Re} z = k$. It then follows that the resolvent

$$\begin{aligned} R(\cdot)f\ :\ Q &\longrightarrow \mathcal{L}^{2+}_{-s}(G),\\ z &\longmapsto (-\Delta-z^2)^{-1}f \end{aligned}$$

can be continuously continued to

$$\bar R(\cdot)f\ :\ \bar Q\longrightarrow\mathcal{L}^{2+}_{-s}(G),$$

and that an estimate

$$\exists\,\gamma>0\quad\forall\,z\in\bar Q\quad\forall\,f\in\mathcal{L}^2_s(G)\qquad \|\bar R(z)f\|_{\mathcal{L}^{2+}_{-s}}\le\gamma\,\|f\|_{\mathcal{L}^2_s}$$

holds.

The question naturally arisis, whether one can optimize the value of s. This is of interest, for instance, looking at the Schrödinger equation with long-range potentials. To do so Agmon 1990 [1] and Hörmander 1983 ([33], Vol. II) introduced spaces $\mathcal{B}$, $\mathcal{B}^*$ and $\mathring{\mathcal{B}}^*$ with

$$\mathcal{L}^2_s\hookrightarrow\mathcal{B}\hookrightarrow\mathcal{L}^2_{1/2}\hookrightarrow\mathcal{L}^2\hookrightarrow\mathcal{L}^2_{-1/2}\hookrightarrow\mathring{\mathcal{B}}^*\hookrightarrow\mathcal{B}^*\hookrightarrow\mathcal{L}^2_{-s}.$$

The continuous spectrum of A is absolutely continuous as a consequence of the principle of limiting absorption, saying that to $u,v\in\mathcal{L}^2(G)$ a $F_{u,v}\in\mathcal{L}^1(\mathbb{R})$ exists such that

$$d(P(\lambda)u,v) = F_{u,v}(\lambda)\,d\lambda.$$

This is the starting point for deriving further properties of the solution from representation (6.1).

Let me add a remark concerning other initial boundary-value problems in mathematical physics. Some results I have presented in [54]. The solution theories are quite similar generally speaking. An essential difference and a distinct difficulty only arises proving the corresponding selection theorems. Dealing with Maxwell's equations in domains with non-smooth boundaries, it was clear in the beginning that solutions generally do not belong to $\mathcal{H}_1(G)$ so that Rellich's theorem could not be used for such domains. What one needs is a statement like

$$\mathcal{D}(G) \cap \mathring{\mathcal{R}}(G) \hookrightarrow\hookrightarrow (\mathcal{L}^2(G))^3 \quad \text{or} \quad \mathring{\mathcal{D}}(G) \cap \mathcal{R}(G) \hookrightarrow\hookrightarrow (\mathcal{L}^2(G))^3,$$

where $\mathcal{D}(G)$ and $\mathcal{R}(G)$ denote the Hilbert spaces of fields E with $\operatorname{div} E \in \mathcal{L}^2(G)$ rsp. $\operatorname{curl} E \in (\mathcal{L}^2(G))^3$ and G is bounded. Such theorems have been proved little by little and, finally, in 1993 by Witsch for domains with the *p-cusp property* [113]. Cp. also [115].

A similar problem arises in elasticity. Here Rellich's theorem was used over the years, and in order to do so the estimate

$$\exists\, p > 0 \quad \forall\, U \in \mathcal{H}_1(G) \qquad \|U\|_1^2 \le p\,\{\|U\|^2 + \sum_{j,k} \|\partial_j U_k + \partial_k U_j\|^2\}$$

was taken. This is *Korn's second inequality*. It was formulated by Arthur Korn, 1870–1945, in 1909. An extensive literature on Korn's inequality exists. Its proof is by no means simple. Friedrichs worked on it in 1947 [23]. A proof for domains with strict cone property was given by Fichera in 1972 [17].

Real progress was made only a few years ago. In 1994 Weck was able to prove a local compactness theorem for a larger class of domains without using Korn's inequality [100]. Korn's inequality thus is unnecessary for establishing the standard solution theory. And Weck's prove is much simpler. He uses the idea of Witsch, assumes the strict p-cusp property for $1 \le p < 2$ (for $p = 1$ this is the strict cone property), and gives a representation of vector fields, similar to Sobolev's representation formula, integrating them over a cusp. Locally

$$\mathcal{E}(G) \hookrightarrow\hookrightarrow (\mathcal{L}^2(G))^n$$

then follows, where in $\mathbb{R}^n$

$$\mathcal{E}(G) := \{U \in (C_1(G))^n \cap (\mathcal{W}_1(G))^n;\ \|\cdot\|_{\mathcal{E}}\}^{\sim} \text{ and } \|U\|_{\mathcal{E}}^2 := \|U\|^2 + \sum_{j,k} \|\partial_j U_k + \partial_k U_j\|^2.$$

7 Asymptotic Behaviour of Solutions

Having proved the existence of a solution, one wants to obtain qualitative results also. First one can think of regularity statements. We have met them in Weyl's lemma already. Dealing with time-dependent equations of mathematical physics, it is of special concern, however, to discuss the behaviour of a solution with respect to the time variable.

First one may ask the question for which (positive or negative) values of t a solution exists. This is especially important dealing with non-linear equations. One defines the *lifespan* T of a solution. By this one wants to express that a solution which starts at $t = 0$ exists in $[0, T)$ with a certain regularity.

Looking at linear equations it is normally easy to show that a solution exists in $\mathbb{R}_0^+$ or in $\mathbb{R}_0^-$. Here one is especially interested in obtaining its asymptotic behaviour say for $t \to \infty$. In scattering theory one wants to compare a solution asymptotically with the solution of a simpler

known *reference case*. Such a reference case may be a homogeneous isotropic medium in the whole-space $\mathbb{R}^n$ (free solution). More complicated reference cases are of interest also. One may think of half- or quarter-planes, wave guides, or anisotropic media.

In scattering theory one describes the deviation from the reference case by means of *scattering* or *wave operators*. The starting point is the *principle of local energy decay*. Let $E(u, K, t) := \{\|A^{1/2}\, u(t)\|^2_{\mathcal{L}^2(K)} + \|u_t(t)\|^2_{\mathcal{L}^2(K)}\}$ be the energy in K. It then says

$$\forall K \subset G,\ K \Subset \mathbb{R}^n, \quad \lim_{t\to\pm\infty} E(u, K, t) = 0,$$

and it follows immediately from the absolute continuity of the spectrum of A and the Riemann-Lebesgue lemma. One therefore expects that $u(t)$ behaves like a free solution for sufficiently large t. To be more precise, let $J_G : \mathcal{L}^2(G) \longmapsto \mathcal{L}^2(\mathbb{R}^n)$ with $(J_G\, u)(x) := u(x)$ for $x \in G$, null otherwise, $v^0 := u^0 + \mathrm{i}\, A^{-1/2}\, u^1 \in \mathcal{L}^2(G)$, and $\mathrm{e}^{-\mathrm{i}A_0^{1/2}t}\, v_0$ be a free solution. Then one looks for v_0^+ and $v_0^- \in \mathcal{L}^2(\mathbb{R}^n)$ such that

$$\lim_{t\to\pm\infty} \|J_G\, \mathrm{e}^{-\mathrm{i}A^{1/2}t}\, v^0 - \mathrm{e}^{-\mathrm{i}A_0^{1/2}t}\, v_0^\pm\| = \lim_{t\to\pm\infty} \|\mathrm{e}^{\mathrm{i}A_0^{1/2}t}\, J_G\, \mathrm{e}^{-\mathrm{i}A^{1/2}t}\, v^0 - v_0^\pm\| = 0.$$

Using the absolute continuity of the spectrum of A and special properties of the solution, the existence of unitary wave operators $W^\pm : \mathcal{L}^2(G) \longrightarrow \mathcal{L}^2(\mathbb{R}^n)$

$$W^\pm := \operatorname*{s\text{-}lim}_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}A_0^{1/2}t}\, J_G\, \mathrm{e}^{-\mathrm{i}A^{1/2}t}$$

follows. Therefore $v_0^\pm := W^\pm\, v^0$ exist. $S := W^+(W^-)^*$ is the *scattering operator*, and $v_0^+ = S\, v_0^-$ holds.

There are different methods to prove the existence of wave operators. Lax and Phillips use semigroup theory in 1967 [53], Wilcox the expansion with respect to generalized eigenfunctions in 1975 [109], and Kato in 1976 [41], Belopolskii and Birman in 1968 [5], and also Pearson in 1978 [71] make use of results from perturbation theory. For that purpose one has to show that the special problem can be interpreted as a nuclear perturbation of the free case. More results from scattering theory and especially from quantum scattering theory can be found in the four-volume work of Reed and Simon (1972–1979 [76]).

The $\mathcal{L}^2$-asymptotic just sketched is to rough for many applications. *$\mathcal{L}^p - \mathcal{L}^q$-estimates* furnish a better description of how fast $u(t)$ vanishes with respect to large t. For solutions of the wave equation they read

$$\exists\, N_p,\ n(1 - \frac{2}{q}) \le N_p \le n, \quad \forall\, u^0 \in \mathcal{L}^p_{,N_p}(\mathbb{R}^n) \qquad \|U(t)\|_{\mathcal{L}^q} \le c_q\, (1+t)^{-k(1-2/q)}\, \|U^0\|_{\mathcal{L}^p_{,N_p}}$$

where $q \ge 2$, $p := q/(q-1)$, $U := (\partial_t u, \nabla u)$, $U^0 := (u^1, \nabla u^0)$, and $k := (n-1)/2$. The case $q = 2$ reflects energy conservation, the case $q = \infty$ follows from direct representation of u, and the case $2 < q < \infty$ by interpolation.

Of great interest are also sharper estimates of the local energy decay. Let the support of the initial-values u^0 and u^1 be contained in the compact set $K_1 \Subset G$, and let G again be an exterior domain. One then proves an estimate of the form

$$\exists\, p(t) \quad \forall\, K, K_1 \Subset G \quad \exists\, c,\ t_0 \quad \forall\, t \ge t_0 \qquad E(u, K, t) \le c\, p(t)\, E(u, G, 0) \tag{7.1}$$

where

$$p(t) = \begin{cases} \exp(-\alpha\, t) & \text{for uneven } n \text{ (exponential energy decay)} \\ (1+t)^{-2n} & \text{for even } n \end{cases}$$

and $\alpha > 0$.

To prove it, an essential assumption on the shape of the underlying domain is coming in. G should be *non-trapping* which means

$$\forall\, a \geq a_0 \quad \exists\, \tau = \tau(a, G) > 0 \quad \forall\, u^0 \in \mathcal{L}^2(G(a)) \qquad u \in C_\infty([\tau, \infty) \times \bar{G}(a)).$$

Here we have used $u = \frac{\sin(-\Delta)^{1/2}t}{(-\Delta)^{1/2}}\, u^0$ and $G(a) := G \cap B(0, a)$. Let me stress that the geometrical interpretation of this assumption heavily depends on the special problem. For the wave equation it is fulfilled, if G is convex or starlike. In case of the Neumann problem in elasticity, however, Rayleigh waves exist in the exterior of a ball. So the non-trapping condition is violated there.

So far we have indicated how solutions of initial boundary-value problems can be approximated by free space solutions for large times. As was already said, there are other limiting cases where approximation by simpler solutions is possible. An example is the case of high frequencies which provides the limit towards geometrical optics.

A first approach in this connection was given by Kirchhoff in 1891 [43]. To determine the wave reflected by an obstacle $B \subset \mathbb{R}^3$, he makes the ansatz $u(t, x) = \mathrm{e}^{\mathrm{i}kt} v(x)$, uses the representation of v by means of Green's formula

$$v(x) = \frac{1}{4\pi} \int_{\partial B} \Big\{ \frac{\partial v}{\partial n}(y)\, \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{|x-y|} \;-\; v(y)\, \frac{\partial}{\partial n_y} \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{|x-y|} \Big\} dy,$$

and for the missing boundary-values he takes those from geometrical optics.

The next important step was made by Sommerfeld and Runge in 1911 [93]. They started from *plane waves* $\mathrm{e}^{\mathrm{i}k(\omega x - t)}$ with $\omega \in S^2$, and tried to obtain more general solutions using the ansatz $u(t, x) = \mathrm{e}^{\mathrm{i}k(\varphi(x) - t)}\, v(x)$. In this case u solves the wave equation for all $k > 0$ if

$$(\nabla \varphi)^2 = 1 \tag{7.2}$$

and $2(\nabla\varphi)(\nabla v) + (\Delta\varphi)v = \mathrm{i}(\Delta v)/k$ hold. The approximation now consists in simplifying the second equation by only demanding

$$2\,(\nabla\varphi)\,(\nabla v) + (\Delta\varphi)\, v = o. \tag{7.3}$$

Eq. (7.2) is the *Eikonal Equation*. It was introduced by William Rowan Hamilton, 1805–1865, in his papers on optics between 1827–32. He studies a bundle of light rays emerging from a luminous point y and being reflected at an obstacle B. For any point x of this bundle, the ray through x defines a direction which is given by $\nabla\varphi(x)$. And the value $\varphi(x)$ is the distance of that point x from y along the ray.

The approach of Sommerfeld and Runge, just described, is the simplest version of a more general method for solving differential equations by approximation, containing a parameter, namely the method of asymptotic series. Asymptotic series were already known in the last century. The method which is of interest here is the so-called *WKB-method* from 1926, named after Wentzel [102], Kramers [48], and Brillouin ([8].

To describe it, let $G \subset \mathbb{R}^3$ be an exterior domain again, $y \in G$ be fixed, and $R(t, x, y)$ the corresponding fundamental solution of the Dirichlet problem for the wave equation in G. We want to represent R in the following form

$$R(t, x, y) \sim S^3(t, x - y) + \sum_{n=0}^{\infty} H_{n-1}(t - \varphi(x, y))\, z_n(x, y), \tag{7.4}$$

where $H_{-1}(s) := \delta(s)$ and $H_n(s) := s^n/n!$ for $s \geq 0$, null elsewhere. The H_n satisfy $H_n' = H_{n-1}$. The functions φ and z_n in (7.4) have to be determined in such a way that the differential equation and the boundary condition are fulfilled. Substituting and making each term zero yields $(\nabla\varphi)^2 = 1$ again and $2(\nabla\varphi)(\nabla z_n) + (\Delta\varphi)z_n = -\Delta z_{n-1}$ with $z_{-1} := o$. The boundary condition reads $\varphi(x, y) = |x - y|$, $z_0(x, y) = -1/(4\pi|x - y|)$, and $z_n(x, y) = 0$, $n > 0$, for $x \in \partial G$.

Thus one again obtains equations similar to (7.2–3). Of course, generally the series on the right-hand side of (7.4) will not converge. One is rather concerned with proving a difference $R - R_k$ to be sufficiently regular, where $R_k := r + \sum H_{n-1}(t - \varphi) \cdot z_n$, $0 \leq n \leq k$.

The connection with the Helmholtz equation is obtained through Fourier transformation with respect to t. Thus

$$G(k, x, y) := \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kt} R(t, x, y)\, dt$$

furnishes Green's function to the Helmholtz equation, and its corresponding expansion reads

$$G(k, x, y) \sim \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{4\pi|x - y|} + \mathrm{e}^{\mathrm{i}k\varphi(x,y)} \sum_{n=0}^{\infty} \left(\frac{\mathrm{i}}{k}\right)^n z_n(x, y).$$

For convex obstacles this asymptotic expansion holds for all x which do not lie on a tangential ray. This follows from results of Morawetz and Ludwig in 1968 [59], and from Morawetz, Ralston and Strauss in 1977 [60]. The asymptotic has been studied on tangential rays also. Further details can be found in the survey article of Alber and the author from 1988 [3].

With these and other techniques, methods for solving inverse scattering problems have been obtained also. We refer to the survey articles of Sleeman from 1982 [88] and Colton 1984 [10].

8 Non-Linear Problems

Much has been reported on linear problems so far. But in non-linear analysis many beautiful results have also been achieved, only two of them will be mentioned here. First we shall study simple non-linear conservation laws, and afterwards treat the question of existence of global smooth solutions of non-linear wave equations.

In $\mathbb{R}^1$

$$\partial_t^2 - \partial_x^2 = (\partial_t + \partial_x)(\partial_t - \partial_x)$$

holds. In order to keep the presentation simple we only look at the first factor,

$$u_t + u_x = o.$$

Essential effects can be seen from that. The initial condition reads $u(0, x) = u^0(x)$. Such equations can be solved easily calculating the characteristic curves. Choosing $u^0 := 1$ for $0 < x < 1$, zero otherwise, we obtain $u(t, x) = 1$ for $t < x < t + 1$, zero otherwise. The solution already shows the typical behaviour of a solution of the wave equation. We speak of *contact singularities*. A contact singularity thus is a linear effect.

More general *conservation laws* are of special interest,

$$u_t + f(u)_x = o. \tag{8.1}$$

They usually express that a quantity like energy, mass or impulse will be conserved. Eq. (8.1) and corresponding systems can quite often be found when studying wave phenomena without dissipation. They may possess discontinuous solutions also. This means that we have to give a

suitable concept of solutions first. The theory of such equations goes back to E. Hopf in 1950 [31]. Papers of Oleinik, especially from 1957 [70], and Lax 1954, 1957 follow [50, 51]. A detailed description can be found in Smoller's book from 1983 [89].

In the linear case, $f(u) = u$, we already met contact singularities. More interesting here is the quadratic case, e.g. $f(u) := u^2/2$. Eq. (8.1) with this f is called *Burgers equation*, more precisely *Burgers equation without viscosity*, named after Johannes Martinus Burgers, 1895–1981.

Equations of type (8.1), or with $a(u) := f'(u)$,

$$u_t + a(u)\, u_x = o \tag{8.2}$$

can again be solved looking at their characteristic differential equations. Let $a(u) := u$. Choosing $u^0 := -1$ for $x < 0$ and $u^0 := 1$ for $0 < x$, we obtain $u(t, x) = -1$ for $x < -t$ and $u(t, x) = 1$ for $t < x$. Notice that in $|x| < t$ there are no characteristic lines.

A second typical example is obtained by choosing $u^0 := 1$ for $x < 0$ and $u^0 := -1$ for $0 < x$. The solution now reads $u(t, x) = 1$ for $x < t$ and $u(t, x) = -1$ for $-t < x$. Notice that within $|x| < t$ the characteristic lines meet. In both cases many piecewise smooth solutions in $\mathcal{L}^\infty(\mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R})$ can be constructed.

To get on with this difficulty, one defines *weak solutions* as is usual by

$$\forall \varphi \in \mathring{\mathcal{C}}_\infty(\mathbb{R} \times \mathbb{R}) \qquad \int_{\mathbb{R}^+ \times \mathbb{R}} \{u\, \varphi_t + f(u)\, \varphi_x\} = -\int_{\mathbb{R}} u^0\, \varphi(0, \cdot). \tag{8.3}$$

Here we assume u^0 and u to be bounded and measurable. Weak piecewise smooth solutions cannot jump arbitrarily. Namely, let u jump along the smooth curve $x = x(t)$. From (8.3)

$$s\,[u] = [f(u)] \tag{8.4}$$

then follows by partial integration where $s := dx/dt$ and $[u] := u_l - u_r$, the difference of values from the left and the right. This condition with respect to the jumping-behaviour of solutions has long been known in fluid- and gas dynamics (since 1870–80). It is named *Rankine-Hugoniot condition*.

In case of the Burgers equation it reads $s = (u_l + u_r)/2$. The examples given, however, show that condition (8.4) does not guarantee uniqueness. For this purpose a second condition is necessary. In $\mathbb{R}^1$ one can obtain it in different ways:

1. Using the viscosity method. The solution $u(t, x)$ is defined by $u := \lim u_\varepsilon$ as $\varepsilon \downarrow 0$ where $u_{\varepsilon t} + f(u_\varepsilon)_x = \varepsilon u_{\varepsilon xx}$.
2. The second formulation goes back to Lax in 1957 [51]. It says that characteristics are allowed to meet along discontinuities, but not to bifurcate (with respect to growing time). Thus only *shock waves* are permitted as discontinuities. Let be $a'(u) \neq 0$ to contrast with the linear case. The *shock condition* then reads

$$a(u_l) > s > a(u_r). \tag{8.5}$$

3. In physics *entropy conditions* exist which also lead to condition (8.5).

In $\mathbb{R}^1$ existence and uniqueness of weak solutions (with shock condition) can be proved for $u^0 \in \mathcal{L}^\infty(\mathbb{R})$ transferring the problem into a difference equation. The method of difference approximation for solving partial differential equations was first used 1928 in the famous paper of Courant, Friedrichs, and Lewy [12]. Lax gave such an existence proof for non-linear conservation laws in 1954 [50].

Dealing with a system of two equations in $\mathbb{R}^1$, especially the Riemann problem is of great interest. It originated from gas dynamics. The initial values are constant here, say for $x < 0$ and $x > 0$ respectively. Using the Riemann invariants one can explicitly solve this problem. Shock conditions can be formulated, and uniqueness holds if the difference of the initial values $|u_l^0 - u_r^0|$ is sufficiently small. Similar results hold for corresponding systems of m equations.

Glimm in 1965 proved existence for variable initial values with small total variation and systems in $\mathbb{R}^1$, again using difference approximation [25]. The question of uniqueness does not seem to be completely clarified. For further details we once more refer to the book of Smoller [89] and that of Majda from 1984 [56]. A global existence theory for systems in more than one space-dimension is still missing.

Looking at our examples, the first solution develops a *rarefaction wave* in $|x| < t$, namely $u(t, x) = x/t$, and the second a shock wave, namely $x = 0$. Its solution thus reads $u(t, x) = 1$ for $x < 0$ and $u(t, x) = -1$ for $0 < x$.

Let me stress that non-linear equations may develop singularities after some time although the initial-values are continuous or even smooth. A well known example encountering in practical life is the *traffic equation*

$$N_t + (1 - N)N_x = o \tag{8.6}$$

with $N^0 := 0$ for $x < 0$, $:= x$ for $0 < x < 1$, and $:= 1$ for $1 < x$. The solution is continuous in $0 \le t < 1$, and develops a shock wave in $(1, 1)$.

In the second part of this chapter we want to report on the existence of global smooth solutions of non-linear wave equations. Fritz John, 1910–1994, and his students proved interesting results in this field. Let us look at the equation $y_{tt} - \Delta y = f(Dy, \nabla Dy)$ with $y(0) = y^0$ and $y_t(0) = y^1$ in $\mathbb{R}^n$. We have set $D := (\partial_t, \nabla)$, and also use $u := Dy$ and $u^0 := (y^1, \nabla y^0)$. Thus we obtain $u(0) = u^0$ and

$$u_t - \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix} u = \begin{pmatrix} f(u, \nabla u) \\ 0 \end{pmatrix}. \tag{8.7}$$

In $\mathbb{R}^1$ solutions of the non-linear wave equation always develop singularities as was shown by John in 1974 and 1976 [34, 35]. For large data this generally also holds in $\mathbb{R}^n$. John defined the *life span* of such solutions in 1981 and estimated it [36]. On the other hand Konrad Jörgens, 1926–1974, already showed global existence in $\mathbb{R}^3$ for special $f = f(y)$ with cubic growth and for large data in 1961 [38]. And also there is an example given 1980 by Nirenberg showing global existence in $\mathbb{R}^3$ for an f with quadratic growth and for small data (in [44, p. 45]).

Klainerman proved global existence of smooth solutions for small data and a large class of right-hand sides of Eq. (8.7) in 1980 and 1982 [44, 45]. To do so, he used a local existence theorem going back to Schauder in 1935 [85] and Kato 1975 [40] on the one hand, and then first a relatively complicated iteration scheme with Nash-Moser technique (see Moser 1961 [61] and Hörmander 1976 [32]). In another paper jointly written with Ponce in 1983, they then used $\mathcal{L}^p - \mathcal{L}^q$*-estimates* [47]. The idea is to choose the initial data so small that the life span is large enough, until the linear influence of spreading dominates, and prevents the solution from exploding. This way one can furnish an elegant analytical existence proof. Assuming $f \in C_\infty(\mathbb{R}^{(n+1)^2}, \mathbb{R})$ with

$$|f(u, \nabla u)| = O\Big((|u| + |\nabla u|)^{\alpha+1}\Big) \quad \text{for } |u| + |\nabla u| \to 0$$

and $\alpha = \alpha(n) \in \mathbb{N}$, one can allow $\alpha = 1$ for $n \ge 6$. In case $n = 3$ this only holds for special f satisfying a *null-condition* (Klainerman 1986 [46]). Generally we have $\alpha(5) = \alpha(4) = \alpha(3) = 2$

and $\alpha(2) = 3$. The energy estimate for local solutions used here looks like

$$\forall\, s \in \mathbb{N}_0,\ \exists\, c_s \in \mathbb{R}^+,\ \forall\, t \in [0,T] \quad \|u(t)\|_{\mathcal{L}^2_s} \le c_s \|u^0\|_{\mathcal{L}^2_s} \exp(c_s \int_0^t \left|Du(\tau)\right|^\alpha d\tau).$$

More details can also be found in Kato (1985 [42]), Strauss (1989 [96]), John (1990 [37]), and Racke (1992 [75]).

These results have been extended to initial boundary-value problems. Vainberg uses the meromorphic behaviour of the resolvent of the exterior boundary-value problem in 1989 to prove local energy decay corresponding to Est. (7.1) [97]. Global existence theorems for non-linear equations and smooth boundaries follow. Non-smooth boundaries have also been considered by Cheeger and Taylor in 1982 [9] and Witt in 1995 [114].

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