Lecture notes on my webpage
&
arXiv 2108.08347

The gradient flow structure of <u>mean curvature flow</u>

Heidelberg, (1)
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<u>Lecture 1</u>: Overview, basic examples, motivation from physics & data science

<u>I. Mean curvature flow</u>

Geometric evolution eq. for surfaces / interfaces $(\Sigma_t)_{t\in[0,T))}$

$\Sigma_t = \partial\Omega_t$ for some smooth open bdd set $\Omega_t \subseteq \mathbb{R}^d$, $t\in[0,T)$;

<u>Def.</u>: $(\Sigma_t)_{t\in[0,T)}$ is called a (classical) <u>mean curvature flow</u> if for all $t\in(0,T)$ and all $x\in\Sigma_t$ we have

$$V(x,t) = -H(x,t),$$

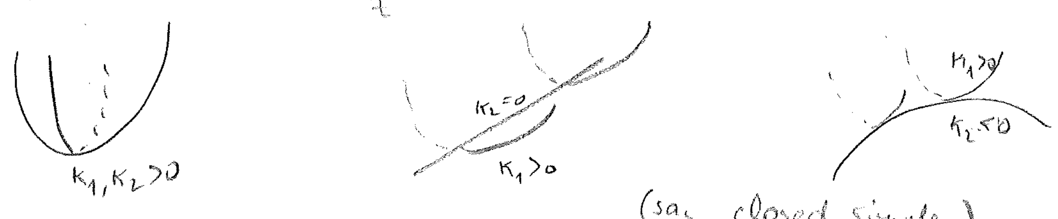
where $V(x,t)$ denotes the <u>normal velocity field</u> and $H(x,t)$ denotes the <u>mean curvature</u> of Σ_t at x.

In terms of a one-parameter family of parametrizations

$F(\cdot,t) : M \longrightarrow \mathbb{R}^d$, s.t. $F(M,t) = \Sigma_t$, this means

$$\nu_{\Sigma_t}\cdot\partial_t F = -\sum_{i=1}^{d-1}\kappa_i$$

(exterior) unit normal to Σ_t ; principal curvatures of Σ_t



<u>Exercise</u>: Compute this for $d=2$, i.e., (say, closed, simple) planar curves.

$$\gamma(\cdot,t) : S^1 \to \mathbb{R}^2,\quad x\mapsto\gamma(x,t)$$

s = arc length $\Rightarrow$ $\tau = \partial_s\gamma$

$\nu = J\partial_s\gamma$

rotation by 90° $J = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$

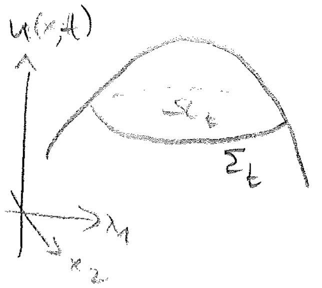
$\kappa = \partial_s\nu\cdot\tau$ ...

You'll see something like the heat equation!

Another way of computing these quantities is via a level set function $u : \mathbb{R}^d \times [0,T) \to \mathbb{R}$:

$$\Sigma_t^{(s)} = \{u(\cdot,t) = s\}, \quad \Omega_t^{(s)} = \{u(\cdot,t) > s\}$$

$$\Rightarrow \quad \nu = -\frac{Du}{|Du|}$$



$$V = \frac{\partial_t u}{|Du|}$$

$$H = -\nabla\cdot\left(\frac{Du}{|Du|}\right)$$

If each level set moves by MCF:

$$\partial_t u = \nabla\cdot\left(\frac{Du}{|Du|}\right)|Du| = \underbrace{\Delta u - \frac{Du}{|Du|}\cdot D^2 u \,\frac{Du}{|Du|}}_{\text{degenerately elliptic operator}}.$$

$$\underbrace{\left(I - \frac{Du}{|Du|}\otimes\frac{Du}{|Du|}\right)}_{\text{has an eigenvalue } = 0} : D^2 u$$

$\rightsquigarrow$ no finite speed of propagation in $\frac{Du}{|Du|}$-direction!

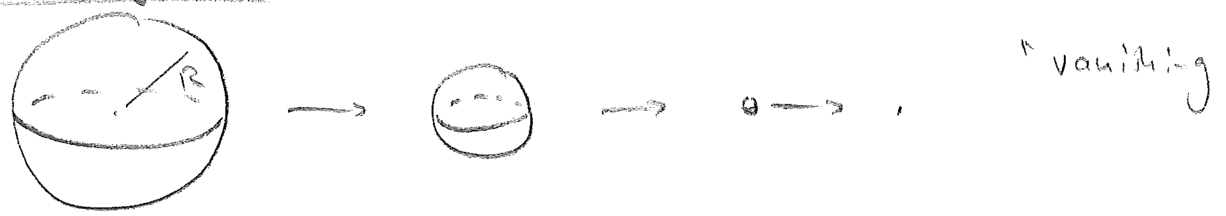
alternative proof: Let $x(t) \in \Sigma_t^{(s)}$ $\forall t$, then

$$0 = \frac{d}{dt} u(x(t),t) = \partial_t u + \dot{x}\cdot Du = \partial_t u - \dot{x}\cdot\nu\,|Du| = \partial_t u + H\,|Du| = \partial_t u - \nabla\cdot\left(\frac{Du}{|Du|}\right)|Du|.$$

Exercise: Suppose $\Sigma_t = \operatorname{graph} f(\cdot,t)$, $f : \mathbb{R}^{d-1}\times[0,T) \to \mathbb{R}$.
If Σ_t is a MCF, which PDE does f solve?
Hint: express ν in terms of Df, ...

Examples

1 Shrinking sphere



$\Omega_0 = B_{R_0}(0)$ "$\Longrightarrow$" $\Omega_t = B_{R(t)}(0)$

$\Sigma_0 = \partial B_{R_0}(0)$ $\qquad$ $\Sigma_t = \partial B_{R(t)}(0)$

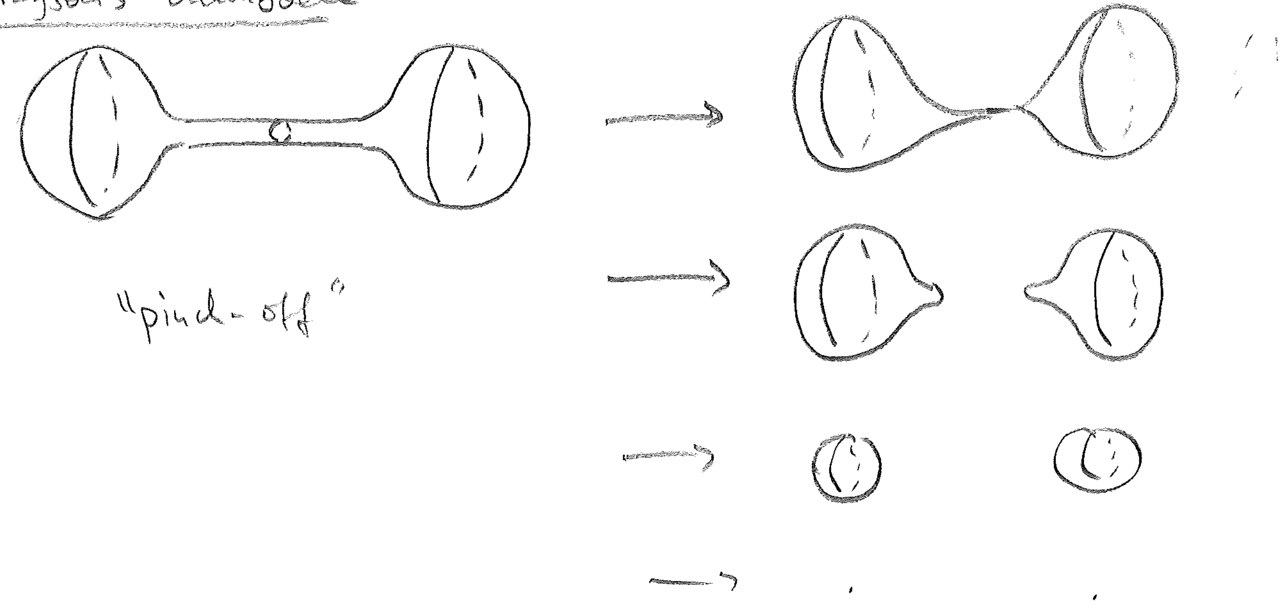
There is something to prove here, but of course it's intuitive that the symmetry is preserved

Q: What's the ODE for $R(t)$?

$$\begin{cases} \dot{R} = -\dfrac{d-1}{R}, & t \in [0, \dots) \\ R(0) = R_0 \end{cases}$$

A: $R(t) = \sqrt{R_0^2 - 2(d-1)t}$

2 Grayson's dumbbell



II. MCF – a gradient flow?

<u>Folklore</u>: MCF is "the" gradient flow of the area functional.

Rayleigh, ..., De Giorgi, Otto, ...

A <u>gradient flow</u> is a steepest descent in an energy landscape:

$$E : X \to \mathbb{R} \qquad (\text{think of } X = \mathbb{R}^N \text{ for the moment})$$

Let $x : (0,T) \to X$ be a solution to

$$(1) \quad \begin{cases} \dot{x}(t) = -\nabla E(x(t)), & t \in (0,T), \\ x(0) = x_0. \end{cases}$$

Then

$$\frac{d}{dt} E(x(t)) = \nabla E(x(t)) \cdot \dot{x}(t) = -|\nabla E(x(t))|^2 \leq 0$$

or

$$\frac{d}{dt} E(x(t)) = -\frac{1}{2}|\dot{x}(t)|^2 - \frac{1}{2}|\nabla E(x(t))|^2,$$

hence

$$(2) \quad E(x(T)) + \int_0^T \frac{1}{2}\left(|\dot{x}(t)|^2 + |\nabla E(x(t))|^2\right) dt \leq E(x_0).$$

I.e., (1) $\Rightarrow$ (2). Remarkably, (2) $\Rightarrow$ (1)!

Indeed, (2) implies

$$0 \geq \int_0^T \Big(\underbrace{\frac{d}{dt} E(x(t))}_{= \dot{x}(t) \cdot \nabla E(x(t))} + \frac{1}{2}|\dot{x}(t)|^2 + \frac{1}{2}|\nabla E(x(t))|^2 \Big) dt$$

$$= \int_0^T |\underbrace{\dot{x} + \nabla E(x(t))}| ^2 \, dt.$$

$\Rightarrow = 0$ for (almost) every $t \in (0,T)$.

$X = (M, g) \leadsto \nabla E(x)$ has to be a tangent vector to M at x.

$$\leadsto \underbrace{\langle \nabla E(x), v \rangle}_{= g(\nabla E(x), v)} = dE(x) \cdot v$$

So ∇E depends on E but also on g!

Back to MCF: Formally,

$$\mathcal{M} = \{ \Sigma = \partial\Omega \text{ smooth surface in } \mathbb{R}^d \}$$

$$E(\Sigma) = \text{Area}(\Sigma) = \mathcal{H}^{d-1}(\Sigma) = P(\Omega)$$

$$T_\Sigma \mathcal{M} \cong \{ V : \Sigma \to \mathbb{R} \}$$

$$g_\Sigma(V,W) = \int_\Sigma VW \, dS = \int_\Sigma V(x) W(x) \, d\mathcal{H}^{d-1}(x).$$

Indeed, then – formally – MCF is the gradient flow of E in $(\mathcal{M}, g)$: in general if $(\Sigma(t))$ is smoothly evolving w/ normal velocity V, then

$$\frac{d}{dt} \text{Area}(\Sigma(t)) = \int_\Sigma VH \, dS.$$

Exercise: Check this for
1) planar curves
2) graphs

So, if $(\Sigma(t))_{t\in[0,T)}$ is a MCF, then

$$\frac{d}{dt} E(\Sigma(t)) = -\frac{1}{2}\int_\Sigma V^2 dS - \frac{1}{2}\int_\Sigma H^2 dS.$$

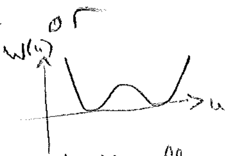
This formal picture can – to some extent – be made rigorous: This lecture series.

III. Motivation from physics & data science

PHYSICS APPLICATIONS

MCF appears in a lot of continuum mechanics models with a (slow) relaxation due to surface tension:

- appears as a scaling limit of several micro- or mesoscopic models



double-well potential

  - Allen-Cahn eq.: $\partial_t u_\varepsilon = \Delta u_\varepsilon - \frac{1}{\varepsilon^2} W'(u_\varepsilon)$
  - other phase transitions in vectorial models (e.g. isotropic – nematic phase transition in liquid crystals),
  - energy concentration measures (e.g. Landau-Lifshitz eq.)
  - Ising model (at subcritical temperature

- Was first introduced as a model to describe the slow relaxation of grain boundaries in polycrystals under heat treatment [Mullins]

DATA SCIENCE APPLICATIONS

- Finding optimal cuts in large graphs built on unstructured data clouds to partition a data set into "clusters" is a basic task in data science. For large graphs, these cuts correspond to (local) minimizers of (a weighted) area functional. To find such minimizers, one often uses some kind of gradient descent — which will share lots of features w/ MCF