# Lecture 3: Weak-strong uniqueness, relative energy and gradient flow calibrations

Last time we've seen weak solution concepts. The key ingredient to proving the last theorem is the following relative energy

$$\mathcal{E}[\chi \mid \xi] := E[\chi] + \int_{\mathbb{R}^d} \xi \cdot \nabla \chi$$

$$= \int_{\mathbb{R}^d} (1 - \xi \cdot \nu) |\nabla \chi|$$

$$= \int_{\mathbb{R}^d} \frac{1}{2} |\nu - \xi|^2 |\nabla \chi| + \int_{\mathbb{R}^d} (1 - |\xi|^2) |\nabla \chi|.$$

The key are the following two statements.

<u>Lemma 1</u> If $\chi \in BV$ and $\varepsilon > 0$ then $\exists\, \xi \in C_c^1(\mathbb{R}^d)^d$ s.t.

$$\mathcal{E}[\chi \mid \xi] < \varepsilon.$$

<u>Lemma 2</u> If $\chi_n \to \chi$ in $L^1$ and $E[\chi_n] \to E[\chi]$, then

$$\mathcal{E}[\chi_n \mid \xi] \xrightarrow{n \to \infty} \mathcal{E}[\chi \mid \xi]$$

for any $\xi \in C_c^1(\mathbb{R}^d)^d$.

Today, we want to exploit $\mathcal{E}$ to prove the uniqueness:

$\boxed{\begin{array}{l}\text{\underline{Theorem} If } (\Sigma(t) = \partial\Omega(t))_{t\in[0,T)} \text{ is a smooth MCF and} \\ \chi \text{ is a De Giorgi sol. to MCF on } [0,T) \text{ w/ initial} \\ \text{data } \chi_0 = \chi_{\Omega_0}, \text{ then} \\ \qquad \chi(x,t) = \chi_{\Omega_t}(x) \text{ for a.e. } (x,t)\in\mathbb{R}^d\times(0,T).\end{array}}$

The key idea is to use the smoothness of $\Sigma(t)$ to construct a suitable vector field $\xi(x,t)$ s.t.

$$(*) \qquad \frac{d}{dt}\mathcal{E}[\chi|\xi] \lesssim \mathcal{E}[\chi|\xi].$$

This is a lengthy computation — see the arXiv notes — but the idea is very simple. Let's first get some intuition from the static case.

<u>Theorem</u> Suppose $\Sigma = \partial\Omega$ is a smooth surface and $\exists\, \xi \in C^\infty(\bar{D})^d$ s.t.

$$\nabla\cdot\xi = 0 \quad \text{in } \bar{D}$$

"$\xi$ = calibration"

$$\xi = \nu \quad \text{on } \Sigma$$

$$|\xi| \le 1 \quad \text{in } \bar{D}$$

$\int_D |\nabla\chi|$ $\|$

Then $E_D[\chi_\Omega] \le E[\chi]$ $\forall\, \chi$ w/ $\chi = \chi_\Omega$ on $\partial D$.

<u>Pf</u>: $E[\chi_\Omega] = \int_D 1\,|\nabla\chi_\Omega| = \int_D \xi\cdot\nu\,|\nabla\chi_\Omega| = -\int_D \xi\cdot\nabla\chi_\Omega$

$\chi$ $\quad \geq$

$$= \int_D \chi_\Omega \nabla\cdot\xi\,dx + \int_{\partial D}\chi_\Omega\,\nu_D\cdot\xi\,d\mathcal{H}^{d-1}.$$

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The idea is to construct a dynamic version of a calibration to prove (*). Let's at least start the computation:

$$\frac{d}{dt}\mathcal{E}[\chi|\xi] = \underbrace{\frac{d}{dt}E[\chi]}_{\substack{\le -\frac{1}{2}\int V^2|D\chi| \\ -\frac{1}{2}\int H^2|D\chi|}} - \underbrace{\frac{d}{dt}\int_{\mathbb{R}^d}\chi(\nabla\cdot\xi)\,dx}_{\substack{=(\partial_t\chi,\,\nabla\cdot\xi) \\ =\int(\nabla\cdot\xi)V|D\chi| \\ +\int\partial_t\xi\cdot\nu|D\chi|}}$$

$$\le \int\left(-\frac{1}{2}V^2+(\nabla\cdot\xi)V\right)|D\chi|$$

$$+\int -\frac{1}{2}H^2|D\chi|$$

$$+\int\partial_t\xi\cdot\nu\,|D\chi|$$

Idea: $\xi$ = extension of $\nu_\Omega$ = (cutoff fct) $\cdot\,\nabla \mathrm{sdist}(x,\Sigma_t)$

Because the computation is too long, we'll prove a simpler, but weaker, statement.

**Theorem** (Inclusion principle) Suppose $\left(\Sigma(t)=\partial\Omega(t)\right)_{t\in[0,T)}$ is a smooth MCF and suppose $\chi:\mathbb{R}^d\times(0,T)\to\{0,1\}$ satisfies

$$\frac{d}{dt}\int_{\mathbb{R}^d}\varphi\,|D\chi| \le \int_{\mathbb{R}^d}\left(-\varphi H^2 + H\nu\cdot\nabla\varphi + \partial_t\varphi\right)|D\chi|$$

for all smooth $\varphi\ge 0$.

If $|D\chi_{\Omega_0}| = |D\chi(\cdot,0)|$, then $|D\chi(\cdot,t)|\le|D\chi_{\Omega_t}|$ for a.e. $t\in(0,T)$.

**Proof:** • Key idea: Take $\varphi \simeq \frac{1}{2}\mathrm{dist}^2(x, \Sigma_t)$, and show that

$$\tilde{\mathcal{E}} = \int \varphi \, |DX|$$

satisfies

$$\frac{d}{dt}\tilde{\mathcal{E}} \le C\tilde{\mathcal{E}}.$$

Exercise: $s(x,t) := \mathrm{sdist}(x, \Sigma_t)$ solves

$$\partial_t s - \Delta s = 0 \quad \text{on } \Sigma_t$$

and $\varphi$ solves

- $\partial_t \varphi - \Delta\varphi \le -1 + C\varphi$ in a nbhd of $\Sigma_t$.
- $D^2\varphi \le I$ —— " ——

Then we compute

$$\frac{d}{dt}\tilde{\mathcal{E}} \le \int_{\mathbb{R}^d} \left(-\varphi H^2 + H\nu\cdot D\varphi + \partial_t\varphi\right)|DX|$$

$$= \int_{\mathbb{R}^d} \Big(\underbrace{-\varphi H^2}_{\le 0} - \Delta\varphi + \underbrace{\nu\cdot D^2\varphi\,\nu}_{\le 1} + \partial_t\varphi\Big)|DX|$$

$\le -1 + C\varphi$

$$\le \int C\varphi |DX| = C\tilde{\mathcal{E}}. \quad \blacksquare$$

However, the functional $\tilde{\mathcal{E}}$ does not enjoy good coercivity properties. In fact, the above proof applies to any Brakke flow (in place of $X$) so that the argument cannot prove uniqueness.

A better functional would be *

$$\mathcal{F}[\chi|\Omega] := \int_{\mathbb{R}^d} \mathrm{sdist}(\cdot,\partial\Omega)\,(\chi-\chi_\Omega)\,dx$$

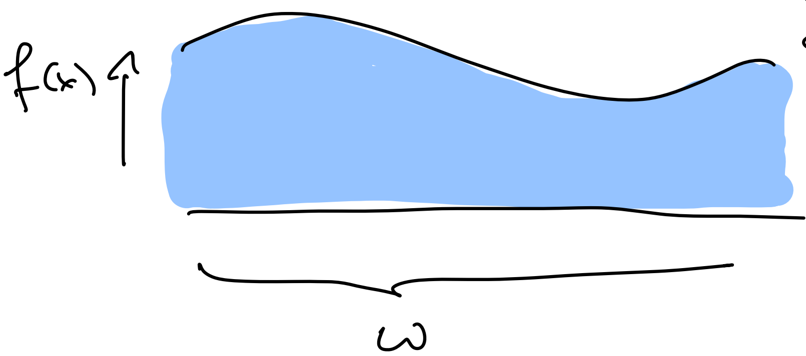
$$= \int_{\mathbb{R}^d} |\mathrm{sdist}(\cdot,\partial\Omega)|\,|\chi-\chi_\Omega|\,dx$$

*: $\mathrm{sdist}(\cdot,\partial\Omega)$ has to be truncated to get a smooth fct — omitted here.

because

$$\mathcal{F}[\chi|\Omega] \implies \chi = \chi_\Omega \text{ a.e. in } \mathbb{R}^d.$$

Also this functional is a proxy for the squared $L^2$-distance of surfaces




In this case

$$2\mathcal{F} = \int_\omega \int_0^{f(x')} 2r\,dr\,dx' = \int_\omega f^2(x')\,dx',$$

just like $\tilde{\mathcal{E}}$, which in this case reads

$$\tilde{\mathcal{E}} = \int_\omega f^2(x')\,dx'.$$

Let's compute

$$\frac{d}{dt} \mathcal{F}[\chi \mid \Omega](t) = \int_{\mathbb{R}^d} \partial_t \, \mathrm{sdist}(x, \partial\Omega(t)) \, (\chi - \chi_\Omega)(x) \, dx$$

$$+ \int_{\mathbb{R}^d} \mathrm{sdist}(x, \partial\Omega(t)) \, V \, |\nabla \chi|$$

$B = H\nu$

$$\leq \int_{\mathbb{R}^d} (B \cdot \nabla) \mathrm{sdist}(x, \partial\Omega(t)) \, (\chi - \chi_\Omega) \, dx$$

$$+ \; C \int_{\mathbb{R}^d} |\mathrm{sdist}(x, \partial\Omega(t))| \, |\chi - \chi_\Omega| \, dx$$

$$+ \int_{\mathbb{R}^d} \mathrm{sdist}(x, \partial\Omega(t)) \, V \, |\nabla \chi|$$

$$\overset{\text{IBP}}{=} \int_{\mathbb{R}^d} \mathrm{sdist}(x, \partial\Omega(t)) \underbrace{(V - B \cdot \nu)}_{\substack{\text{Can be absorbed} \\ \text{into dissipation} \\ \text{of } \mathcal{E}.}} |\nabla \chi|$$

$$+ \; C \, \mathcal{F}$$

$$\Longrightarrow \frac{d}{dt} (\mathcal{E} + \mathcal{F}) \lesssim \mathcal{E} + \mathcal{F}.$$