Lecture 2 Weak solutions, (conditional) closure thm, relating different concepts (1)

* Comparison principle $\rightsquigarrow$ a) <u>Viscosity solution</u> $\partial_t u = \Delta u - \frac{\nabla u}{|\nabla u|}\cdot \nabla^2 u \frac{\nabla u}{|\nabla u|}$

construction via vanishing viscosity $|\cdot| \rightsquigarrow \sqrt{\cdot^2+\varepsilon^2}$

* gradient flow structure $\rightsquigarrow$ b) <u>Brakke flow</u>: $\frac{d}{dt}\int_{\Sigma_t}\varphi\, dS \le \int\left(-\varphi H^2 + H\nu\cdot\nabla\varphi + \partial_t\varphi\right)dS$

(varifolds)

c) <u>distributional sol's</u>: define V & H via distr. eq.'s and impose $V=-H$

(sets of finite perimeter)

d) <u>De Giorgi sol's</u>: define V & H via distr. eq's and impose optimal energy dissipation rate

varifolds or sets of finite perimeter

$$\frac{d}{dt}\mathrm{Area}(\Sigma_t) \le -\frac{1}{2}\int_{\Sigma_t} V^2\, dS - \frac{1}{2}\int_{\Sigma_t} H^2\, dS$$

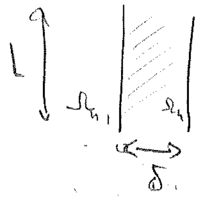
Rmk's

- Almost every level set of a) solves b), c), d) [Evans-Spruck], [L.-Ullrich]
- No connection between b)–d) ((c)+(d) $\overset{?}{\Rightarrow}$ b))
- Drawback of a): It's very restrictive: only one interface, does not apply to more general eq's, ...

- Ways to construct weak sol's: * <u>Implicit time discretization</u>

$h>0$, Ω_0 given, for $n=1,2,\dots$ solve

$$\Omega_n \in \arg\min\left\{\mathcal{H}^{d-1}(\partial\Omega) + \frac{1}{2h}\underbrace{\int_{\Omega\Delta\Omega_{n-1}} 2\,\mathrm{dist}(x,\partial\Omega_{n-1})\,dx}_{\text{proxy for } d^2}\right\}$$

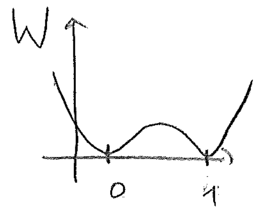


$= L^{d-1}\int_0^\delta 2r\,dr = L^{d-1} r^2$

* <u>Thresholding scheme</u>

* <u>Allen-Cahn equation</u>

$$\begin{cases} \partial_t u_\varepsilon = \Delta u_\varepsilon - \frac{1}{\varepsilon^2}W'(u_\varepsilon), & t>0 \\ u_\varepsilon(\cdot,0) = u_{\varepsilon,0} \end{cases}$$



$u_{\varepsilon,0} \approx 1$ in Ω_0

$u_{\varepsilon,0} \approx 0$ in Ω_0^c

$$\Rightarrow \begin{cases} u_\varepsilon(\cdot,t) \approx 1 \text{ in } \Omega_t \\ u_\varepsilon(\cdot,t) \approx 0 \text{ in } \Omega_t^c \end{cases}$$

How to encode V & H?

* $$\frac{d}{dt}\int_{\Omega_t} \zeta\, dx = \int_{\Omega_t} \partial_t \zeta\, dx + \int_{\Sigma_t} \zeta V \, d\mathcal{H}^{d-1} \quad \forall \zeta \in C_c^1(\mathbb{R}^d \times [0,T])$$

* $$\int_{\Sigma_t} (\nabla\cdot B - \nu\cdot \nabla B\, \nu)\, d\mathcal{H}^{d-1} = \int_{\Sigma_t} H \nu\cdot B \, d\mathcal{H}^{d-1} \quad \forall t \in (0,T),\ \forall B \in C_c^1(\mathbb{R}^d)^d.$$

Can be formalized for less regular objects, too.
For the full story, we would need to introduce varifolds.
Let's instead focus on the simpler class of sets of finite perimeter, or BV functions.

$\Omega_t \rightsquigarrow \chi(\cdot,t) : [0,1)^d \longrightarrow \{0,1\}$

← "flat torus", $[0,1)^d$ equipped w/ periodic bc's

We'll consider

$$\chi : [0,1)^d \times (0,T) \longrightarrow \{0,1\} \quad \text{measurable}$$

but want to understand under which conditions $\{\chi(\cdot,t)=1\}$ is a reasonable family of sets.

Define

$$\int |\nabla \chi| = \int_{[0,1)^d\times(0,T)} |\nabla\chi| := \sup_{\xi} \int_{[0,1)^d\times(0,T)} \chi\, \nabla\cdot\xi \, dx\, dt$$

where the sup runs over all $\xi \in C_c^1([0,1)^d \times (0,T))^d$ with $\sup_{[0,1)^d\times(0,T)} |\xi| \le 1$.

For a.e. $t \in (0,T)$, we also define

$$\int_{[0,1)^d} |\nabla\chi(\cdot,t)| := \sup_{\xi} \int \chi(\cdot,t)\, \nabla\cdot\xi\, dx$$

where the sup runs over all $\xi \in C_c^1([0,1)^d)^d$ w/ $\sup_{[0,1)^d} |\xi| \le 1$.

Def.: We say, χ is of finite (spatial) perimeter if $\int |\nabla\chi| < \infty$.
In that case, $\nabla\chi$ is a (vector-valued) Radon measure, and $|\nabla\chi|$, its total variation, is a finite Radon measure.
We call $\nu := -\nabla\chi/|\nabla\chi|$ (the Radon-Nikodym derivative) the measure theoretic normal.

Exercise: If $\chi = \chi_\Omega$ for some open, bdd set Ω with C^1 boundary.

Show that $\int |D\chi| < \infty$ and

$$\int \zeta |D\chi| = \int_{\partial\Omega} \zeta \, d\mathcal{H}^{d-1} \qquad \forall \zeta \in C([0,1)^d).$$

$$\int \xi \cdot D\chi = \int_{\partial\Omega} \xi \cdot \nu_{\partial\Omega} \, d\mathcal{H}^{d-1} \qquad \forall \xi \in C([0,1)^d).$$

Definition (De Giorgi sol)

$\chi : [0,1)^d \times (0,T) \to \{0,1\}$ measurable is a De Giorgi sol. of MCF w/ initial condition $\chi_0 : [0,1)^d \to \{0,1\}$ if

$$(1) \qquad \operatorname*{ess\,sup}_{t \in (0,T)} \int_{[0,1)^d} |D\chi(\cdot,t)| < \infty$$

and there exists two $|D\chi|$-measurable fct $H, V : [0,1)^d \times (0,T) \to \mathbb{R}$ with

$$(2) \qquad \int V^2 |D\chi| < \infty, \qquad \int H^2 |D\chi| < \infty$$

such that the following holds:

i) V = normal velocity $\quad \forall \zeta \in C^1_c([0,1)^d \times (0,T))$ and a.e. $T' \in (0,T)$

$$(3) \qquad \int_{[0,1)^d} \zeta(\cdot,T') \chi(\cdot,T') dx - \int_{[0,1)^d} \zeta(\cdot,0) \chi_0 \, dx$$
$$= \int \chi \, \partial_t \zeta \, dx dt + \int V \zeta |D\chi|$$

ii) H = mean curvature $\quad \forall B \in C^1_c([0,1)^d \times (0,T))^d$

$$(4) \qquad \int (\nabla \cdot B - \nu \cdot DB \, \nu) |D\chi| = \int H \nu \cdot B \, |D\chi|$$

iii) De Giorgi's ineq. For a.e. $T' \in (0,T)$

$$(5) \qquad \int |D\chi(\cdot,T')| + \frac{1}{2} \int_0^{T'} \int_{[0,1)^d} V^2 |D\chi| + \frac{1}{2} \int_0^{T'} \int_{[0,1)} H^2 |D\chi| \le \int |D\chi_0|$$

Rmk: Initial conditions are encoded weakly in (3) & (5).

Exercise (Consistency): Suppose $\chi = \chi_{\Omega_t}(x)$ for some smoothly evolving set Ω_t. Show that if it is a De Giorgi sol., then it is a classical sol. to MCF.

Exercise (Approximating normal of finite perimeter sets)

Suppose χ is of bounded variation. Show that for any $\varepsilon > 0$ there exists $\xi \in C^1([0,1)^d)^d$ s.t. $|\xi| \leq 1$ in $[0,1)^d$ and

$$\int |\xi - \nu|^2 \, |D\chi| < \varepsilon.$$

Hint: ν is a $|D\chi|$-measurable vector field.

The main result of today's lecture is the following (conditional) closure theorem. It's a ~~the~~ pivot to show convergence of approximations (like implicit time discretization, Allen-Cahn, thresholding, ...).

Theorem Let $\{\chi_n\}_n$ be a seq. of De Giorgi sol's to MCF on a common time interval $(0,T)$ w/ initial data $\chi_{0,n}$ s.t.

$$\chi_{0,n} \longrightarrow \chi_0 \quad \text{in } L^1([0,1)^d)$$

and

$$\int_{[0,1)^d} |D\chi_{0,n}| \longrightarrow \int_{[0,1)^d} |D\chi_0| < \infty.$$

① Then $\{\chi_n\}$ is precompact in $L^1([0,1)^d \times (0,T))$.

② If $\chi_n \to \chi$ in $L^1([0,1)^d \times (0,T))$ and $\int_{[0,1)^d \times (0,T)} |D\chi_n| \longrightarrow \int_{[0,1)^d \times (0,T)} |D\chi|$,

then χ is a De Giorgi sol.

Was developed for showing that thresholding converges to MCF [L-Otto, L-Lelmi]

but also extended to a varifold setting to prove unconditional convergence of Allen-Cahn [Hensel-L].

Proof sketch (Full proof follows arXiv-lecture notes)

1. Compactness follows from the energy-dissipation estimate (5).

The key ingredients: For

- For a.e. $t \in (0,T)$ and all $z \in \mathbb{R}^d$
$$\int_{[0,1)^d} |\chi(x+z,t) - \chi(x,t)|\, dx \leq |z| \int_{[0,1)^d} |\nabla \chi|$$

- For almost all $0 < s < t < T$
$$\int_{[0,1)^d} |\chi(x,t) - \chi(x,s)|\, dx \leq \sqrt{E_0}\,\sqrt{t-s}$$
$E_0 = \int |\nabla \chi_0|$

2. Only have to show
$$\int |\nabla \chi(\cdot,t)| \leq \liminf_{n\to\infty} \int |\nabla \chi_n(\cdot,t)| \quad \text{for a.e. } t$$

a) $V_n |\nabla \chi_n| \overset{*}{\rightharpoonup} V |\nabla \chi|$ & $\int V^2 |\nabla \chi| \leq \liminf_{n\to\infty} \int V_n^2 |\nabla \chi_n|$

b) $H_n \nu_n |\nabla \chi_n| \overset{*}{\rightharpoonup} H\nu |\nabla \chi|$ & $\int H^2 |\nabla \chi| \leq \liminf \int H_n^2 |\nabla \chi_n|$

In our present setting, a) is easier. This is usually the main difficulty because e.g. in the implicit time discretizations we have to work with a proxy for the metric term!
The second statement, b), follows from the following proposition.

Proposition If $\chi_n \to \chi$ in L^1 and $\int |\nabla \chi_n| \to \int |\nabla \chi|$,
then
$$\int (\nabla \cdot B - \nu_n \cdot \nabla B\, \nu_n) |\nabla \chi_n| \longrightarrow \int (\nabla \cdot B - \nu \cdot \nabla B\, \nu) |\nabla \chi|.$$

This is not obvious since
$$\nabla \chi_n = -\nu_n |\nabla \chi_n| \overset{*}{\rightharpoonup} -\nu |\nabla \chi| = \nabla \chi$$
but we have a nonlinear fct of the normal: $\nu_n \cdot \nabla B\, \nu_n |\nabla \chi_n|$

The key is that $\int |\nabla \chi_n| \to \int |\nabla \chi|$ implies strong convergence of the normal!

The main tool is the following relative energy

$$\begin{aligned}\mathcal{E}(\chi,\xi) &:= E(\chi) + \int \xi\cdot D\chi \\ &= \int (1-\xi\cdot\nu)\,|D\chi| \\ &= \int \tfrac{1}{2}|\xi-\nu|^2\,|D\chi| + \int (1-|\xi|^2)\,|D\chi|\end{aligned}$$

which is defined for $\chi : [0,1)^d \to \{0,1\} \in BV$ and $\xi \in C([0,1)^d)^d$.

Exercise: For $\chi \in BV$ as above, and for $\varepsilon > 0$ $\exists\, \xi \in C^\infty([0,1)^d)^d$:

$$\mathcal{E}(\chi,\xi) < \varepsilon.$$

Hint: ν is $|D\chi|$-measurable ...

Lemma If $\{\chi_n\}$ and χ as in the theorem ($\chi_n \to \chi$ in L^1, $\int|D\chi_n| \to \int|D\chi|$ ...) then

$$\mathcal{E}(\chi_n,\xi) \longrightarrow \mathcal{E}(\chi,\xi)$$

for any $\xi \in C^1([0,1)^d)^d$.

Pf:
$$\int \xi\cdot D\chi_n = -\int(\nabla\cdot\xi)\chi_n \longrightarrow -\int(\nabla\cdot\xi)\chi = \int \xi\cdot D\chi_n,$$
$$E(\chi_n) = \int|D\chi_n| \longrightarrow \int|D\chi| = E(\chi). \qquad \square$$

Proof of proposition: Know: $\chi_n \to \chi$ in L^1, $D\chi_n \xrightarrow{*} D\chi$ and $|D\chi_n| \xrightarrow{*} |D\chi|$.

So $\int(\nabla\cdot B)\,|D\chi_n| \longrightarrow \int(\nabla\cdot B)\,|D\chi|$.

The difficulty is the term $\int \nu_n\cdot DB\,\nu_n\,|D\chi_n| \overset{?}{\longrightarrow} \int \nu\cdot DB\,\nu\,|D\chi|$.

Using $\mathcal{E}$, we "freeze" the normal:

$$\int \nu_n\cdot DB\,\nu_n\,|D\chi_n| = \underbrace{\int (\xi\cdot D)B\cdot\nu_n\,|D\chi_n|}_{= -\int(\xi\cdot D)B\cdot D\chi_n \,\to\, -\int(\xi\cdot D)B\cdot D\chi} + O\left(\|DB\|_\infty\,\mathcal{E}^{1/2}(\chi_n,\xi)\right) \qquad \square$$