Lecture 4: The sharp-interface limit of the Allen-Cahn eq.

Today we'll see that the techniques from the previous lectures can be used to answer questions that arise in applications in physics, numerics or data science.

The simplest question related to this is the following:

We consider sol's to the reaction-diffusion equation

$$
(1) \quad \begin{cases} \partial_t u_\varepsilon = \Delta u_\varepsilon - \frac{1}{\varepsilon^2} W'(u_\varepsilon) \\ u_\varepsilon(\cdot, 0) = u_{\varepsilon,0}, \end{cases}
$$

with $W: \mathbb{R} \to \mathbb{R}$ a double well potential, say, $W(u) = u^2(1-u)^2$

called the Allen-Cahn eq. It's the gradient flow of the Cahn-Hilliard energy

$$
E_\varepsilon(u) := \int_{\mathbb{R}^d} \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \, dx
$$

with respect to the scaled L^2-metric

$$
(\delta u, \delta u) := \int_{\mathbb{R}^d} \varepsilon (\delta u)^2 \, dx.
$$

Indeed,

$$
\begin{aligned}
\frac{d}{dt} E_\varepsilon(u_\varepsilon(\cdot, t)) &= \int_{\mathbb{R}^d} \varepsilon \nabla u_\varepsilon \cdot \nabla \partial_t u_\varepsilon + \frac{1}{\varepsilon} W'(u_\varepsilon) \partial_t u_\varepsilon \, dx \\
&= -\int_{\mathbb{R}^d} \varepsilon \left(\Delta u_\varepsilon - \frac{1}{\varepsilon^2} W'(u_\varepsilon) \right) \partial_t u_\varepsilon \, dx \\
&= -\int_{\mathbb{R}^d} \varepsilon (\partial_t u_\varepsilon)^2 \, dx.
\end{aligned}
$$

Heuristics

We see that

$$u_\varepsilon \approx f\left(\frac{\mathrm{sdist}(x, \Sigma_t)}{\varepsilon}\right).$$

If we make the Ansatz

$$u_\varepsilon(x,t) = f\left(\frac{\mathrm{sdist}(x, \Sigma_t)}{\varepsilon}\right),$$

plug it into (1) and collect terms of equal order in ε, we get

$$\partial_t u_\varepsilon = \tfrac{1}{\varepsilon} f' \partial_t s$$
$$\nabla u_\varepsilon = \tfrac{1}{\varepsilon} f' \nabla s$$
$$\Delta u_\varepsilon = \tfrac{1}{\varepsilon^2} (f')^2 |\nabla s|^2 + \tfrac{1}{\varepsilon} f' \Delta s$$

$$\frac{1}{\varepsilon} f'\left(\frac{s}{\varepsilon}\right) \partial_t s = \frac{1}{\varepsilon^2} f''\left(\frac{s}{\varepsilon}\right) + \frac{1}{\varepsilon} f'\left(\frac{s}{\varepsilon}\right) \Delta s - \frac{1}{\varepsilon^2} W'\left(f\left(\frac{s}{\varepsilon}\right)\right)$$

$O\left(\frac{1}{\varepsilon^2}\right)$: $\quad = f'' - W'(f) \rightsquigarrow f$ is the "optimal profile", i.e., f solves the static 1-d problem.

$O\left(\frac{1}{\varepsilon}\right)$: $\quad \partial_t s = \Delta s \rightsquigarrow \Sigma_t$ is a MCF

There are several possibilities to make this argument rigorous. In this class, we'll do this using only the gradient flow structure.

(I) Weak convergence methods

- Ilmanen '91 : Brakke flow (unconditional)
- L-Simon '18 : distributional sol. also vectorial AC (conditional)
- L '20 : optimal energy-dissipation relation (conditional)
- Hensel-L '21 : De Giorgi type varifold sol. (unconditional)
- Steinke '23 : De Giorgi BV sol. & vectorial AC (conditional)

Idea:

$$\frac{d}{dt} E_\varepsilon(u_\varepsilon) = -\frac{1}{2}\int_{\mathbb{R}^d} \varepsilon\left(\partial_t u_\varepsilon\right)^2 dx - \frac{1}{2}\int_{\mathbb{R}^d} \varepsilon\left(\Delta u_\varepsilon - \frac{1}{\varepsilon^2} W'(u_\varepsilon)\right)^2 dx$$

Integrate

$$E_\varepsilon(u_\varepsilon(\cdot,T)) + \frac{1}{2}\int_0^T\int_{\mathbb{R}^d} \varepsilon\left(\partial_t u_\varepsilon\right)^2 dx\,dt + \frac{1}{2}\int_0^T\int_{\mathbb{R}^d} \varepsilon\left(\Delta u_\varepsilon - \frac{1}{\varepsilon^2}W'(u_\varepsilon)\right)^2 dx\,dt$$

$$\leq E_\varepsilon(u_{\varepsilon,0}).$$

This gives compactness via the Modica-Mortola trick

$$\int_{\mathbb{R}^d} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon)\, dx \geq \int_{\mathbb{R}^d} |\nabla u_\varepsilon|\sqrt{2W(u_\varepsilon)}\, dx$$

$$= \int |\nabla \phi\circ u_\varepsilon|\, dx,$$

where $\phi: \mathbb{R} \to \mathbb{R}$ s.t. $\phi'(u) = \sqrt{2W(u)}$.

Similarly,

$$\int_0^T\int \frac{\varepsilon}{2}(\partial_t u_\varepsilon)^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\, dx\, dt \geq \int_0^T\int |\partial_t \phi\circ u_\varepsilon|\, dx\, dt,$$

so $w_\varepsilon := \phi\circ u_\varepsilon \longrightarrow w = \phi\circ u$ in $L^1(\mathbb{R}^d\times(0,T))$.

ϕ is 1-1

Post-process to p-w a.e. & then $u_\varepsilon \longrightarrow u$ in L^1.
& $u = \chi: \mathbb{R}^d\times(0,T) \to \{0,1\}$.

Now only need to pass to the limit in the energy-dissipation inequality:

a) $\liminf_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) \geq \sigma \int_{\mathbb{R}^d} |\nabla \chi|$

b) $\liminf_{\varepsilon \to 0} \int_{\mathbb{R}^d} \varepsilon \left(\Delta u_\varepsilon - \frac{1}{\varepsilon^2} W'(u_\varepsilon) \right)^2 dx \geq \sigma \int_{\mathbb{R}^d} H^2 |\nabla \chi|$

c) $\liminf_{\varepsilon \to 0} \int_0^T \int_{\mathbb{R}^d} \varepsilon (\partial_t u_\varepsilon)^2 \, dx \, dt \geq \sigma \int_0^T \int V^2 |\nabla \chi| \, dt$

d) $\lim_{\varepsilon \to 0} E_\varepsilon(u_{\varepsilon,0}) = \sigma \int |\nabla \chi|$.

To pass to the limit in the nonlinearities, we use the following relative energy

$$\mathcal{E}_\varepsilon [u_\varepsilon | \xi] := E_\varepsilon(u_\varepsilon) + \int_{\mathbb{R}^d} \xi \cdot \nabla(\phi \circ u_\varepsilon) \, dx$$

$$= \int_{\mathbb{R}^d} \left(\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) - |\nabla(\phi \circ u_\varepsilon)| \right) dx$$

$$+ \int_{\mathbb{R}^d} \Big(1 - \xi \cdot \underbrace{\nu_\varepsilon}_{\frac{\nabla(\phi \circ u_\varepsilon)}{|\nabla(\phi \circ u_\varepsilon)|}}\Big) |\nabla(\phi \circ u_\varepsilon)| \, dx ,$$

$$\geq \int_{\mathbb{R}^d} \frac{1}{2} \left(\sqrt{\varepsilon} |\nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(u_\varepsilon)} \right)^2 dx$$

$$+ \int_{\mathbb{R}^d} \frac{1}{2} |\nu_\varepsilon - \xi|^2 |\nabla(\phi \circ u_\varepsilon)| \, dx$$

which again controls oscillations of the normal ν_ε.

In addition, it gives control on the

"equipartition of energy",
namely whether both terms $\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2$ and $\frac{1}{\varepsilon}W(u_\varepsilon)$
carry the same amount of energy.

Thm: For well-prepared initial conditions, i.e.,
(L-Simon, L, Steinke)
- $u_{\varepsilon,0} \longrightarrow u_0 = \chi_{\Omega_0}$ in L^1
- $E_\varepsilon(u_{\varepsilon,0}) \longrightarrow \sigma \int |\nabla \chi_{\Omega_0}|$,

the sol. u_ε to (1) is precompact in $L^1_{loc}(\mathbb{R}^d \times (0,\infty))$. Furthermore,
if $u_\varepsilon \to u = \chi$ and $\int_0^T E_\varepsilon(u_\varepsilon(\cdot,t))\,dt \to \int_0^T \sigma \int |\nabla\chi|\,dt$, then χ is
a De Giorgi sol. to MCF.

② Convergence rate via relative energy inequality

- Fischer - L - Simon '20
- L - Liu '20
- Fischer - Marveggio '22
- L - Stinson - Ullrich '22

Thm: If $\Sigma_t = \partial\Omega_t$, $t \in [0,T]$ is a smooth MCF,
(FLS)
for strongly well-prepared initial data
$\left(\text{e.g., } u_{\varepsilon,0} = f\left(\frac{\mathrm{sdist}(x,\partial\Omega_0)}{\varepsilon}\right)\right)$, and if
the sol. u_ε converges to χ_{Ω_t} in the
quantitative sense

$$\sup_{t\in[0,T]} \| \phi \circ u_\varepsilon - \sigma\chi \|_{L^1} \le C\,\varepsilon.$$

depends on smoothness of $(\Sigma_t)_{t\in[0,T]}$.

Main idea: Show

$$\frac{d}{dt} \mathcal{E}_\varepsilon [u_\varepsilon | \xi] \leq C\, \mathcal{E}_\varepsilon [u_\varepsilon | \xi].$$

Again, the computation is relatively elementary but takes some time. Let's instead try again the "classical" approach

$$\frac{d}{dt} \tilde{\mathcal{E}}_\varepsilon [u_\varepsilon | \varphi] \leq C\, \tilde{\mathcal{E}}_\varepsilon [u_\varepsilon | \varphi],$$

where $\varphi = \frac{1}{2} \operatorname{dist}^2(x, \Sigma_t)$ and

$$\tilde{\mathcal{E}}_\varepsilon [u_\varepsilon | \varphi] := \int_{\mathbb{R}^d} \varphi \underbrace{\left(\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \right)}_{=: e_\varepsilon} dx$$

We compute

$$\begin{aligned}
\partial_t e_\varepsilon &= \varepsilon \nabla u_\varepsilon \cdot \nabla \partial_t u_\varepsilon + \frac{1}{\varepsilon} W'(u_\varepsilon)\, \partial_t u_\varepsilon \\
&= \nabla \cdot \left(\varepsilon \partial_t u_\varepsilon \nabla u_\varepsilon \right) - \left(\varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} W'(u_\varepsilon) \right) \partial_t u_\varepsilon \\
&= \nabla \cdot \left(\underbrace{\left(\varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} W'(u_\varepsilon) \right) \nabla u_\varepsilon}_{\substack{= \varepsilon \Delta u_\varepsilon \nabla u_\varepsilon - \nabla \left(\frac{1}{\varepsilon} W(u_\varepsilon) \right) \\ = \nabla \cdot (\varepsilon \nabla u_\varepsilon \otimes \nabla u_\varepsilon) - \nabla \left(\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 \right)}} \right) - \varepsilon (\partial_t u_\varepsilon)^2 \\
&= -\Delta e_\varepsilon + \nabla \cdot \nabla \cdot \left(\varepsilon \nabla u_\varepsilon \otimes \nabla u_\varepsilon \right) - \varepsilon (\partial_t u_\varepsilon)^2 .
\end{aligned}$$

Recall

$$\partial_t \varphi - \Delta \varphi \le -1 + C\varphi$$

$$D^2 \varphi \le I_d .$$

Then

$$\frac{d}{dt} \tilde{\mathcal{E}}_\varepsilon [u_\varepsilon | \varphi] = \int_{\mathbb{R}^d} \varphi \, \partial_t e_\varepsilon + \partial_t \varphi \, e_\varepsilon \, dx$$

$$= \int_{\mathbb{R}^d} (\partial_t \varphi - \Delta \varphi) e_\varepsilon - D^2 \varphi : \varepsilon \nabla u_\varepsilon \otimes \nabla u_\varepsilon - \varepsilon (\partial_t u_\varepsilon)^2 \, dx$$

$$\le \int_{\mathbb{R}^d} - e_\varepsilon + \varepsilon |\nabla u_\varepsilon|^2 - \varepsilon (\partial_t u_\varepsilon)^2 + C\varphi \, dx.$$

So we get $\le \tilde{\mathcal{E}}_\varepsilon [u_\varepsilon | \varphi]$, provided

$$\varepsilon |\nabla u_\varepsilon|^2 \le e_\varepsilon = \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon} W(u_\varepsilon), \quad \text{or}$$

$$P_\varepsilon := \boxed{\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon} W(u_\varepsilon) \le 0}$$

This can be done by maximum principle for P_ε:

$$(\partial_t - \Delta) P_\varepsilon \le \dots$$