Lecture notes for the course V5B4 - Selected Topics in PDE and Mathematical Models -

Spectral theory and random differential operators

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In this script, the content of the lectures is given in a compressed form and may include misprints. Corrections are welcome and may be sent to the email address of the lecturer:

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This script cannot substitute textbooks, monographs, and research papers that are the base of the course.

Lighting barrage [O].

The main books and papers that were used in the preparation of this course are:

[AG] Akhiezer, N.I. and Glazman, I.M., 1993. Theory of linear operators in Hilbert space. Dover.

[DZ] Dyatlov, S. and Zworski, M., 2019. Mathematical theory of scattering resonances (Vol. 200). American Mathematical Soc..

[KMN19] Kaminaga, M., Mine, T., Nakano, F., 2019. A self-adjointness criterion for the Schrödinger operator with infinitely many point interactions and its application to random operators, F. Ann. Henri Poincaré 21 (2), 405-435.

[Kato] Kato, T., Perturbation theory for linear operators. Springer Science & Business Media, 2013.

[LP] Last, G. and Penrose, M., 2017. Lectures on the Poisson process. Cambridge University Press.

[MR] Meester, R. and Roy, R., 1996. Continuum percolation. Cambridge University Press.

[RS1] Reed, M. and Simon, B., 1980. Methods of modern mathematical physics. Vol. I: Functional analysis. Academic Press.

[RS4] Reed, M. and Simon, B. Methods of modern mathematical physics. IV: Analysis of operators. Academic Press, New York, 1978.

The list of all references used in the course is given at the end of these notes.

Contents

1 Overview. Random wave equations and stochastic point processes (SPPs).

1.1 Two types of stochastic models for resonators.

One can consider several types of models for wave equations in uncertain or random structures:

- Type 1. A stochastic medium in a bounded domain G is surrounded in $\mathbb{R}^d \backslash G$ by a deterministic isotropic homogeneous outer medium.
- Type 2. A deterministic structure in G is surrounded by uncertain (or unknown) structure that is modeled stochastically.
- Type 3. The medium is stochastic in the whole $\mathbb{R}^d$. In the contexts of self-averaging and Anderson localization such models are much better studied than Type 1 and Type 2.

Sometimes spectral properties of a structure of Type 1, Type 2, or Type 3 can be described via a stochastic point process (in short, SPP) in $\mathbb{R}$ or in $\mathbb{C} \cong \mathbb{R}^2$ or via a family of such SPPs. These SPPs will represent random (multi-)sets of random eigenvalues or random resonances.

Since one of the ways to define stochastic media also involve SPPs in $\mathbb{R}^d$, SPPs become one of the main topics of the course. We will pay a considerable attention to related definitions, examples, and basic properties.

1.2 Main ideas about (stochastic) point processes.

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be our underlying (complete) probability space.

SPPs in $\mathbb{R}^d$ are one of the rigorous ways to describe random collection of points in $\mathbb{R}^d$, see, e.g., the textbook [LP].

Denote by $\mathfrak{N}_{<\infty}$ the set of all $\mathbb{N}_0$-valued Borel measures in $\mathbb{R}^d$, where $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. Let

$$\widehat{\mathbb{N}}_0 := \mathbb{N}_0 \cup \{+\infty\}.$$

Let $\mathfrak{N}$ be the set of all $\widehat{\mathbb{N}}_0$-valued measures that can be written as at most countable sums of measures from the set $\mathfrak{N}_{<\infty}$.

Definition 1.1 (SPP)**.**
A (stochastic) point processes in $\mathbb{R}^d$ is a random element of $\mathfrak{N}$ (random here means measurable in an appropriate probabilistic sense that will be specified later in this course).

This short definition is a bit too abstract and too general for most of our needs. We mainly use the following subclass of SPPs.

Definition 1.2 (proper SPP)**.**
A proper point processes in $\mathbb{R}^d$ is a random measure $\eta : \omega \mapsto \eta_\omega$, $\omega \in \Omega$, of the form

$$\eta_\omega = \sum_{j=1}^{\varkappa} \delta_{y_j(\omega)} = \sum_{j=1}^{\varkappa} \delta(x - y_j(\omega)), \qquad \omega \in \Omega,$$

where $y_1, y_2, \dots,$ are $\mathbb{R}^d$-valued random variables, $\varkappa$ is an $\widehat{\mathbb{N}}_0$-valued random variable, and $\delta_z = \delta(x - z)$ is a Dirac measure placed at the site $z \in \mathbb{R}^d$.

We use the convention that $\sum_{j=1}^{0} = 0$. The random variable $\varkappa$ in Definition 3.3 represents the random number of points in the proper SPP η. Since $\varkappa$ is probabilistically measurable, the set $\{\varkappa = 0\} = \{\omega \in \Omega : \varkappa(\omega) = 0\}$ belongs to the σ-algebra $\mathbb{F}$ of all events. For $\eta = \sum_{j=1}^{\varkappa} \delta_{y_j}$, the convention $\sum_{j=1}^{0} = 0$ means that we have $\eta_\omega = 0\mathrm{d}x$ (the zero measure) for all $\omega \in \{\varkappa = 0\}$.

Example 1.1 (homogeneous Poisson process on S with $|S|_d < \infty$)**.**
Let $S \subset \mathbb{R}^d$ be a Borel subset with a finite (d-dimensional Lebesgue) measure $|S|_d \in \mathbb{R}_+ = (0, +\infty)$. Let $y_1, y_2, \dots$ be independent random variables uniformly distributed in S. Let $\varkappa$ be an $\mathbb{N}_0$-valued random variable with the Poisson distribution $\mathrm{Po}(\gamma)$, where $\gamma \geqslant 0$ is a rate- (or intensity-) parameter. That is,

$$\mathbb{P}\{\varkappa = n\} = \frac{\gamma^n}{n!}\mathrm{e}^{-\gamma}, \qquad n \in \mathbb{N}_0.$$

Then $\eta = \sum_{j=1}^{\varkappa} \delta_{y_j}$ is a homogeneous Poisson point process with the rate (or intensity) equal to $\frac{\gamma}{|S|_d}$.

Remark 1.1.
Strictly speaking, Example 1.1 is not a definition. However, it is almost a definition (up to equality in distributions), see Lecture 5.

1.3 Examples of random Schrödinger operators of Type 1.

We denote by $\mathbb{R}_+ = (0, +\infty)$ the positive half-line and put $\overline{\mathbb{R}}_+ := [0, +\infty)$.

Example 1.2 (1-d model of an amorphous solid slab, cf. [LGP, Sections 1.1.1 and 7])**.**
Let $d = 1$ and let $G = (x_-, x_+)$ be a bounded interval in $\mathbb{R}$. Let $V_0 \in L^\infty_{\mathrm{comp}}(\mathbb{R}, \overline{\mathbb{R}}_+)$ be an $\overline{\mathbb{R}}_+$-valued L^∞-potential with a compact support $\operatorname{supp} V_0$ in $\mathbb{R}$. Let η be a homogeneous Poisson SPP from Example 1.1 with a certain rate $\gamma > 0$. Let us introduce a *random potential* $V(x) = V_\omega(x)$, $\omega \in \Omega$, by

$$V_\omega(x) = \int V_0(x - y)\eta_\omega(\mathrm{d}y) = \sum_{j=1}^{\varkappa} V_0(x - y_j(\omega)).$$

The associated random 1-d Schrödinger operator $\mathcal{H}_\omega = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\omega$, $\omega \in \Omega$, is defined by

$$\mathcal{H}_\omega u(x) = -u''(x) + V_\omega(x)u(x), \quad u \in \operatorname{dom}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right) = H^2(\mathbb{R}),$$

where by $\operatorname{dom} A$ we denote the domain (of definition) for a possibly unbounded operator

$$A : \operatorname{dom} A \subseteq X \to X$$

in a Hilbert or Banach space X. In our case, $X = L^2(\mathbb{R}) = L^2(\mathbb{R}, \mathbb{C})$ is a complex Hilbert space.

Recall that the Sobolev space

$$H^n(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : f^{(j)} \in L^2(\mathbb{R}), 0 \leqslant j \leqslant n\}$$

is a Hilbert space with the norm given by $\|f\|^2_{H^n(\mathbb{R})} = \sum_{j=0}^n \int_{\mathbb{R}} |f^{(j)}|^2 \mathrm{d}x$.

In Example 1.2, G is bounded and V_0 has a compact support. So, the support $\operatorname{supp} V_\omega \subseteq \overline{G} + \operatorname{supp} V_0$ is almost surely (a.s.) compact. That is, a.s. $V_\omega(x) = 0$ for all $x \notin \mathcal{I}$ for a certain compact interval $\mathcal{I}$. This means that outer medium in $\mathbb{R}\backslash\mathcal{I}$ is homogeneous, while the structure $V_\omega(x)$, $x \in \mathcal{I}$, of the medium inside $\mathcal{I}$ is stochastic.

Example 1.3.
It is possible to simplify this construction to another random Schrödinger operator

$$\mathcal{H}_\omega = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + c\eta = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + c\sum_{j=1}^{\varkappa} \delta_{y_j}$$

with the potential V given by a sum of randomly placed δ-functions multiplied on a coupling constant $c > 0$.

Remark 1.2.
Example 1.2 can be without essential changes generalized for $\mathbb{R}^d$ with any $d \in \mathbb{N}$. Example 1.3 can be generalized to $\mathbb{R}^d$ with $d = 2$ and $d = 3$ with some changes needed for the interpretation of δ-potentials. For $d \geqslant 4$, δ-potentials supported at separated points are not well-defined (at least in the sense of Schrödinger operators in $L^2(\mathbb{R}^d)$).

For models of Type 1, it is important that the random potential has a compact support. Let us say a few words about some cases where the support of V is not compact.

Remark 1.3.

(a) Assume that we replace bounded G with $\widetilde{G} = \mathbb{R}^d$ in Example 1.2. Then we obtain a well-known Poisson-Anderson model for random media with strong structural disorder [LGP, S95, BS01]. This model is used for amorphous (or non-crystalline) solids. For small intensity-parameter $\gamma > 0$, the model can also be used for the description of random impurities in a homogeneous medium.

(b) Let us concentrate on the simplest case where $d = 1$ and $\widetilde{G} = \mathbb{R}$. Let η be a homogeneous Poisson SPP with rate $\gamma > 0$ on the whole $\mathbb{R}$. Then, it is known that for 1-d Poisson-Anderson operator $\mathcal{H}_\omega = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\omega$ there exists a deterministic set $S \subset \overline{\mathbb{R}}_+$ such that $S = \sigma(\mathcal{H}_\omega)$ with probability 1. Moreover, the complete spectral Anderson localization takes place. This means that a.s. $S = \sigma(\mathcal{H}_\omega) = \sigma_{\mathrm{pp}}(\mathcal{H}_\omega)$, $\sigma_{\mathrm{ac}}(H_\omega) = \sigma_{\mathrm{sc}}(H_\omega) = \varnothing$, and all eigenfunctions of $\mathcal{H}_\omega$ decay exponentially as $x \to \pm\infty$.

(c) The statement in (b) that a.s. $\mathcal{H}_\omega$ has deterministic purely point spectrum *S (i.e.,a.s. $S = \sigma(H_\omega) = \sigma_{\mathrm{pp}}(\mathcal{H}_\omega)$ and $\sigma_{\mathrm{ac}}(H_\omega) = \sigma_{\mathrm{sc}}(H_\omega) = \varnothing$) can be reformulated in the following more elementary way. For almost all (a.a.) $\omega \in \Omega$, there exists an orthonormal basis of eigenfunctions $\{u^j_\omega\}_{j\in\mathbb{N}}$ of H_ω such that*

$$\mathcal{H}_\omega u = \sum_{j\in\mathbb{N}} k_j(\omega)(u|u^j_\omega)_{L^2(\mathbb{R})} u^j_\omega \qquad \forall u \in \operatorname{dom}(\mathcal{H}_\omega) = H^2(\mathbb{R})$$

and a.s. $\overline{\{k_j(\omega)\}_{j\in\mathbb{N}}} = S$. The complete spectral Anderson localization includes additionally the statement that a.s. all eigenfunctions $u^j_\omega(x)$ decay exponentially as $x \to \pm\infty$.

(d) The complete Anderson localization for 1-d Poisson-Anderson operator $\mathcal{H}_\omega$ follows

from the papers [S95, BS01]. First mathematically rigorous results on spectral Anderson localization go back to [GMP77].

Formally, we have not defined yet Poisson SPP on $\mathbb{R}$ or $\mathbb{R}^d$. This will be done later in the course. Roughly speaking, it is a sum of independent Poisson SPP on the unit sells $x + [0,1)^d$, $x \in \mathbb{Z}^d$ (see Lemma 6.1).

1.4 Spectral properties of open systems and (continuation) resonances.

The spectral theory is another component of this course. We will address some basics of the spectral theory of selfadjoint operators.

Let $A : \operatorname{dom} A \subseteq X_1 \to X_2$ be an abstract operator between Hilbert or Banach spaces X_1 and X_2. The Banach space of all bounded operators A from X_1 to X_2 with $\operatorname{dom} A = X_1$ will be denoted $\mathcal{L}(X_1, X_2)$. If we write $A : X_1 \to X_2$, this means that $\operatorname{dom} A = X_1$.

Consider now the case $X_1 = X_2 = X$. In this case, one says that $A : \operatorname{dom} A \subseteq X \to X$ is an operator in X. We use the notation $\mathcal{L}(X) := \mathcal{L}(X, X)$. The set $\rho(A)$ of all $k \in \mathbb{C}$ such that the inverse operator $(A-k)^{-1} := (A-kI)^{-1}$ exists and belongs to $\mathcal{L}(X)$ is called the resolvent set of A. The $\mathcal{L}(X)$-valued function $k \mapsto (A-k)^{-1}$ defined for $k \in \rho(A)$ is called the resolvent of A. The resolvent $(A-k)^{-1} : \rho(A) \to \mathcal{L}(X)$ is an analytic $\mathcal{L}(X)$-valued function, see, e.g., [Kato, RS1].

The spectrum $\sigma(A)$ of A is defined as

$$\sigma(A) := \mathbb{C} \backslash \rho(A).$$

The set of eigenvalues $\sigma_{\mathrm{p}}(A)$ of A is a subset of $\sigma(A)$.

If $A = A^*$ in a Hilbert space X, one can define the absolutely continuous spectrum $\sigma_{\mathrm{ac}}(A)$, the singular continuous spectrum $\sigma_{\mathrm{sc}}(A)$ and the pure point spectrum $\sigma_{\mathrm{pp}}(A)$, see Lecture 13. In this case, the following equalities are valid

$$\overline{\sigma_{\mathrm{p}}(A)} = \sigma_{\mathrm{pp}}(A)$$

and

$$\sigma(A) = \sigma_{\mathrm{ac}}(A) \cup \sigma_{\mathrm{sc}}(A) \cup \sigma_{\mathrm{pp}}(A).$$

(Sometimes, e.g., in [RS1], the pure point spectrum $\sigma_{\mathrm{pp}}(A)$ is defined as the set of eigenvalues; in this case, the last equality is written as $\sigma(A) = \sigma_{\mathrm{ac}}(A) \cup \sigma_{\mathrm{sc}}(A) \cup \overline{\sigma_{\mathrm{pp}}(A)}$).

Exercise 1.1.
Let us take the Hilbert space $X = L^2(0,1)$. Let $\mathcal{M}_f : L^2(0,1) \to L^2(0,1)$ be the operator of multiplication on $f \in L^\infty(\mathbb{R})$, i.e., $\mathcal{M}_f : u \mapsto fu$. In the case $f(x) = x$, $x \in [0,1]$, we obtain the operator $\mathcal{M}_x$, i.e., $\mathcal{M}_x\, u(x) = xu(x)$ for all $u \in L^2(0,1)$. Show that $\sigma_p(\mathcal{M}_x) = \varnothing$, but $\sigma(\mathcal{M}_x) = [0,1]$.

Exercise 1.2.
In the Hilbert space $\ell^2(\mathbb{N}) = \ell^2(\mathbb{N}, \mathbb{C})$, let us consider the operator A defined by $A(u_n)_{n\in\mathbb{N}} = (u_n/n)_{n\in\mathbb{N}}$. Then $\sigma_p(A) = \{1/n\}_{n\in\mathbb{N}}$, but $\sigma(A) = \{0\} \cup \{1/n\}_{n\in\mathbb{N}}$.

Returning to the examples of Schrödinger operators, consider a deterministic finite collection of points $y_1, \ldots, y_\varkappa \in \mathbb{R}$ and the associated Schrödinger operator defined as

above

$$\mathcal{H} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{j=1}^{\varkappa} V_0(x - y_j).$$

Since we have assumed that $V_0 \in L^\infty_{\mathrm{comp}}(\mathbb{R}, \overline{\mathbb{R}}_+)$, we see that $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}, \overline{\mathbb{R}}_+)$.

Theorem 1.1.
Assume that $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}, \overline{\mathbb{R}}_+)$. *Consider* $\mathcal{H} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V$ *with* $\operatorname{dom}\mathcal{H} = H^2(\mathbb{R})$. *Then* $\mathcal{H} = \mathcal{H}^*$ *and*

$$\sigma(\mathcal{H}) = \overline{\mathbb{R}}_+ = \sigma_{\mathrm{ac}}(\mathcal{H}).$$

This result follows from much more general [RS3, Theorem XI.30].

In other words, in the case $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}, \overline{\mathbb{R}}_+)$, the spectrum of $\mathcal{H} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V$ gives not so much of information about $\mathcal{H}$. To get more interesting spectral properties for $\mathcal{H}$, let us consider (continuation) resonances.

For simplicity, we keep the assumption $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}, \overline{\mathbb{R}}_+)$ (many of the subsequent basic statements about resonances can be adapted with some small changes to the cases $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^d)$, $V \in L^1_{\mathrm{comp}}(\mathbb{R})$, or even to V with a sufficiently fast decay at ∞).

We replace the spectral parameter k with $k = \lambda^2$ and consider for $\lambda \in \mathbb{C}_+ := \{\operatorname{Im} z > 0\}$ one more version of the resolvent-function

$$\mathcal{R}_{\mathcal{H}}(\lambda) = (\mathcal{H} - \lambda^2)^{-1}, \qquad \mathcal{R}_{\mathcal{H}} : \mathbb{C}_+ \to \mathcal{L}(L^2(\mathbb{R})).$$

By Theorem 1.1, $\mathcal{R}_{\mathcal{H}}(\lambda)$ does not exists as an $\mathcal{L}(L^2(\mathbb{R}))$-valued function for $\lambda \in \mathbb{R}$. However, it is possible to continue $\mathcal{R}_{\mathcal{H}}(\lambda)$ from $\mathbb{C}_+$ meromorphically through $\mathbb{R}$ to $\mathbb{C}_- = \{\operatorname{Im} z < 0\}$ in a certain generalized way.

Theorem 1.2 ([DZ])**.**
Assume that $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}, \overline{\mathbb{R}}_+)$ *and* $\mathcal{H} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V$. *Then* $\mathcal{R}_{\mathcal{H}}(\lambda)$, $\lambda \in \mathbb{C}_+$, *can be continued to the whole* $\mathbb{C}$ *meromorphically as an* $\mathcal{L}(L^2_{\mathrm{comp}}(\mathbb{R}), L^2_{\mathrm{loc}}(\mathbb{R}))$*-valued function* $\mathcal{R}^{\mathrm{cont}}_{\mathcal{H}}(\lambda)$.

The resonances are the poles of this generalized meromorphic continuation.

The Physics meaning of resonances is connected with the description of the long-time behaviors of solutions and with the rate of decay of energy contained inside of the resonator (i.e., inside of G or inside of the support $\operatorname{supp} V$ of the potential).

If the operator $H_\omega = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\omega$ is randomized in a reasonable way, the set of resonances becomes random and can be sometimes described by a locally finite SPP on $\mathbb{C}$.

References for Lecture 1.

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2 Overview (continuation). Eigenvalues and resonances in random and deterministic open systems.

2.1 Elementary definition of resonances.

There is a great variety of resonance type effects in Mathematical Physics. One particular type that describe spectral properties of open systems and is called continuation resonances (in the sequel, simply *resonances*).

In Lecture 1, the discussion of resonances for a particular type of deterministic Schrödinger operators has been started. We consider a deterministic finite collection of points $y_1, \dots, y_\varkappa \in \mathbb{R}$ and the Schrödinger operator

$$\mathcal{H} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{j=1}^{\varkappa} V_0(x - y_j),$$

where $V_0 \in L^\infty_{\mathrm{comp}}(\mathbb{R}, \overline{\mathbb{R}}_+)$. Hence

$$V(x) = \sum_{j=1}^{\varkappa} V_0(x - y_j) \in L^\infty_{\mathrm{comp}}(\mathbb{R}, \overline{\mathbb{R}}_+).$$

We have replaced the spectral parameter k with $k = \lambda^2$ in the resolvent $(\mathcal{H} - k)^{-1}$ of the operator $\mathcal{H}$ and considered for $\lambda \in \mathbb{C}_+ := \{\operatorname{Im} z > 0\}$ one more version of the resolvent-function

$$\mathcal{R}_{\mathcal{H}}(\lambda) = (\mathcal{H} - \lambda^2)^{-1}, \qquad \mathcal{R}_{\mathcal{H}} : \mathbb{C}_+ \to \mathcal{L}(L^2(\mathbb{R})).$$

Theorem 1.2 has continued $\mathcal{R}_{\mathcal{H}}(\lambda)$ from $\mathbb{C}_+$ meromorphically to the whole $\mathbb{C}$ as an $\mathcal{L}(L^2_{\mathrm{comp}}(\mathbb{R}), L^2_{\mathrm{loc}}(\mathbb{R}))$-valued function $\mathcal{R}^{\mathrm{cont}}_{\mathcal{H}}(\lambda)$. The resonances were defined in Lecture 1 as the poles of this meromorphic continuation.

Let us consider now another, more elementary, way to continue $\mathcal{R}_{\mathcal{H}}$ through $\mathbb{R}$ and to define the (multi-)set $\Sigma(\mathcal{H})$ of the resonances associated with the operator $\mathcal{H}$. This approach is based on the construction of *cut-off resolvent* [S, DZ].

Let $\mathbb{B}_r = \{x \in \mathbb{R}^d : |x| < r\}$. In our present case $d = 1$, so $\mathbb{B}_r = (-r, r)$. Let

$$\chi_r(x) = \chi_{\mathbb{B}_r}(x) := \begin{cases} 1, & x \in \mathbb{B}_r \\ 0, & x \notin \mathbb{B}_r \end{cases}$$

be the indicator function of $\mathbb{B}_r$. By $\mathcal{M}_{\chi_r}$ we denote the multiplication operator on χ_r,

$$\mathcal{M}_{\chi_r}\, u(x) = \chi_r(x) u(x), \qquad u \in L^2(\mathbb{R}).$$

Exercise 2.1.
Show that $\mathcal{M}_{\chi_r} \in \mathcal{L}(L^2(\mathbb{R}))$. Find $\mathcal{M}^*_{\chi_r}$, $\rho(\mathcal{M}_{\chi_r})$, and $\sigma(\mathcal{M}_{\chi_r})$. What is $\| \mathcal{M}_{\chi_r} \|$?

In what follows, we assume that r is large enough in such a way that $\operatorname{supp} V \subset \mathbb{B}_r$. We define the cut-off resolvent as

$$\widetilde{\mathcal{R}}(\lambda) = \widetilde{\mathcal{R}}_{\mathcal{H},r}(\lambda) := \mathcal{M}_{\chi_r} (\mathcal{H} - \lambda^2)^{-1} \mathcal{M}_{\chi_r}, \qquad \lambda \in \mathbb{C}_+.$$

Theorem 2.1 ([S, DZ])**.**
The $\mathcal{L}(L^2(\mathbb{R}))$-valued function $\widetilde{\mathcal{R}}(\lambda)$, $\lambda \in \mathbb{C}_+$, can be continued on $\mathbb{C}$ as a meromorphic $\mathcal{L}(L^2(\mathbb{R}))$-valued function $\widetilde{\mathcal{R}}^{\rm cont}(\lambda) = \widetilde{\mathcal{R}}^{\rm cont}_{\mathcal{H},r}(\lambda)$.

Definition 2.1 (Vainberg, Melrose, see [S, DZ])**.**
The poles of $\widetilde{\mathcal{R}}^{\rm cont}(\lambda)$ in $\mathbb{C}$ (equivalently, the poles of $\mathcal{R}^{\rm cont}_{\mathcal{H}}(\lambda)$) are called *resonances* associated with $\mathcal{H}$. We denote by $\Sigma(\mathcal{H})$ the (multi-)set consisting of these poles (taking their multiplicities into account).

Remark 2.1.

(a) The poles of $\widetilde{\mathcal{R}}^{\rm cont}_{\mathcal{H},r}(\lambda)$ do not depend on r if r is large enough [S]. Therefore resonances are well-defined.

(b) This definition works in $\mathbb{R}^d$ with every odd $d \in \mathbb{N}$.

(c) For even d the analytic continuation has a branching point at $\lambda = 0$ (even if $V \equiv 0$). So, the meromorphic continuation to the whole $\mathbb{C}$ is not possible. In this case, the definition and the physical meaning of resonances are somewhat different [AGHH, DZ].

Remark 2.2.
Consider the case where $V \in L^1_{\rm comp}(\mathbb{R}, \overline{\mathbb{R}}_+)$ or $V \in L^\infty_{\rm comp}(\mathbb{R}^d, \overline{\mathbb{R}}_+)$ with odd $d \in \mathbb{N}$.

(a) Since $\sigma(\mathcal{H}) = \overline{\mathbb{R}}_+$ for $\mathcal{H} = -\Delta + V$, we see that $\mathcal{R}_{\mathcal{H}}(\lambda) = (\mathcal{H} - \lambda^2)^{-1}$ is analytic for $\lambda \in \mathbb{C}_+$ and so are $\widetilde{\mathcal{R}}(\lambda) = \mathcal{M}_{\chi_r}\, \mathcal{R}_{\mathcal{H}}(\lambda)\, \mathcal{M}_{\chi_r}$ and $\widetilde{\mathcal{R}}(\lambda)^{\rm cont}$. Thus, the set of resonances $\Sigma(\mathcal{H})$ is a subset of $\overline{\mathbb{C}}_- = \{\operatorname{Im} z \leqslant 0\}$. Actually, Rellich's uniqueness theorem implies that $\Sigma(\mathcal{H}) \subset \mathbb{C}_- \cup \{0\}$ (see [DZ, Theorem 3.33]).

(b) If we drop the assumption of nonnegativity of V, resonances may appear in $\mathbb{C}_+$. For $V \in L^\infty_{\rm comp}(\mathbb{R}^d, \mathbb{R})$, the set $\Sigma(\mathcal{H}) \cap \mathbb{C}_+ = \{\sqrt{k_n^-}\}_{n=1}^{N_-}$ is at most finite. The values k_n^- are exactly negative eigenvalues of $\mathcal{H}$. In this case, the resonances have physical interpretation somewhat different from that of the next subsection.

In the case $d = 1$, it is possible to define resonances via the eigenvalue problem with specific nonstandard boundary conditions. These boundary conditions include the spectral parameter λ. Namely, the set of resonances $\Sigma(\mathcal{H})$ is the set of eigenvalues λ for the eigenproblem

$$-u''(x) + V(x)u(x) = \lambda^2 u(x), \qquad x \in (-r, r), \tag{2.1}$$

$$u'(r) = \mathrm{i}\lambda u(r), \qquad u'(-r) = -\mathrm{i}\lambda u(r), \tag{2.2}$$

where we can choose any $r > 0$ that is large enough to ensure $\operatorname{supp} V \subset [-r, r]$. It is rather difficult to generalize this definition with special boundary conditions to multidimensional cases.

2.2 Physical meaning of deterministic and random resonances.

Remark 2.3.
If one considers the operator $\mathcal{H}^{\rm D} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$ with the Dirichlet boundary condition

$$u(x) = 0, \qquad x \in \partial G,$$

then the waves modeled by the associated wave equation will reflect from the boundary ∂G and the energy will be conserved inside of G. In this case, one says that that $\mathcal{H}^{\mathcal{D}}$ models

a conservative system/resonator. The spectral properties of the conservative resonator are described exactly by the spectrum $\sigma(\mathcal{H}^{\mathrm{D}})$ of $\mathcal{H}^{\mathrm{D}}$ and the construction with continuation resonances is not needed. For bounded domains G and $V \in L^{\infty}_{\mathrm{comp}}(G, \overline{\mathbb{R}}_+)$, the spectrum

$$\sigma(\mathcal{H}^{\mathrm{D}}) = \sigma_{\mathrm{disc}}(\mathcal{H}^{\mathrm{D}}) = \{k_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$$

is purely discrete, i.e., every point k_n of $\sigma(\mathcal{H}^{\mathrm{D}})$ is isolated and is an eigenvalue of finite algebraic multiplicity. These eigenvalues k_n (or $\lambda_n = \pm\sqrt{k_n}$) are also sometimes called resonances. Eigenvalues k_n are resonances or eigenfrequencies of a conservative system. However, this is a different type of resonances, and we will keep for them the name eigenfrequencies.

The Physics meaning of (continuation) resonances is connected with the description of the long-time behaviors of solutions of wave equations inside the resonator, i.e., inside of a bounded domain G or inside of $\mathbb{B}_r$, and with the corresponding rate of (exponential) decay of the energy. In such settings, $\mathbb{R}^d\backslash G$ or $\mathbb{R}^d\backslash\mathbb{B}_r$ is considered as the outer medium where the energy escapes in the form of waves going to ∞. One says that the resonator is leaky, lossy, or open.

Theorem 2.2.
Assume that $V \in L^{\infty}_{\mathrm{comp}}(\mathbb{R}, \overline{\mathbb{R}}_+)$, $\mathcal{H} = -\Delta + V$. Assume additionally that all resonances associated with $\mathcal{H}$ are simple, i.e., of algebraic multiplicity 1. Consider the solution w of the acoustic-type wave equation

$$\partial_t^2 w(t,x) - \partial_x^2 w(t,x) + V(x)w(t,x) = 0$$

with initial data

$$w(0,\cdot) = u_0(\cdot) \in H^1_{\mathrm{comp}}(\mathbb{R}),$$
$$\partial_t w(0,\cdot) = u_1(\cdot) \in L^2_{\mathrm{comp}}(\mathbb{R}).$$

Then, for every $\beta > 0$, the following representation of w is valid

$$w(t,x) = \sum_{\substack{\operatorname{Im}\lambda_j > -\beta \\ \lambda_j \in \Sigma(\mathcal{H})}} e^{-\mathrm{i}\lambda_j t} w_j(x) + A_\beta(x,t),$$

where the sum is finite, the functions w_j are $L^2_{\mathrm{loc}}(\mathbb{R})$-solutions of

$$\mathcal{H} w_j = \lambda_j w_j$$

in the distributional sense, and, for every $r > 0$ such that $\operatorname{supp} u_0 \cup \operatorname{supp} u_1 \subset \mathbb{B}_r$, there are constants $C_{r,\beta} > 0$ and $T_{r,\beta} > 0$ such that the remainder term A_β is estimated by

$$\|A_\beta(\cdot,t)\|_{H^2(\mathbb{B}_r)} \leqslant C_{r,\beta}\ \mathrm{e}^{-t\beta}\left(\|u_0\|_{H^1(\mathbb{R})} + \|u_1\|_{L^2(\mathbb{R})}\right)$$

for all $t \geqslant T_{r,\beta}$.

In this theorem, $e^{-\mathrm{i}\lambda_j t} w_j(x)$ is an exponentially decaying dissipative eigenoscillation corresponding to the resonance λ_j. In the case $w_j \not\equiv 0$, its (exponential) rate of decay is

$$\beta_j = -\operatorname{Im}\lambda_j > 0.$$

The real part $\alpha_j = \operatorname{Re}\lambda_j$ is the (real) frequency of eigenoscillations.

The spectral abscissa

$$\operatorname{Absc}(\mathcal{H}) = \sup_{\lambda\in\Sigma(\mathcal{H})} \operatorname{Im}\lambda_j$$

corresponds to the lowest horizontal line $\mathbb{R}+\mathrm{i}\operatorname{Absc}(\mathcal{H})$ such that the whole set of resonances is in the closed half-plane below this line. The value $(-1)\operatorname{Absc}(\mathcal{H})$ is the optimal estimate from below on the decay rates of all eigenoscillations. In some cases, $(-1)\operatorname{Absc}(\mathcal{H})$ is the decay rate of the energy in $\mathbb{B}_r$ for generic initial data $\{u_0, u_1\}$ [CZ95].

One more quantity is of interest in engineering applications. Let $\mathcal{I} = [\alpha_1, \alpha_2] \subset \mathbb{R}$ be the interval of real frequencies in that the resonator is supposed to operate. Then the interesting quantity is

$$\operatorname{dist}(\mathcal{I}, \Sigma(\mathcal{H})) = \inf\{|\lambda - \alpha| \ : \ \lambda \in \Sigma(\mathcal{H}),\ \alpha \in [\alpha_1, \alpha_2]\}.$$

Consider now the stochastic model with a random operator $\mathcal{H}_\omega$, e.g., the operator from Lecture 1

$$\mathcal{H}_\omega = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\omega(x) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{j=1}^{\varkappa(\omega)} V_0(x - y_j(\omega)), \qquad \omega \in \Omega,$$

generated by a certain SPP $\{y_j\}_{j=1}^{\varkappa}$. The quantities of interest then the random variables $\operatorname{Absc}(\mathcal{H}_\omega)$ and $\operatorname{dist}(\mathcal{I}, \Sigma(\mathcal{H}_\omega))$, their distributions, expectations $\mathbb{E}(\operatorname{Absc}(\mathcal{H}_\omega))$, $\mathbb{E}(\operatorname{dist}(\mathcal{I}, \Sigma(\mathcal{H}_\omega)))$, and other values describing these random variables.

However, except of some almost trivial cases, it is very difficult to study directly $\operatorname{Absc}(\mathcal{H}_\omega)$ and $\operatorname{dist}(\mathcal{I}, \Sigma(\mathcal{H}_\omega))$ even in the simplest 1-d settings of lecture 1. The existing studies usually include into the stochastic models $\mathcal{H}_\omega$ some parameter going to ∞ or to 0, and investigate certain limiting properties of random sets $\Sigma(\mathcal{H}_\omega)$ [S14, K16, AK21].

2.3 Random resonances for 1-dimensional model with Anderson cut-off potential

One of the models for random resonances [K16] is connected with the 1-d discrete Anderson model

$$(\mathcal{H}_\omega u)(n) = (\Delta_{\mathrm{disc}} u)(n) + V_\omega(n)u(n) = u(n+1) + u(n-1) + V_\omega(n)u(n)$$

in the Hilbert space $\ell^2(\mathbb{Z}) = \ell^2(\mathbb{Z}, \mathbb{C})$. Here $V_\omega(n)$ are independent identically distributed (i.i.d.) nondeterministic random variables.

The operator

$$(\Delta_{\mathrm{disc}} u)(n) = u(n+1) + u(n-1), \qquad n \in \mathbb{Z}, \quad (u(n))_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}),$$

is (shifted) discrete Laplacian (or shifted graph Laplacian).

Exercise 2.2.

(a) $\Delta_{\mathrm{disc}} = \Delta_{\mathrm{disc}}^*$ in $\ell^2(\mathbb{Z})$.

(b) Find $\sigma(\Delta_{\mathrm{disc}})$.

(c) Prove that $\sigma_p(\Delta_{\rm disc}) = \varnothing$.

Actually, it is easy to prove that $\sigma(\Delta_{\rm disc}) = \sigma_{\rm ac}(\Delta_{\rm disc})$ and $\varnothing = \sigma_{\rm pp}(\mathcal{H}_\omega) = \sigma_{\rm sc}(\mathcal{H}_\omega)$.

As soon as the random potential $V_\omega(n)$ is added, the spectral properties of $\mathcal{H}_\omega = \Delta_{\rm disc} + V_\omega$ change substantially. Under very weak assumption on the common distribution measure μ of the random variables $V_\omega(n)$, the complete Anderson localization is proved for $\mathcal{H}_\omega$.

Theorem 2.3 ([CKM87])**.**
Let $\mathcal{H}_\omega = \Delta_{\rm disc} + V_\omega$ with i.i.d. random variables $V_\omega(n)$, $n \in \mathbb{Z}$, having a distribution measure μ. Suppose that $\operatorname{supp}\mu$ *is not concentrated in a single point of $\mathbb{R}$ (i.e., random variables $V_\omega(n)$ are not deterministic). Assume additionally that*

$$\int_{\mathbb{R}} |x|^\delta \mathrm{d}\mu(x) < \infty$$

for a certain $\delta > 0$.

Then, with probability 1,

$$\sigma(\mathcal{H}_\omega) = \sigma_{\rm pp}(\mathcal{H}_\omega), \qquad \varnothing = \sigma_{\rm ac}(\mathcal{H}_\omega) = \sigma_{\rm sc}(\mathcal{H}_\omega),$$

and all (discrete) eigenfunctions of $\mathcal{H}_\omega$ are exponentially decaying as $n \to \pm\infty$.

Klopp [K16] studied SPPs generated by the resonances of operators with cut-off random potentials

$$V_\omega^L(n) = \begin{cases} V_\omega(n) & \text{if} -L+1 \leqslant n \leqslant L \\ 0, & \text{otherwise} \end{cases}$$

in the limit $L \to \infty$.

The paper [AK21] considered random resonances associated with 3-d Schrödinger operators

$$H_\omega = -\Delta + \text{“} \sum_{j=1}^{\varkappa(\omega)} \mathfrak{m}(a)\delta(x - y_j(\omega)) \text{“},$$

where a generalized potential is defined by the SPP $\{y_j\}_{j=1}^{\varkappa}$ consisting of random positions of δ-potentials.

A stochastic model of Type 2 was considered in [K24], where the leakage of energy into the stochastic outer environment is modeled by certain random dissipative boundary conditions on ∂G for multidimensional bounded domains $G \subset \mathbb{R}^d$. These random boundary conditions can be seen as a multi-dimensional generalization and randomization of damping boundary conditions (2.2).

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3 Point processes.

3.1 Basic definitions concerning (stochastic) point processes.

A (stochastic) point process (SPP) in a measurable space $(\mathbb{X}, \mathcal{F}_{\mathbb{X}})$ is a rigorous way to describe a random collection of points in $\mathbb{R}^d$, see, e.g., [LP].

Let us recall that a measurable space is a pair $(\mathbb{X}, \mathcal{F}_{\mathbb{X}})$ consisting of a set $\mathbb{X}$ and a certain σ-field $\mathcal{F}_{\mathbb{X}}$ of subsets of $\mathbb{X}$. A class $\mathcal{F}$ of subsets of $\mathbb{X}$ is called a field (on $\mathbb{X}$) if the following conditions are satisfied:

- $\mathbb{X} \in \mathcal{F}$,
- $A, B \in \mathcal{F}$ implies $A \backslash B \in \mathcal{F}$ and $A \cup B \in \mathcal{F}$.

A field $\mathcal{F}$ closed w.r.t. countable unions is called a σ-field. That is, a field $\mathcal{F}$ is called a σ-field if $\{A_j\}_{j\in\mathbb{N}} \subset \mathcal{F}$ implies $\bigcup_{j\in\mathbb{N}} A_j \in \mathcal{F}$.

One of the most important examples of a measurable space $(\mathbb{X}, \mathcal{F}_{\mathbb{X}})$ is an arbitrary metric space $\mathbb{X}$ equipped with the Borel σ-field $\mathcal{F} = \mathcal{B}(\mathbb{X})$ (which by definition is generated by open subsets of $\mathbb{X}$). We will mainly work with the measurable spaces $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ or $(S, \mathcal{B}(S))$, where $S \in \mathcal{B}(\mathbb{R}^d)$. If it is not explicitly stated otherwise, for a metric space $\mathbb{X}$, we take the σ-field $\mathcal{F}_{\mathbb{X}} = \mathcal{B}(\mathbb{X})$ of all Borel subsets of $\mathbb{X}$ as the associated σ-field for the measurable space.

Let

$$\mathbb{N}_0 := \{0\} \cup \mathbb{N}, \quad \widehat{\mathbb{N}}_0 := \mathbb{N}_0 \cup \{+\infty\}, \quad \text{and } \widehat{\mathbb{N}} := \mathbb{N} \cup \{+\infty\}.$$

Definition 3.1.
Denote by $\mathfrak{N}_{<\infty} = \mathfrak{N}_{<\infty}(\mathbb{X}) = \mathfrak{N}_{<\infty}(\mathbb{X}, \mathcal{F}_{\mathbb{X}})$ the set of all $\mathbb{N}_0$-valued measures $\mu : \mathcal{F}_{\mathbb{X}} \to \mathbb{N}_0$ on $\mathbb{X}$. Let $\mathfrak{N} = \mathfrak{N}(\mathbb{X})$ be the set of all $\widehat{\mathbb{N}}_0$-valued measures that can be written as at most countable sums of measures from the set $\mathfrak{N}_{<\infty}$.

An SPP in $\mathbb{X}$ is a random element of $\mathfrak{N}(\mathbb{X})$. However, we need to explain rigorously the meaning of "random" here.

Let $\mathcal{F}_{\mathfrak{N}} = \mathcal{F}_{\mathfrak{N}}(\mathbb{X})$ be the σ-field on $\mathfrak{N}(\mathbb{X})$ generated by all subsets of $\mathfrak{N}$ of the form

$$\{\mu \in \mathfrak{N} \ : \mu(A) = k\}, \qquad A \in \mathcal{F}_{\mathbb{X}}, \quad k \in \mathbb{N}_0.$$

Exercise 3.1.
The σ-field $\mathcal{F}_{\mathfrak{N}}$ is the smallest σ-field on $\mathfrak{N}$ such that

$$f^A : \mu \mapsto \mu(A), \qquad f^A : \mathfrak{N} \to \widehat{\mathbb{R}},$$

is a measurable function from $(\mathfrak{N}, \mathcal{F}_{\mathfrak{N}})$ to $(\mathbb{R}, \mathcal{B}(\widehat{\mathbb{R}}))$ for every $A \in \mathcal{F}_{\mathbb{X}}$, where $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is the standard compactification of the metric space $\mathbb{R}$.

Let $(\Omega, \mathcal{F}_\Omega, \mathbb{P})$ be our underlying (complete) probability space.

Definition 3.2 (SPP)**.**
A point processes η on $\mathbb{X}$ is a random variable with the values in $(\mathfrak{N}, \mathcal{F}_{\mathfrak{N}})$, i.e., $\eta : \Omega \to \mathfrak{N}$ is a measurable mapping (w.r.t. the σ-fields $\mathcal{F}_\Omega$ and $\mathcal{F}_{\mathfrak{N}}$).

Remark 3.1 (s-finite measures)**.**
A measure μ on $\mathbb{X}$ with values in $\widehat{\mathbb{R}}_+ = [0, +\infty]$ is called s-finite if μ is a countable sum of finite measures (i.e., of measures with values in $\overline{\mathbb{R}}_+ := [0, +\infty)$). By definition, $\mu \in \mathfrak{N}$ implies that μ is s-finite. However, an s-finite $\widehat{\mathbb{N}}_0$-valued measure on a measurable space $\mathbb{X}$ does not necessarily belongs to $\mathfrak{N}(\mathbb{X})$, as it is shown by the next exercise.

Exercise 3.2.
Let $\mathbb{X} = [0, 1]$ and let μ be $\widehat{\mathbb{N}}_0$-valued measure on $[0, 1]$ defined by

$$\mu(A) = \begin{cases} 0, & |A|_1 = 0 \\ +\infty, & |A|_1 > 0 \end{cases}.$$

Show that μ is s-finite, but $\mu \notin \mathfrak{N}([0, 1])$.

Let us explain Definition 3.2 of a point process η. For $A \in \mathcal{F}_{\mathbb{X}}$, let us denote by $\eta(A)$ a function

$$\omega \mapsto \eta(\omega, A) = \eta(\omega)(A), \qquad \omega \in \Omega.$$

Definition 3.2 means that $\eta(A)$ is an $\widehat{\mathbb{N}}_0$-valued random variable for every $A \in \mathcal{F}_{\mathbb{X}}$. The random variable $\eta(A)$ is called the number of points in A.

Example 3.1.
Let $m \in \mathbb{N}_0$ and let μ be a probability measure on $\mathbb{X}$. Let $y_1, \ldots, y_m$ be i.i.d. random variables with values in $\mathbb{X}$ and the distribution μ. Then the sum η of Dirac measures placed at the random sites $y_1(\omega), \ldots, y_m(\omega)$,

$$\eta = \delta_{y_1} + \cdots + \delta_{y_m},$$

is an SPP on $\mathbb{X}$ and

$$\mathbb{P}(\eta(A) = k) = \binom{m}{k} \mu(A)^k (1 - \mu(A))^{m-k}, \qquad k = 0, \ldots, m, \quad A \in \mathcal{F}_{\mathbb{X}}.$$

This SPP is called *a binomial process with sample size m and sampling distribution μ.*

Recall that a random variable $\xi : \Omega \to \mathbb{R}$ is said to have a binomial distribution $\mathrm{Bi}(m, p)$ with parameters $m \in \mathbb{N}_0$ and $p \in [0, 1]$ if

$$\mathbb{P}(\xi = k) = \binom{m}{k} p^k (1 - p)^{m-k}, \qquad k = 0, \ldots, m.$$

where $0^0 = 1$. So, in Example 3.1, $\eta(A) \in \mathrm{Bi}(m, \mu(A))$ for every $A \in \mathcal{F}_{\mathbb{X}}$.

Example 3.2.
Assume that $\mathbb{X} \in \mathcal{B}(\mathbb{R}^d)$ with $0 < |\mathbb{X}|_d < \infty$. The uniform distribution $\mathbb{U} = \mathbb{U}_{\mathbb{X}}$ on $\mathbb{X}$ corresponds to the probability measure $\mathbb{U}(A) = |A|_d / |\mathbb{X}|_d$. Taking $\mu = \mathbb{U}$ in Example 3.1 one obtains on $\mathbb{X}$ *a uniform binomial process with sample size m.*

Binomial processes are simplest nontrivial examples of proper SPP arising in applications.

3.2 Proper point processes and simple point processes

Definition 3.3 (proper SPP)**.**
An SPP η is called a proper point processes if there exist $\mathbb{X}$-valued random variables y_1, y_2, ..., and a $\widehat{\mathbb{N}}_0$-valued random variable $\varkappa$ such that

$$\eta = \sum_{j=1}^{\varkappa} \delta_{y_j}$$

almost surely (a.s.).

Exercise 3.3.

(a) Let η_1, η_2, ... be a sequence of SPPs. Then $\eta = \eta_1 + \eta_2 + \dots$ is an SPP.

(b) Let η_1, η_2, ... be a sequence of proper SPPs. Then $\eta = \eta_1 + \eta_2 + \dots$ is a proper SPP.

The question is under what conditions we can represent an SPP as at most countable sums of δ_{y_j} with certain $\mathbb{X}$-valued random variables y_j. It is easier to approach this question in deterministic settings.

Exercise 3.4.
Let $\mathbb{X} = [0, 1]$. A measure μ belongs to $\mathfrak{N}([0, 1])$ if and only if μ is at most countable sum of Dirac measures $\mu = \sum_{j=1}^{k} \delta_{y_j}$ with a certain $k \in \widehat{\mathbb{N}}_0$ and certain $y_j \in [0, 1]$ for $1 \leqslant j \leqslant k$.

Let us discuss to what extent Exercise 3.4 can be generalized.

Let $\#A$ be a cardinal number of elements in a set A. Assume that $Y = \{y_n\}_{n=1}^{k}$ with $k \in \widehat{\mathbb{N}}_0$ is at most countable subset of $\mathbb{X}$. Then the mapping $\eta_Y : \mathcal{F}_{\mathbb{X}} \to \widehat{\mathbb{N}}_0$ defined by $\eta_Y(A) = \#(Y \cap A)$ for every $A \subseteq \mathbb{X}$ is a measure on $\mathbb{X}$, which is called *a counting measure supported by the set* Y.

At most countable sums

$$\mu = \sum_{j=1}^{k} \delta_{y_j}, \qquad k \in \widehat{\mathbb{N}}_0, \tag{3.1}$$

of δ-functions are measures on $\mathbb{X}$ belonging to $\mathfrak{N}$. They are counting measures of $Y = \{y_n\}_{n=1}^{k}$ if and only if all points y_j are distinct. Otherwise, some of the points y of the set Y are multiple with multiplicities $\mu(\{y\}) \in \widehat{\mathbb{N}}$. One can say that a measure of the form (3.1) is a counting measure for a multi-set $Y = \{y_n\}_{n=1}^{k}$ (or simply a counting measure with multiplicities).

A multi-set Y is a set where a point $y \in Y$ can be repeated and the corresponding multiplicities $\mathrm{mult}(y)$ are taken into account for the comparison of the sets (and possibly some other set operations).

Definition 3.4.

(a) A measure $\mu \in \mathfrak{N}(\mathbb{X})$ is said to be simple if $\mu(\{x\}) \leqslant 1$ for very $x \in \mathbb{X}$. The set of all simple measures is denoted by $\mathfrak{N}_s(\mathbb{X})$.

(b) An SPP η is said to be simple if $\mathbb{P}(\eta \in \mathfrak{N}_s(\mathbb{X})) = 1$.

Remark 3.2.
The set $\mathfrak{N}_s(\mathbb{X})$ is measurable, i.e., $\mathfrak{N}_s(\mathbb{X}) \in \mathcal{F}_{\mathfrak{N}}(\mathbb{X})$ (see [LP, Proposition 6.7]). So, $\{\eta \in \mathfrak{N}_s(\mathbb{X}))\} \in \mathcal{F}_\Omega$ and $\mathbb{P}(\eta \in \mathfrak{N}_s(\mathbb{X}))$ is well-defined.

Exercise 3.4 says that every measure $\mu \in \mathfrak{N}([0,1])$ is a counting measure with multiplicities. For a general measurable space $\mathbb{X}$, this is not necessarily true.

Example 3.3.
There are (rather pathological) examples of measurable spaces $\mathbb{X}$ and $\mu \in \mathfrak{N}(\mathbb{X})$ such that μ cannot be written in the form (3.1). Let us take $\mathbb{X} = [0,1]$ with the following non-standard σ-field $\mathcal{F}$. Let $\mathcal{F}$ consists of all at most countable subsets of $[0,1]$ and their complements in $[0,1]$ (this σ-field is the smallest σ-field generated by finite subsets of $[0,1]$). Let $\mu(A) = |A|_1$ for all $A \in \mathcal{F}$. Then μ is $\mathbb{N}_0$-valued, and so, $\mu \in \mathfrak{N}_{<\infty} \subseteq \mathfrak{N}$, but $\mu(\mathbb{X}) = \mu([0,1]) = 1$ and there exists no y such that $\mu = \delta_y$. Hence, μ cannot be written in the form (3.1).

Remark 3.3.
Example 3.3 immediately implies that there are measurable spaces $\mathbb{X}$ such that not all SPP on $\mathbb{X}$ are proper.

3.3 Proper point processes on Borel spaces

Definition 3.5.
A Borel space is a measurable space $(\mathbb{X}, \mathcal{F}_{\mathbb{X}})$ such that there exists a Borel measurable bijection f from $\mathbb{X}$ to a Borel subset $S \subseteq [0,1]$ such that the inverse bijection f^{-1} is also measurable.

The important and practical class of Borel spaces is described by the next theorem.

Let $\mathbb{X}$ be a Borel subset of a complete separable metric space (CSMS). Let us consider $\mathbb{X}$ as a metric space with the induced metric and let $\mathcal{F}_{\mathbb{X}} = \mathcal{B}(\mathbb{X})$ be the σ-field generated by open subsets of $\mathbb{X}$. The measurable space $(\mathbb{X}, \mathcal{F}_{\mathbb{X}})$ is called a measurable Borel subspace of $\mathbb{X}$.

Theorem 3.1 (e.g., [K02, Theorem A.1.2], [K17, Theorem 1.1], and [LP, Theorem A.19])**.**
Let $\mathbb{X}$ be a Borel subset of a CSMS. Then the measurable Borel subspace $(\mathbb{X}, \mathcal{F}_{\mathbb{X}})$ of $\mathbb{X}$ is a Borel space.

Corollary 3.1.
Any Borel subset $\mathbb{X}$ of $\mathbb{R}^d$ is a Borel space.

Recall that a measure μ on $\mathbb{X}$ is said to be σ-finite if there is a sequence $A_j \in \mathcal{F}_{\mathbb{X}}$, $j \in \mathbb{N}$, such that $\mu(A_j) < \infty$ for all j and $\bigcup_{j\in\mathbb{N}} A_j = \mathbb{X}$. The Lebesgue measure $|\cdot|_d$ on $\mathbb{R}^d$ is σ-finite, but is not finite. A simple example of a measure μ that is not σ-finite is given by (3.1) with $k = +\infty$ if $y_j = y$ for all $j \in \mathbb{N}$ (i.e., if all points y_j, $j \in \mathbb{N}$, coincide).

Exercise 3.5.
Every $\widehat{\mathbb{N}}_0$-valued σ-finite measure belongs to $\mathfrak{N}(\mathbb{X})$.

Definition 3.6.
An SPP η on $\mathbb{X}$ is said to be uniformly σ-finite if there exist $A_j \in \mathcal{F}_{\mathbb{X}}$, $j \in \mathbb{N}$, such that $\bigcup_{j\in\mathbb{N}} A_j = \mathbb{X}$ and

$$\mathbb{P}(\eta(A_j) < \infty) = 1$$

for all $j \in \mathbb{N}$.

Theorem 3.2 (see, e.g., [LP, Corollary 6.5])**.**
A uniformly σ-finite SPP on a Borel space is a proper SPP.

Definition 3.7.
Let $(\mathbb{X}, \rho)$ be a metric space and let $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ be the corresponding measurable space.

(a) A set $B \subseteq \mathbb{X}$ is called bounded if $B = \varnothing$ or its diameter

$$\operatorname{diam}(B) := \sup\{\rho(x, y) : x, y \in B\}$$

is finite.

(b) A measure μ is said to be locally finite if $\mu(B) < \infty$ for every bounded $B \in \mathcal{B}(\mathbb{X})$.

(c) An SPP η on $\mathbb{X}$ is said to be locally finite if $\mathbb{P}(\eta(B) < \infty) = 1$ for every bounded $B \in \mathcal{B}(\mathbb{X})$.

Exercise 3.6.
A locally finite SPP on a metric space $\mathbb{X}$ is uniformly σ-finite.

Corollary 3.2.

(a) A locally finite SPP on a Borel subset $\mathbb{X}$ of CSMS is a proper SPP.

(b) A locally finite SPP on a Borel subset $\mathbb{X} \subseteq \mathbb{R}^d$ is a proper SPP.

Proof. Satetement (a) follows from Theorems 3.1 and 3.2 combined with Exercise 3.6. Statement (b) follows from (a). □

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4 Schrödinger operators with δ-interactions. Abstract symmetric and selfadjoint operators, restrictions and extensions.

Our next goal is to define rigorously random 3-d Schrödinger operators

$$\mathcal{H}_\omega u(x) = -\Delta u(x) + \text{“} \sum_{j=1}^{\varkappa(\omega)} \mathfrak{m}(a_0)\delta(x-y_j(\omega))u(x) \text{“}, \qquad u \in \mathrm{dom}(\mathcal{H}_\omega) \subset L^2(\mathbb{R}^3), \quad \omega \in \Omega,$$

with point interactions of the quasi-strength $1/a_0 \in \mathbb{R} \cup \{\infty\}$ placed at the random sites $y_j(\omega) \in \mathbb{R}^3$. If we have a proper SPP on $\mathbb{R}^3$

$$\Upsilon = \sum_{j=1}^{\varkappa} \delta_{y_j}$$

and fix a certain deterministic $a_0 \in \mathbb{R}$, then $\mathcal{H}_\omega$ becomes the Schrödinger operator associated with this SPP and with the inverse strength parameter a_0.

In this lecture, we start from the deterministic version of $\mathcal{H}_\omega$. Moreover, we consider first the 1-d case as an introduction to the main definitions. A detailed exposition of a related deterministic and sotchastic theories until 1988 is given in the monograph of Albeverio, Gesztesy, Hoegh-Krohn, & Holden [AGHH]. An appendix to this monograph written by Exner gives an overview of the developments until 2005.

4.1 Point interactions in the 1-d case.

Let $N \in \widehat{\mathbb{N}}$, $b = \{b_j\}_{j=1}^N \subset \mathbb{C}$, and let $Y = \{y_j\}_{j=1}^N \subset \mathbb{R}$ be a collection of distinct deterministic real numbers without finite accumulation points.

The operator

$$\mathcal{H} = \mathcal{H}_{Y,b} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{j=1}^{N} b_j \delta_{y_j}$$

in the Hilbert space $L^2(\mathbb{R})$ is defined by the differential operation

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} : u \mapsto -u''$$

and the glue-type boundary conditions

$$u'(y_j + 0) - u'(y_j - 0) = b_j u(y_j), \qquad y_j \in Y. \tag{4.1}$$

Rigorously this means that $\mathcal{H}u = -u''$ for all u in the domain of $\mathcal{H}$

$$\mathrm{dom}\,\mathcal{H} = \{u \in H^1(\mathbb{R}) \cap H^2(\mathbb{R}\backslash Y) \ : \ (5.2) \text{ is satisfied } \}.$$

Example 4.1.
Assume that $b_j = 0$ for all j. Then $\mathrm{dom}\,\mathcal{H} = H^2(\mathbb{R})$ and $\mathcal{H}$ is the standard nonnegative 1-d Laplacian $(-1)\frac{\mathrm{d}^2}{\mathrm{d}x^2} = -\Delta$ with the natural domain in $L^2(\mathbb{R})$. We denote this operator by $\mathcal{H}^0$ and consider it as the unperturbed case. One can say that $\mathcal{H}^0$ is the same as the

operator in the case $N = 0$, i.e., there are no point interactions at all. With the use of the 1-d Fourier transformation

$$Fg(x) = \frac{1}{\sqrt{2\pi}} \text{s-}\lim_{r\to\infty} \int_{-r}^{r} \mathrm{e}^{-\mathrm{i}xy} g(y)\mathrm{d}y,$$

which is a unitary operator in $L^2(\mathbb{R})$, we can write

$$F\mathcal{H}^0F^{-1} = \mathcal{M}_{x^2} = (\mathcal{M}_{x^2})^*.$$

Here

s- lim means the limit in the sense of the norm (i.e., w.r.t. the strong convergence).

So $\mathcal{H}^0 = (\mathcal{H}^0)^*$. Moreover,

$$\sigma(\mathcal{H}^0) = \sigma(\mathcal{M}_{x^2}) = [0, +\infty) \qquad \text{(an exercise)}.$$

The associated quadratic form equals

$$(\mathcal{H}^0u|u)_{L^2} = -\int_{\mathbb{R}} u''\overline{u}\mathrm{d}x = \int_{\mathbb{R}} |u'|\mathrm{d}x.$$

Definition 4.1.
An operator $A : \operatorname{dom} A \subseteq X \to X$ is called nonnegative if the associated quadratic form is nonnegative, i.e., if $(Au|u)_X \geqslant 0$ for all $u \in \operatorname{dom} A$.

Example 4.2.
Let $N = 1$ and $Y = \{y_1\} = \{0\}$. Using integration by parts, let us consider the quadratic form associated with $\mathcal{H} = \mathcal{H}_{Y,b}$,

$$(\mathcal{H}u|u)_{L^2} = \int_{\mathbb{R}\setminus\{0\}} u''\overline{u}\mathrm{d}x = \int_{\mathbb{R}} |u'|^2 + b_1|u(0)|^2,$$

which on a formal (nonrigorous) level looks like

$$(\mathcal{H}^0u|u)_{L^2} + b_1 \int_{\mathbb{R}} \delta(x)|u(x)|^2\mathrm{d}x.$$

This justifies the expression

$$\mathcal{H} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b_1\delta_{y_1}.$$

In the 1-d case, the parameter $b_1 \in \mathbb{R}$ is the strength of the point interaction at y_1.

Remark 4.1.
Example 4.2 can be easily extended to the case $N < \infty$. If additionally $b_j \in \mathbb{R}$ for all j, it can be transformed into an alternative rigorous definition of the selfadjoint operator $\mathcal{H}_{Y,b}$ with the use of the theory of (unbounded) symmetric bilinear forms, see [AGHH, Kato, RS2] (especially , see the Kato-Lax-Milgram-Nelson (KLMN) theorem in [RS2] and [RS2, Example X.2.3]).

The case $N = \infty$ (as well as the 2-d and 3-d cases) are substantially more difficult.

Theorem 4.1 (Gesztesy & Kirsch[GK85], see also [AGHH, KM10])**.**
Assume that $b = \{b_j\}_{j=1}^N \subset \mathbb{R}$.

(a) If additionally the following uniform discreteness condition holds

$$\inf_{j\neq n} |y_j - y_n| > 0, \tag{4.2}$$

then $\mathcal{H}_{Y,b} = \mathcal{H}^*_{Y,b}$ in $L^2(\mathbb{R})$.

(b) Note that $N < \infty$ implies (4.2). So if $N < \infty$, then $\mathcal{H}_{Y,b} = \mathcal{H}^*_{Y,b}$ in $L^2(\mathbb{R})$.

We will use the notation

$$d_* = d_*(Y) := \inf_{j\neq n} |y_j - y_n|.$$

With this notation, the uniform discreteness condition (4.2) can be written as $d_*(Y) > 0$.

The uniform discreteness condition handles the cases of deterministic lattices and lattices with small random displacement. For a homogeneous Poisson point process Y_ω, (4.2) is too restrictive.

Remark 4.2.
Let $b_j = b_0 \in \mathbb{R}$ for all j. Let $Y_\omega = \{y_j(\omega)\}_{j\in\mathbb{N}}$, $\omega \in \Omega$, be a homogeneous Poisson SPP. Then it is easy to see that (4.2) holds with probability 0. However, the operator $\mathcal{H}_{Y_\omega,b} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b_0 \sum_{j\in\mathbb{N}} \delta_{y_j(\omega)}$ is selfadjoint with probability 1 (this follows from the results of [M88], see also [KMN19] and Theorem 10.2 of this lecture series). Note that we have used the fact that a homogeneous Poisson SPP on $\mathbb{R}$ is proper, which follows from the fact that homogeneous Poisson SPP is locally finite and Corollary 3.2 (b).

The uniform discreteness assumption can be relaxed, but cannot be dropped in Theorem 4.1. This can be seen from the following result of [KM10] (in a slightly reformulated form).

Theorem 4.2 (Kostenko & Malamud [KM10])**.**
Assume that Y can be renumbered as $\widetilde{Y} = \{y_j\}_{j\in\mathbb{Z}}$ in strictly increasing order such that $d_j = y_j - y_{j-1} > 0$ for all $j \in \mathbb{Z}$. Then:

(a) In the case $\sum d_j^2 = \infty$, one has $\mathcal{H}_{\widetilde{Y},b} = \mathcal{H}^*_{\widetilde{Y},b}$ for every $b \subset \mathbb{R}$.

(b) Let $\sum d_j^2 < \infty$ and $d_{j-1}d_{j+1} \geqslant d_j^2$ for all $j \geqslant 0$. Then there exists $b = \{b_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}$ such that $\mathcal{H}_{\widetilde{Y},b}$ is symmetric, but $\mathcal{H}_{\widetilde{Y},b} \neq \mathcal{H}^*_{\widetilde{Y},b}$.

In this very advanced result, one can see a much simpler effect that the selfadjointness of unbounded operators is a more tricky property, than the selfadjointness of bounded operators. In particular, symmetric operators are not necessarily selfadjoint.

4.2 Abstract symmetric and selfadjoint operators

Let X_1 and X_2 be Hilbert spaces with inner products $(\cdot|\cdot)_{X_{1,2}}$. Mainly we work with complex Hilbert spaces, but some of material can be adapted to the real case and to Banach spaces.

Recall that a (linear) operator A from X_1 to X_2 with a domain $\operatorname{dom} A \subseteq X_1$ is a linear map defined on a linear subspace $\operatorname{dom} A$ of X_1. The corresponding notation is $A : \operatorname{dom} A \subseteq X_1 \to X_2$. If $\operatorname{dom} A = X_1$, one can write $A : X_1 \to X_2$. The Banach space of all bounded operators $A : X_1 \to X_2$ (with the standard operator norm) is denoted by $\mathcal{L}(X_1, X_2)$.

If $X_1 = X_2 = X$, one says that $A : \operatorname{dom} A \subseteq X \to X$ is an operator in X. Besides, $\mathcal{L}(X) = \mathcal{L}(X, X)$.

For unbounded operators in X, $\operatorname{dom} A$ is typically not a closed subspace of X (except some pathological cases). If $\operatorname{dom} A$ is dense in X (i.e., $\overline{\operatorname{dom} A} = X$), one says that A is densely defined in X. Differential operators in $X = L^2(G)$ (with open $G \subseteq \mathbb{R}^d$) are typically densely defined.

Definition 4.2.
Assume that $A : \operatorname{dom} A \subseteq X_1 \to X_2$ is densely defined. Then the adjoint operator $A^* : \operatorname{dom} A^* \subseteq X_2 \to X_1$ is defined in the following way:

(a) $\operatorname{dom} A^*$ consists of all $v \in X_2$ with the property that there exists $f \in X_1$ such that

$$(v|Au)_{X_2} = (f|u)_{X_1} \qquad \forall u \in \operatorname{dom} A; \tag{4.3}$$

(b) $A^* v = f$.

The assumption $\overline{\operatorname{dom} A} = X_1$ in Definition 4.2 ensures that f in (9.2) is unique.

Definition 4.3.

(a) An operator $A : \operatorname{dom} A \subseteq X \to X$ is called selfadjoint if $A = A^*$ (in particular, this assumes $\operatorname{dom} A = \operatorname{dom} A^*$).

(b) An operator $A : \operatorname{dom} A \subseteq X \to X$ is called symmetric if

$$(Au|v)_X = (u|Av)_X \qquad \forall u, v \in \operatorname{dom} A.$$

Remark 4.3.

(a) Equality (9.2) *implies that*

$$(Au|v)_{X_2} = (u|A^* v)_{X_1}, \qquad u \in \operatorname{dom} A, \quad v \in \operatorname{dom} A^*. \tag{4.4}$$

(b) Equality (4.4) *implies that every selfadjoint operator is symmetric.*

(c) If $A \in \mathcal{L}(X)$, then A is selfadjoint if and only if it is symmetric (an exercise).

Not every unbounded symmetric operator is selfadjoint, as we have seen in Theorem 4.2. A much simpler example of symmetric non-selfajoint operator in $L^2(0,1)$ is provided by the 1-d Laplacian operator $-\Delta^{\mathrm{D,N}}$ defined by the differential expression $(-1)\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ and a combination of Dirichlet and Neumann boundary conditions. That is

$$\operatorname{dom} \Delta^{\mathrm{D,N}} = H_0^2(0,1),$$

where $H_0^2(G)$ is the closure of $C_0^\infty(G)$ in $H^2(G)$. Typically symmetric nonselfadjoint boundary operators appear if one put too much boundary conditions.

Let us consider the PDE version of the last example.

Example 4.3.
Let $G \subseteq \mathbb{R}^d$ be open and nonempty. Let

$$Au = -\Delta u, \qquad A : C_0^\infty(G) \subset L^2(G) \to L^2(G).$$

Then A is symmetric, but $A \neq A^*$ since $\operatorname{dom}(A^*) \supsetneqq \operatorname{dom}(A) = C_0^\infty(G)$. Indeed, by Green's formula $\operatorname{dom}(A^*) \supseteq H^2(G)$. In the case $d = 1$, $\operatorname{dom}(A^*) = H^2(G)$.

The selfadjointness of the differential operators A is important, in partucluar, because the equation

$$\mathrm{i}\partial_t u = Au \tag{4.5}$$

in a Hilbert space X has good properties. Namely, $A = A^*$ if and only if $(-\mathrm{i}A)$ is a generator of unitary group. This means that, in the case $A = A^*$, the unique solution $u(t) = e^{-\mathrm{i}At}u_0$ to the initial value problem $u|_{t=0} = u_0 \in X$ for (4.5) exists in a certain reasonable sense for all $t \in \mathbb{R}$ and $\|u(t)\|_X = \|u_0\|_X$ for all $t \in \mathbb{R}$, see [Kato, RS1]. In this case, the unitary operators $e^{-\mathrm{i}At}$, $t \in \mathbb{R}$, build a group.

Theorem 4.3 (e.g., [Kato, RS1])**.**
If $A = A^*$, then $\sigma(A) \subseteq \mathbb{R}$.

If the operator A is symmetric, but is not selfadjoint, then $\sigma(A) \nsubseteq \mathbb{R}$. The properties of solutions to $\mathrm{i}\partial_t u = Au$ are (softly speaking) not so good. For some classes of operators, there are no nontrivial solutions on sufficiently large time intervals $t \in [0, T]$. For differential operators, the reason is typically that too much boundary conditions are imposed.

If $A = -\Delta + V(x)$ is Schrödinger operator in the classical sense, or in the generalized sense of this lecture, then (4.5) is a time-dependent Schrödinger equation that describes, e.g., a dynamic of a quantum particle in a certain environment represented by a potential V depending on the position. For example, the case of generalized potential $V = b_0 \sum_{n\in\mathbb{Z}} \delta(x-n)$ on $\mathbb{R}$ or a singular case of Kronig-Penny model, which is a simplified model of a nonrelativistic electron moving in a fixed crystal lattice. Theorem 4.1 implies that the related operator $A = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b_0 \sum_{n\in\mathbb{Z}} \delta(x-n)$ in $L^2(\mathbb{R})$ is selfadjoint and there exists an associated unitary group $\{e^{-\mathrm{i}At}\}_{t\in\mathbb{R}}$.

4.3 Restrictions and extensions of operators

Definition 4.4.
The graph $\mathrm{Gr}\, A$ of an operator $A : \mathrm{dom}\, A \subseteq X_1 \to X_2$ is a subspace of $X_1 \oplus X_2$ defined by

$$\mathrm{Gr}\, A = \{\ \{u, Au\}\ :\ u \in \mathrm{dom}\, A\}.$$

Definition 4.5.
Let $A : \mathrm{dom}\, A \subseteq X_1 \to X_2$ and $B : \mathrm{dom}\, B \subseteq X_1 \to X_2$ be connected by $\mathrm{Gr}\, A \subseteq \mathrm{Gr}\, B$. Then an operator B is called an extension of A, and the operator A is said to be a restriction of B. One writes $A = B \restriction_{\mathrm{dom}\, A}$.

Note that $\mathrm{Gr}\, A \subseteq \mathrm{Gr}\, B$ is equivalent to

$$\mathrm{dom}\, A \subseteq \mathrm{dom}\, B \text{ and } Au = Bu \quad \forall u \in \mathrm{dom}\, A.$$

In operator theory, operators are often identified with their graphs and so one writes $\mathrm{Gr}\, A \subseteq \mathrm{Gr}\, B$ shorter $A \subseteq B$.

Proposition 4.1.

(a) If $\mathrm{Gr}\, A \subseteq \mathrm{Gr}\, B$, then $\mathrm{Gr}\, B^* \subseteq \mathrm{Gr}\, A^*$.

(b) A densely defined operator A in X is symmetric if and only if $\mathrm{Gr}\, A \subseteq \mathrm{Gr}\, A^*$.

(c) A restriction of a symmetric operator is a symmetric operator. In particular, a restriction of a selfadjoint operator is a symmetric operator.

(d) Every restriction of a nonnegative operator is a nonnegative operator.

(e) Every densely defined nonnegative operator is symmetric.

Proof. The proof of (a) easily follows from Definition 4.2. Statement (b) follows from (a) and Definition 4.3. Statement (c) follows from (b). Statement (d) follows from Definition 4.5.

Statement (e) follows from the polarization principle, see [Kato, formulae (I.6.11) and (VI.1.1)]. □

Theorem 4.4 (von Neumann, see, e.g., [AG, Section 107] and [Kato])**.**
A densely defined nonnegative operator A has at least one selfadjoint extension $\widehat{A}$. Every such selfadjoint extension $\widehat{A}$ is a restriction of A^*.

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5 Schrödinger operators with δ-interactions in $\mathbb{R}^3$ and Poisson processes.

In the 2-dimensional and 3-dimensional case, it is difficult to define point interactions via quadratic forms. That is why we consider two other methods for the 3-d case.

The 1st method is based on restrictions and extensions of operators. The 2nd method introduces Schrödinger operators with δ-interactions $\mathcal{H}$ via their resolvent $(\mathcal{H}-k)^{-1}$, $k \in \rho(\mathcal{H}) \subset \mathbb{C}$, and is convenient for the study of the spectrum $\sigma(\mathcal{H}) = \mathbb{C}\backslash\rho(\mathcal{H})$.

It should be pointed out that, in $\mathbb{R}^d$ with $d \geqslant 4$, Schrödinger operators with δ-interactions at isolated points do not exist at all.

5.1 Point interactions in the 3-d case defined via restrictions, extensions, and boundary conditions.

Recall that $A : \operatorname{dom} A \subseteq X \to X$ is a restriction of $B : \operatorname{dom} B \subseteq X \to X$ if $\operatorname{Gr} A \subset \operatorname{Gr} B$. In this case, B is called an extension of A and one writes $A = B \restriction_{\operatorname{dom} A}$. Every densely defined nonnegative operator A in a Hilbert space X has at least one selfadjoint extension $\widehat{A}$, and every such selfadjoint extension $\widehat{A}$ is a restriction of A^* (see Theorem 4.4).

In the Hilbert space $L^2(\mathbb{R}^3)$, consider the nonnegative Laplace operator

$$\mathcal{H}^0 := -\Delta, \qquad \operatorname{dom} \mathcal{H}^0 = H^2(\mathbb{R}^3).$$

Then $\mathcal{H}^0 = (\mathcal{H}^0)^*$, which can be proved using the reduction to a multiplication operator with the use of the Fourier transform.

Let $N \in \widehat{\mathbb{N}}$, $a = \{a_j\}_{j=1}^N \subset \mathbb{C}$, and let $Y = \{y_j\}_{j=1}^N \subset \mathbb{R}^3$ be a collection of distinct deterministic points in $\mathbb{R}^3$. Assume additionally that

$$Y \text{ has no finite accumulation points}$$

In the terminology of Definition 3.7, the last assumption is equivalent to the statement that the counting measure of Y is locally finite.

Consider the following restriction of $\mathcal{H}^0$

$$\mathcal{H}_Y^{\min} := \mathcal{H}^0 \restriction_{\operatorname{dom} \mathcal{H}_Y^{\min}}, \quad \operatorname{dom} \mathcal{H}_Y^{\min} = C_0^\infty(\mathbb{R}^3\backslash Y).$$

It follows from Proposition 4.1 that $\mathcal{H}_Y^{\min}$ is symmetric and nonnegative as a restriction of the nonnegative selfadjoint operator $\mathcal{H}^0$.

By Theorem 4.4, there exist selfadjoint extensions $\widehat{\mathcal{H}}$ of $\mathcal{H}_Y^{\min}$ and these extensions are restrictions of the operator

$$\mathcal{H}_Y^{\max} := (\mathcal{H}_Y^{\min})^*.$$

The definition of the adjoint operator (Definition 4.2) implies that

$$\mathcal{H}_Y^{\max} u = -\Delta_{\mathbb{R}^3\backslash Y} u \qquad \text{for all } u \in \operatorname{dom} \mathcal{H}_Y^{\max} = \{u \in L^2(\mathbb{R}^3) \ : \ \Delta_{\mathbb{R}^3\backslash Y} u \in L^2(\mathbb{R}^3)\},$$

where the Laplacian $\Delta_{\mathbb{R}^3\backslash Y}$ is understood in the sense of the space of distributions $\mathcal{D}'(\mathbb{R}^3\backslash Y)$ in the open set $\mathbb{R}^3\backslash Y$.

The domain $\operatorname{dom}\mathcal{H}_Y^{\min}$ can be described in another way with the use of the elliptic inner regularity:

$$\operatorname{dom}\mathcal{H}_Y^{\max} = \{u \in L^2(\mathbb{R}^3) \cap H^2_{\mathrm{loc}}(\mathbb{R}^3\backslash Y) \ : \ \Delta u \in L^2(\mathbb{R}^3)\},$$

where Δ is understood in the sense of the Laplacian in $H^2_{\mathrm{loc}}(\mathbb{R}^3\backslash Y)$ (see, e.g., [A, Theorem 6.3] and [KMN19, Proposition 8]).

Let $\mathbb{B}_r(y) = y + \mathbb{B}_r = \{x \in \mathbb{R}^d : |x-y| < r\}$.

Proposition 5.1 ([KMN19, Proposition 8])**.**
Let $y \in Y$ and $\varepsilon > 0$ be such that $\mathbb{B}_\varepsilon(y) \cap Y = \{y\}$. Then, for every $u \in \operatorname{dom}\mathcal{H}_Y^{\max}$, there exist unique constants $u_y^0, u_y^1 \in \mathbb{C}$ and a function $\widetilde{u} \in H^2(\mathbb{B}_\varepsilon(y))$ such that $\widetilde{u}(y) = 0$ and

$$u(x) = u_y^0|x-y|^{-1} + u_y^1 + \widetilde{u}(x), \qquad x \in \mathbb{B}_\varepsilon(y).$$

The complex constants u_y^0 and u_y^1 play the role of generalized boundary values (traces) at y. The operator

$$\mathcal{H}_{Y,a} = -\Delta + \text{“} \sum_{j=1}^N \mathfrak{m}(a_j)\delta_{y_j} \text{“}$$

is defined as the restriction of $\mathcal{H}_Y^{\max}$ generated by the boundary conditions

$$u_{y_j}^1 - 4\pi a_j u_{y_j}^0 = 0, \qquad 1 \leqslant j \leqslant N. \tag{5.1}$$

In other words, $\mathcal{H}_{Y,a}u = -\Delta u$ for all $u \in \operatorname{dom}\mathcal{H}_{Y,a}$, where

$$\operatorname{dom}\mathcal{H}_{Y,a} = \{u \in \operatorname{dom}\mathcal{H}_Y^{\max} \ : \ (10.1) \text{ holds for all } y_j \in Y\}.$$

As before, we put

$$d_* = d_*(Y) := \inf_{j\neq n} |y_j - y_n|.$$

Theorem 5.1 ([AGHH])**.**
Assume that $a = \{a_j\}_{j=1}^N \subset \mathbb{R}$.

(a) If the uniform discreteness condition $d_* > 0$ holds, then $\mathcal{H}_{Y,a} = \mathcal{H}_{Y,a}^*$ in $L^2(\mathbb{R}^3)$.

(b) If $N < \infty$, then $d_* > 0$ and $\mathcal{H}_{Y,a} = \mathcal{H}_{Y,a}^*$ in $L^2(\mathbb{R}^3)$.

Theorem 5.1 is applicable to periodic lattices, e.g., to $Y = \mathbb{Z}^3$.

5.2 Comparison of the strength parameters b_j and $1/a_j$ in the 1-d and 3-d cases and their singular values

The role of parameters a_j for the operators

$$\mathcal{H}_{Y,a} = -\Delta + \text{“} \sum_{j=1}^N \mathfrak{m}(a_j)\delta_{y_j} \text{“}$$

in $L^2(\mathbb{R}^3)$ is different from the role of the strength parameters b_j for the operators

$$\mathcal{H} = \mathcal{H}_{Y,b} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{j=1}^N b_j\delta_{y_j}$$

in $L^2(\mathbb{R}^3)$.

Consider the case of $\mathcal{H}_{Y,b}$ in $L^2(\mathbb{R})$. If $b_j = 0$, the point interaction at y_j disappears. If $b_j = \infty$, then the corresponding boundary condition

$$u'(y_j + 0) - u'(y_j - 0) = b_j u(y_j), \qquad y_j \in Y. \tag{5.2}$$

takes the form of the Dirichlet boundary condition

$$u(y_j) = 0.$$

If all $b_j = \infty$ and $Y = \{y_j\}_{j=1}^N$ is ordered increasingly, then the corresponding operator $\mathcal{H}_{Y,b}$ in $L^2(\mathbb{R}^3)$ is the orthogonal sum of the positive selfadjoint Dirichlet Laplace operators $(-1)\Delta^{\mathrm{D}}_{\mathcal{I}_j}$, where

$$\mathcal{I}_j = (y_j, y_{j+1}, \qquad \Delta^{\mathrm{D}} : \operatorname{dom}\Delta^{\mathrm{D}} \subset L^2(\mathcal{I}_j) \to L^2(\mathcal{I}_j),$$
$$\operatorname{dom}\Delta^{\mathrm{D}} = u \in H^2(\mathcal{I}_j) \cap H^1_0(\mathcal{I}_j).$$

This orthogonal sum is written w.r.t. the orthogonal decomposition

$$L^2(\mathbb{R}) = \bigoplus_j L^2(\mathcal{I}_j).$$

If $d = 3$, the case $a_j = \infty$ corresponds to the disappearance of the point interaction at y_j. In particular, if $a_j = \infty$ for all j then $\mathcal{H}_{Y,a}$ is interpreted as the standard nonnegative Laplace operator $(-1)\Delta$ in $L^2(\mathbb{R}^3)$ (with $\operatorname{dom}\Delta = H^2(\mathbb{R}^3)$). The strength-type parameter is $1/a_j$, but even this parameter $1/a_j$ is not an analogue of the strength b_j for the operator in $L^2(\mathbb{R})$. In the 3-d case, the expression

$$-\Delta + \frac{1}{a_1}\delta_{y_j}$$

has no accurate interpretation, in particular, it has no interpretation in the sense of quadratic forms. That is why it is replaced with

$$\mathcal{H}_{y_1,a_1} = -\Delta + \text{“}\mathfrak{m}(a_1)\delta_{y_1}\text{ “},$$

where the expression $\mathfrak{m}(a_1)\delta_{y_1}$ represents formally a certain renormalization of the δ-potential δ_{y_1}. The value $a_1 \in \mathbb{C}$ is a free parameter that emerges in this renormalization process, see [AGHH, Section 1.1]. We have used here a more transparent approach to via the explicit form of the boundary conditions.

5.3 Poisson point processes and associated Schrödinger operators

Consider the operator $\mathcal{H}_{Y,a}$ in $L^2(\mathbb{R}^3)$. Then the uniform discreteness condition $d_* > 0$ is not necessary for the statement that $\mathcal{H}_{Y,a} = \mathcal{H}^*_{Y,a}$ for every $a = \{a_j\}_{j=1}^N$. We consider a counterexample in the stochastic settings.

Recall that an $\mathbb{N}_0$-valued random variable $\varkappa$ is said to have the Poisson distribution $\mathrm{Po}(\gamma)$ with rate $\gamma \in [0, +\infty)$ if

$$\mathbb{P}(\varkappa = n) = \mathrm{Po}(\gamma, n) := \frac{\gamma^n}{n!}e^{-\gamma}, \qquad n \in \mathbb{N}_0.$$

An $\widehat{\mathbb{N}}_0$-valued random variable $\varkappa$ is said to have the Poisson distribution $\mathrm{Po}(+\infty)$ with rate $\gamma = +\infty$ if

$$\mathbb{P}(\varkappa = +\infty) = \mathrm{Po}(+\infty, +\infty) := 1, \qquad \mathbb{P}(\varkappa = n) = \mathrm{Po}(+\infty, n) := 0 \text{ for } n \in \mathbb{N}_0.$$

Let $(\mathbb{X}, \mathcal{F}_{\mathbb{X}})$ be a measurable space.

Definition 5.1 (Poisson point process)**.**
Let μ be an s-finite measure on $\mathbb{X}$. An SPP η is called a Poisson SPP with intensity measure μ if it has the following properties:

(a) For every $A \in \mathcal{F}_{\mathbb{X}}$, the random variable $\eta(A)$ has the Poisson distribution $\mathrm{Po}(\mu(A))$.

(b) For every $m \in \mathbb{N}$ and pairwise disjoint sets $A_1, \ldots, A_m \in \mathcal{F}_{\mathbb{X}}$, the random variables $\eta(A_1), \ldots, \eta(A_m)$ are independent.

Remark 5.1.
If a certain SPP η has property (b) of Definition 5.1, then η is called completely independent.

Definition 5.2 (homogeneous Poisson point process)**.**
For $\mathbb{X} \in \mathcal{B}(\mathbb{R}^d)$, let us take $(\mathbb{X}, \mathcal{F}_{\mathbb{X}}) = (\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Then a homogeneous (or uniform) Poisson SPP η on $\mathbb{X}$ is a Poisson SPP on $\mathbb{X}$ such that its intensity measure μ is a multiple of the d-dimensional Lebesgue measure $|\cdot|_d$, i.e., $\mu(\cdot) = \gamma|\cdot|_d$ with a certain constant $\gamma \in [0, +\infty)$. In this case, γ is called the intensity of η.

Corollary 5.1.
Let η be a homogeneous Poisson SPP on $\mathbb{X} \in \mathcal{B}(\mathbb{R}^d)$. Then η is a proper SPP.

Proof. By Definition 5.2, η is locally finite. Corollary 3.2 implies that η is proper. □

Let η be a homogeneous Poisson SPP on $\mathbb{R}^3$ with intensity $\gamma > 0$. Note that the random variable $\eta(\mathbb{R}^3)$ has the Poisson distribution $\mathrm{Po}(+\infty)$, i.e.,

$$\mathbb{P}(\eta(\mathbb{R}^3) = \infty) = 1.$$

By Corollary 5.1, η is proper. Thus, there exist a sequence $Y = \{y_j\}_{j\in\mathbb{N}}$ of $\mathbb{R}^3$-valued random variables y_j such that a.s.

$$\eta = \sum_{j=1}^{\infty} \delta_{y_j}.$$

Let $a = \{a_j\}_{j=1}^{+\infty}$ be a sequence of real numbers, or more generally, a sequence of $\mathbb{R}$-valued random variables a_j. Since η is locally finite, we can define for a.a. $\omega \in \Omega$ a randomized Schrödinger operator $H_{Y,a}$

$$\mathcal{H}_{Y_\omega,\alpha_\omega} = -\Delta + \text{“} \sum_{j=1}^{N} \mathfrak{m}(a_j(\omega))\delta_{y_j(\omega)} \text{”}, \qquad \omega \in \Omega, \tag{5.3}$$

using the definition of Section 5.1.

Theorem 5.2 (Kaminaga, Mine, Nakano [KMN19])**.**
Let $Y = \{y_j\}_{j\in\mathbb{N}}$ be a collection of random sites y_j in $\mathbb{R}^3$ associated as above with a certain

homogeneous Poisson SPP η of positive intensity. Let $a = \{a_j\}_{j=1}^{+\infty}$ be an arbitrary sequence of $\mathbb{R}$-valued random variables a_j. Then $H_{Y,a} = H^*_{Y,a}$ with probability 1.

Proposition 5.2.
In the settings of Theorem 5.2, the uniform discreteness condition $d_* > 0$ holds with probability 0. (Recall that $d_* = \inf_{j \neq n} |y_j - y_n|$.)

Proof. Note that d_* is a random variable. Let $\Omega_n = \{d_* > 1/n\}$.

Assume that $\mathbb{P}(d_* > 0) > 0$. Then there exists $n \in \mathbb{N}$ such that $\mathbb{P}(\Omega_n) > 0$. Hence, there exists $m \in \mathbb{N}$ such that $\eta_\omega(z + [0,1)^3) \leqslant m$ for all $\omega \in \Omega_n$ and all $z \in \mathbb{Z}^3$.

By Definitions 5.1 and 5.2, $\{\eta(z + [0,1)^3)\}_{z \in \mathbb{Z}^3}$ is a collection of i.i.d. random variables with $\mathbb{P}(\eta(z + [0,1)^3) \leqslant m) = p < 1$. Hence, $\mathbb{P}(\Omega_n) \leqslant \lim_{k \to +\infty} p^k = 0$. This contradicts the assumption $\mathbb{P}(d_* > 0) > 0$. □

5.4 Equality in distributions for point processes

We consider various additional properties of SPPs and apply them to Poisson SPPs following [LP, Sections 2-3].

Let η be an SPP on a measurable space $(\mathbb{X}, \mathcal{F}_\mathbb{X})$. Then the distribution of η is a probability measure $\mathbb{P}_\eta$ on $(\mathfrak{N}, \mathcal{F}_\mathfrak{N})$ that is the distribution of η as an $\mathfrak{N}$-valued random variable. Recall that the measurable space $(\mathfrak{N}, \mathcal{F}_\mathfrak{N})$ of countable sums of $\mathbb{N}_0$-valued measures was introduced in Lecture 3.

If η and ξ are two SPP on $\mathbb{X}$ such that $\mathbb{P}_\eta = \mathbb{P}_\xi$, then one says that η and ξ have the same distribution and write $\eta \stackrel{\mathrm{d}}{=} \xi$.

Proposition 5.3.
The two following statements are equivalent:

(a) $\eta \stackrel{\mathrm{d}}{=} \xi$

(b) $(\eta(A_1), \dots \eta(A_m)) \stackrel{\mathrm{d}}{=} (\xi(A_1), \dots \xi(A_m))$ *as $\mathbb{R}^m$-valued random variables for all $m \in \mathbb{N}$ and all pairwise disjoint deterministic sets $A_1, \dots, A_m \in \mathcal{F}_\mathbb{X}$.*

Corollary 5.2.
Let η and ξ be two Poisson SPP with the same intensity measure μ. Then $\eta \stackrel{\mathrm{d}}{=} \xi$.

Proof. The statement follows from Definition 5.1 (of Poisson SPP) and Proposition 5.3. □

References for Lecture 5.

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6 Mixed binomial SPPs. Existence of Poisson processes. Intensity measures and transformations for general SPPs.

6.1 Mixed binomial SPPs and the existence of Poisson processes with given intensity measures

Corollary 5.1 state that a homogeneous Poisson SPP on a Borel set $B \subset \mathbb{R}^d$ is proper. An example for Corollary 5.1 is given by Example 1.1 of Lecture 1, which we reformulate now as a proposition.

Proposition 6.1 (homogeneous Poisson process on S with $|S|_d \in \mathbb{R}_+$)**.**
Let $S \subset \mathbb{R}^d$ be a Borel subset with $|S|_d \in \mathbb{R}_+ = (0, +\infty)$. Let y_1, y_2, ... be independent random variables uniformly distributed in S. Let $\varkappa$ be an $\widehat{\mathbb{N}}_0$-valued random variable with the Poisson distribution $\mathrm{Po}(\widetilde{\gamma})$ of rate $\widetilde{\gamma} \geqslant 0$. Then $\eta = \sum_{j=1}^{\varkappa} \delta_{y_j}$ is a homogeneous Poisson point process with the intensity equal to $\gamma = \frac{\widetilde{\gamma}}{|S|_d}$.

This proposition follows from the next theorem (Theorem 6.1) on mixing binomial SPP with Poisson mixing distributions.

Definition 6.1 (mixed binomial SPP)**.**
Let p_{m} and p_{s} be probability measures on $\mathbb{N}_0$ and $\mathbb{X}$, respectively. Let y_1, y_2, ... be i.i.d. $\mathbb{X}$-valued random variables with distribution p_{s}. Let $\varkappa$ be an $\mathbb{N}_0$-valued random variable with distribution p_{m} and assume that $\varkappa$ is independent of $\{y_j\}_{j\in\mathbb{N}}$. Then the proper SPP

$$\eta := \sum_{j=1}^{\varkappa} \delta_{y_j}$$

is called a mixed binomial process with mixing distribution p_m and sampling distribution p_s.

Theorem 6.1.
Let p_{s} be a probability measure on $\mathbb{X}$ and let $\gamma \geqslant 0$. Let η be a a mixed binomial process with mixing distribution $\mathrm{Po}(\gamma)$ and sampling distribution p_{s}. Then η is a Poisson process with intensity measure γp_{s}.

Proof. Step 1. Assume that disjoint sets $B_1, B_2 \in \mathcal{F}_{\mathbb{X}}$ are such that $B_1 \cup B_2 = \mathbb{X}$. Let $k_1, k_2 \in \mathbb{N}_0$ and $k = k_1 + k_2$. Then

$$\begin{aligned}\mathbb{P}(\eta(B_1) = k_1, \eta(B_2) = k_2) &= \mathbb{P}(\varkappa = k)\frac{k!}{k_1!k_2!}p_{\mathrm{s}}(B_1)^{k_1}p_{\mathrm{s}}(B_2)^{k_2} \\ &= \frac{\gamma^k}{k!}\mathrm{e}^{-\gamma}\frac{k!}{k_1!k_2!}p_{\mathrm{s}}(B_1)^{k_1}p_{\mathrm{s}}(B_2)^{k_2} = \prod_{j=1}^{2}\frac{(\gamma p_{\mathrm{s}}(B_j))^{k_j}}{k_j!}\mathrm{e}^{-\gamma p_{\mathrm{s}}(B_j)}. \qquad (6.1)\end{aligned}$$

Summing over all $k_2 \in \mathbb{N}_0$, we get that $\eta(B_1) \in \mathrm{Po}(\gamma p_{\mathrm{s}}(B_1))$. Similarly,

$$\eta(B_2) \in \mathrm{Po}(\gamma p_{\mathrm{s}}(B_2)).$$

Hence, (6.1) implies that $\eta(B_1)$ and $\eta(B_2)$ are independent. We proved the properties (a) and (b) in Definition 5.1 of Poisson SPP for the case of two mutually complementing sets.

Step 2. It remains to note that this argument for the verification of Definition 5.1 can be extended to every $m \in \mathbb{N}$ and disjoint $B_1, \ldots, B_m \in \mathcal{F}_{\mathbb{X}}$ such that $\bigcup_{j=1}^{m} B_j = \mathbb{X}$ with

the use of the fact that

$$\mathbb{P}\left(\eta(B_1) = k_1, \dots, \eta(B_2) = k_m \mid \varkappa = k\right) = \frac{k!}{k_1! \dots k_m!} p_{\mathrm{s}}(B_1)^{k_1} \dots p_{\mathrm{s}}(B_m)^{k_m},$$

where $k = k_1 + \dots + k_m$. □

Exercise 6.1.
Let $m \in \mathbb{N}$ and $\varkappa_1, \dots, \varkappa_m$ are independent random variables with Poisson distributions and rates $\gamma_1, \dots, \gamma_m$, respectively. Then $\sum_{j=1}^m \varkappa_j$ has the Poisson distribution with the rate $\sum_{j=1}^m \gamma_j$.

Lemma 6.1 (superposition lemma)**.**
Let $\{\eta_j\}_{j=1}^m$, $m \in \widehat{\mathbb{N}}$, be a sequence of independent Poisson SPPs η_j with s-finite intensity measures μ_j (on $\mathbb{X}$). Then

$$\eta = \sum_{j=1}^m \eta_j \text{ is a Poisson SPP with intensity measure } \mu = \sum_{j=1}^m \mu_j.$$

Proof. Step 1. By Exercise 3.3 (a), η is an SPP.

Step 2. Let $m < \infty$. In order to prove that η is a Poisson SPP with intensity measure $\mu = \sum_{j=1}^m \mu_j$, it is enough to use Exercise 6.1.

Step 3. The case $m = +\infty$ can be proved by passing to the limit $\widetilde{m} \to \infty$ for nondecreasing sequences $\{\sum_{j=1}^{\widetilde{m}} \eta_j(A)\}_{\widetilde{m} \in \mathbb{N}}$, $A \in \mathcal{F}_{\mathbb{X}}$. □

Theorem 6.2 (existence of Poisson SPPs)**.**
Let μ be an s-finite measure on a measurable space $(\mathbb{X}, \mathcal{F}_{\mathbb{X}})$. Then there exists a proper Poisson SPP on $\mathbb{X}$ with the intensity measure μ.

Proof. Let $\mu = \sum_{j=1}^{+\infty} \mu_j$ be the sum of measures μ_j such that $\mu_j(\mathbb{X}) < \infty$ for all j. For every such μ_j, Theorem 6.1 implies that there exists a proper Poisson SPP $\widetilde{\eta}_j$ with the intensity measure μ_j on a certain probability space $(\Omega_j, \mathbb{P}_j)$. On the product space $\bigtimes_{j \in \mathbb{N}} \Omega_j$, there exists a sequence of induced independent proper Poisson SPPs η_j with intensity measures μ_j, $j \in \mathbb{N}$. It remains to apply Lemma 6.1 to $\sum \eta_j$. □

Remark 6.1.

(a) For every Poisson SPP η_1 on a measure space $\mathbb{X}$, there exists a proper Poisson SPP η_2 on $\mathbb{X}$ such that $\eta_1 \stackrel{\mathrm{d}}{=} \eta_2$. This follows from Corollary 5.2 and the proof of Theorem 6.2.

(b) There exists measures spaces $(\mathbb{X}, \mathcal{F}_{\mathbb{X}})$ and Poisson processes η on $\mathbb{X}$ with intensity measure μ such that $\mu(\mathbb{X}) < \infty$, but η is not proper (see, [LP, Exercise 3.9]). Theorem 3.2 implies that the measure space $(\mathbb{X}, \mathcal{F}_{\mathbb{X}})$ in any such example is pathological in the sense that $(\mathbb{X}, \mathcal{F}_{\mathbb{X}})$ is not a Borel space.

Theorem 6.3 (partial inversion of Theorem 6.1)**.**
Let η_1 be a Poisson process with intensity measure μ such that $\mu(\mathbb{X}) < \infty$. Then:

(a) η_1 has the same distribution as a mixed binomial process η_2 with mixing distribution $\mathrm{Po}(\mu(\mathbb{X}))$ and sampling distribution $p_{\mathrm{s}} = \frac{1}{\mu(\mathbb{X})}\mu$.

(b) For very $m \in \mathbb{N}$, the conditional distribution

$$\mathbb{P}\,(\eta_1 \in \cdot \mid \eta_1(\mathbb{X}) = m)$$

is the distribution of a binomial process of sampling size m and sampling distribution p_{s}.

Proof. (a) follows from Theorem 6.1 combined with Corollary 5.2 (about equality in distributions).

(b) is obvious for η_2. In order to obtain (b) for η_1 it is enough to use the equality $\eta_1 \stackrel{\mathrm{d}}{=} \eta_2$ obtained in the statement (a). □

Definition 6.2 (restrictions of deterministic and random measures)**.**
Let $A \in \mathcal{F}_{\mathbb{X}}$.

(a) Let μ be a measure on $\mathbb{X}$. Then the restriction of μ to A is the measure μ_A on $\mathbb{X}$ defined by

$$\mu_A(B) = \mu(A \cap B), \qquad B \in \mathcal{F}_{\mathbb{X}}.$$

(b) Let η be an SPP on $\mathbb{X}$. Then the map $\eta_A : \omega \mapsto \eta_A(\omega)$, $\omega \in \Omega$ is an SPP on $\mathbb{X}$, which is called the restriction of η on A.

Theorem 6.4.
Let η be a Poisson SPP with intensity measure μ, and let $\{A_j\}_{j\in\mathbb{N}} \subset \mathcal{F}_{\mathbb{X}}$ be a sequence of pairwise disjoint sets such that $\bigcup_{j\in\mathbb{N}} A_j = \mathbb{X}$. Then:

(a) η_{A_1}, η_{A_2}, ... is a sequence of independent Poisson SPPs with intensity measures μ_{A_1}, μ_{A_2}, ..., respectively.

(b) $\eta = \sum_{j\in\mathbb{N}} \eta_{A_j}$.

All statements of Theorem 6.4 are obvious, except the independence of η_{A_1}, η_{A_2}, ... This independence we leave as an exercise (see, e.g., the proof of [LP, Theorem 5.2]).

For every $B \in \mathcal{B}(\mathbb{R}^d)$ with $|B|_d \in \mathbb{R}_+$, Proposition 6.1 allows us to construct explicitly a proper homogeneous Poisson SPP with any given intensity $\gamma \in [0, +\infty)$ (i.e., with the intensity measure $\gamma|\cdot|_d$, where $\gamma \in [0, +\infty)$ is a constant).

Example 6.1 (explicit construction of a homogeneous Poisson SPP on $\mathbb{R}^d$)**.**
Let $\mathbb{R}^d = \bigcup_{j\in\mathbb{N}} B_j$ be a decomposition of $\mathbb{R}^d$ into a pairwise disjoint Borel subsets such that $|B_j|_d < \infty$ for all j. For every j, let $\widetilde{\eta}_j$ be a proper homogeneous Poisson SPP on B_j of intensity $\gamma \in [0, +\infty)$ defined on a probability space Ω_j. The sequence $\{\widetilde{\eta}_j\}_{j\in\mathbb{N}}$ generates a sequence of independent SPPs $\{\eta_j\}_{j\in\mathbb{N}}$ on the product space $\times_{j\in\mathbb{N}} \Omega_j$ with the property that $\eta_j \stackrel{\mathrm{d}}{=} \widetilde{\eta}_j$ for all j. Then $\eta = \sum_{j\in\mathbb{N}} \eta_j$ is a homogeneous Poisson SPP on $\mathbb{R}^d$ of intensity γ (an exercise).

6.2 Intensity measure for a general SPP and Campbell's formula

Definition 6.3 (intensity measure for a general SPP)**.**
Let η be an arbitrary SPP on a measure space $\mathbb{X}$. The intensity measure μ of η is defined by

$$\mu(A) := \mathbb{E}\,\eta(A), \qquad A \in \mathcal{F}_{\mathbb{X}}. \tag{6.2}$$

Remark 6.2.
Our terminology concerning intensity measures is consistent. That is:

(a) The intensity measure is indeed an $[0,+\infty]$-valued measure (an exercise).

(b) Let η be a Poisson SPP with s-finite intensity measure μ defined as in Definition 5.1. Then (6.2) holds true, and so, μ is indeed the intensity measure in the sense of Definition 6.3.

Statement (b) of Remark 6.1 follows from the following basic fact:

$$\text{for } \xi \in \mathrm{Po}(\gamma), \qquad \text{we have } \mathbb{E}\,\xi = \gamma \text{ (an exercise).}$$

We denote by $M(\mathbb{X},\mathbb{V})$ the family of measurable mappings from a measurable space $\mathbb{X}$ to a measurable space $\mathbb{V}$.

Let B be a Borel subset of $\widehat{\mathbb{R}} = [-\infty,+\infty]$, for example, $B = \widehat{\mathbb{R}}_+ := [0,+\infty]$. We denote by $M(\mathbb{X},B)$ the set of all measurable functions $u : \mathbb{X} \to B$. For $u \in M(\mathbb{X},\widehat{\mathbb{R}})$, the functions $u^{\pm} \in M(\mathbb{X},\widehat{\mathbb{R}}_+)$ are defined by

$$u^+(x) := \max\{u(x),0\}, \qquad u^- := \max\{-u(x),0\}.$$

Then

$$u = u^+ - u^-.$$

For every measure μ on $\mathbb{X}$, the integral of $u \in M(\mathbb{X},\widehat{\mathbb{R}})$ w.r.t. μ

$$\int u\mathrm{d}\mu = \int_{\mathbb{X}} u(x)\mu(\mathrm{d}x) =: \langle \mu, u\rangle$$

is defined as

$$\int u^+\mathrm{d}\mu - \int u^-\mathrm{d}\mu \tag{6.3}$$

whenever this expression is not of the form $(+\infty)-(+\infty)$. If (6.3) is of the form $(+\infty)-(+\infty)$, we put $\int u\mathrm{d}\mu := 0$ (following [LP]). If η is an SPP, then $\int u\mathrm{d}\eta = \langle \eta, u\rangle$ is the mapping

$$\omega \mapsto \int u(x)\eta(\omega,\mathrm{d}x), \qquad \omega \in \Omega.$$

Theorem 6.5 (Campbell's formula)**.**
Let η be an SPP on $\mathbb{X}$ with intensity measure μ, and let $u \in M(\mathbb{X},\widehat{\mathbb{R}})$. Then:

(a) $\int u\ \mathrm{d}\eta$ is an $\widehat{\mathbb{R}}$-valued random variable.

(b) If $u \geqslant 0$ or $\int |u|\mathrm{d}\mu < +\infty$,

$$\mathbb{E}\left[\int u\ \mathrm{d}\eta\right] = \int u\ \mathrm{d}\mu. \tag{6.4}$$

Proof. For an indicator-function $u = \chi_A$ with $A \in \mathcal{F}_{\mathbb{X}}$, (6.4) follows from (6.2). Then it is extended in the usual way to $u \in M(\mathbb{X},\widehat{\mathbb{R}}_+)$, and in turn, to all u such that $\int |u|\mathrm{d}\mu < +\infty$, see [LP, Proposition 2.7]. □

6.3 Image of SPPs under measurable mappings

Let $T : \mathbb{X} \to \mathbb{V}$ be a measurable mapping from the measurable space $(\mathbb{X}, \mathcal{F}_{\mathbb{X}})$ to a measurable space $(\mathbb{V}, \mathcal{F}_{\mathbb{V}})$, i.e., $T \in M(\mathbb{X}, \mathbb{V})$.

Definition 6.4.
For any measure μ on $\mathbb{X}$, the image of μ under T (or push-forward of μ) is the measure $T(\mu) = \mu \circ T^{-1}$, i.e.,

$$T(\mu)(A) = \mu(T^{-1}A), \qquad A \in \mathcal{F}_{\mathbb{V}}.$$

Theorem 6.6 (mapping theorem)**.**
Let η be an SPP on $\mathbb{X}$ with intensity measure μ, and let $T : \mathbb{X} \to \mathbb{V}$ be a measurable mapping. Let $T(\eta)$ be defined by

$$\omega \mapsto T(\eta(\omega)), \qquad \omega \in \Omega.$$

Then:

(a) $T(\eta)$ is an SPP with intensity measure $T(\mu)$.

(b) If $\eta : \omega \mapsto \sum_{j=1}^{\varkappa(\omega)} \delta_{y_j(\omega)}$, $\omega \in \Omega$, is a proper SPP on $\mathbb{X}$, then $T(\eta) = \sum_{j=1}^{\varkappa} \delta_{T(y_j)}$ is a proper SPP on $\mathbb{V}$.

(c) If η is a Poisson SPP, then $T(\eta)$ is a Poisson SPP too.

Proof. (a) The probabilistic measurability of $T(\eta)$ follows from the definition of $T(\eta)$ and the probabilistic measurability of η. By definition, the intensity measure of $T(\eta)$ is

$$\mathbb{E}[T(\eta)(A)] = \mathbb{E}[\eta(T^{-1}A)] = \mu[T^{-1}A] = T(\mu)(A), \quad A \in \mathcal{F}_{\mathbb{V}}.$$

(b) follows from the definition of $T(\eta)$ applied pointwise in $\omega \in \Omega$.

(c) If η is a Poisson SPP, it is straightforward to check the properties of Definition 5.1 (of a Poisson process) for $T(\eta)$. □

References for Lecture 6.

[LP] Last, G. and Penrose, M., 2017. Lectures on the Poisson process. Cambridge University Press.

7 Stationarity for general SPPs. Independent markings and the Boolean model.

7.1 Transformations and stationarity for general SPPs

Let $T : \mathbb{X} \to \mathbb{V}$ be a measurable mapping from the measurable space $\mathbb{X}$ to a measurable space $\mathbb{V}$, i.e., $T \in M(\mathbb{X}, \mathbb{V})$. Let η be an SPP on $\mathbb{X}$ with intensity measure μ. Statements (a) and (c) of Theorem 6.6 state that, for an SPP η with intensity measure μ, the mapping $T(\eta) : \Omega \to \mathfrak{N}_{\mathbb{V}}$ defined by

$$\omega \mapsto T(\eta(\omega)), \qquad \omega \in \Omega,$$

is an SPP with intensity measure $T(\mu)$. If η is a Poisson SPP, then $T(\eta)$ is a Poisson SPP too.

Let us fix now our measurable space as $\mathbb{X} = \mathbb{R}^d$.

Example 7.1.
Let η be a homogeneous Poisson SPP on $\mathbb{R}^d$ of intensity $\gamma \geqslant 0$. Let $\alpha \in \mathbb{R}$. Let

$$\mathrm{Dil}_\alpha = \mathrm{Dil}_{\alpha,d}$$

be the transformation of $\mathbb{R}^d$ defined by

$$\mathrm{Dil}_\alpha\, x = \alpha x, \quad x \in \mathbb{R}^d.$$

If $\alpha > 0$, then Dil_α is the dilation with ratio α, and

$$\mathrm{Dil}_\alpha\, \eta \text{ is a homogeneous Poisson SPP of intensity } \alpha^{-d}\gamma.$$

Could you describe $\mathrm{Dil}_\alpha\, \eta$ for $\alpha = 0$ and for $\alpha < 0$?

For arbitrary $y \in \mathbb{R}^d$, consider in $\mathbb{R}^d$ the mapping $\mathrm{Sh}_y : \mathbb{R}^d \to \mathbb{R}^d$ that shifts every site $x \in \mathbb{R}^d$ to $x + y$. Then for a measure μ on $\mathbb{R}^d$,

$$(\mathrm{Sh}_y\, \mu)(A) = \mu(\mathrm{Sh}_{-y}\, A) = \mu(A - y), \qquad A \in \mathcal{B}(\mathbb{R}^d).$$

Let $\mathfrak{N} = \mathfrak{N}_{\mathbb{R}^d}$ be the measurable space of measures on $\mathbb{R}^d$ introduced in Lecture 3. Following [LP], we define the map $\theta_y : \mathfrak{N} \to \mathfrak{N}$ by $\theta_y : \mu \mapsto \mathrm{Sh}_{-y}\, \mu$, i.e.,

$$\theta_y \mu(A) = \mu(A + y), \qquad A \in \mathcal{B}(\mathbb{R}^d).$$

We denote in the same way the induced map $\theta_y : M(\Omega, \mathfrak{N}) \to M(\Omega, \mathfrak{N})$ on the space of SPPs η on $\mathbb{R}^d$, i.e.,

$$\theta_y \eta_\omega(A) = \eta_\omega(A + y), \qquad A \in \mathcal{B}(\mathbb{R}^d), \quad \omega \in \Omega.$$

In particular, $\theta_y \delta_y = \delta_0$ and, for a proper SPP $\eta : \omega \mapsto \sum_{j=1}^{\varkappa(\omega)} \delta_{y_j(\omega)}$, we have

$$T(\eta) = \sum_{j=1}^{\varkappa} \delta_{y_j - y}.$$

For a general SPP η on $\mathbb{R}^d$, we have

$$\int g(x)\ (\theta_y\mu)(\mathrm{d}x) = \int g(x-y)\ \mu(\mathrm{d}x), \qquad \mu \in \mathfrak{N}, \quad g \in M(\mathbb{R}^d, \mathbb{R}_+).$$

The family $\{\theta_y\}_{y\in\mathbb{R}^d}$ builds a commutative group of measurable maps $\theta_y : \mathfrak{N} \to \mathfrak{N}$ with identity θ_0. The induced maps $\theta_y : M(\Omega, \mathfrak{N}) \to M(\Omega, \mathfrak{N})$ build a commutative group $\{\theta_y\}_{y\in\mathbb{R}^d}$ of maps on $M(\Omega, \mathfrak{N})$ with identity θ_0. These statements follow from the obvious *flow property*

$$\theta_y \circ \theta_x = \theta_{x+y}, \qquad x, y \in \mathbb{R}^d.$$

Definition 7.1.
An SPP η on $\mathbb{R}^d$ is said to be stationary if $\theta_x\eta \stackrel{\mathrm{d}}{=} \eta$ for all $x \in \mathbb{R}^d$.

Example 7.2.
A homogeneous Poisson SPP η on $\mathbb{R}^d$ is stationary. This follows from Theorem 6.6 and the invariance of the intensity measure $\gamma|\cdot|_d$ of η under the transformations θ_y.

Example 7.3.
There are examples of non-Poisson stationary SPPs (see [LP, Section 8.1]). Let us consider one such example. Let

$$\mathcal{C}_1 = \mathcal{C}_{1,\mathbb{R}^d} := [0,1)^d$$

be a half-open unit cube. Let $\mu \in \mathfrak{N}_{<\infty}(\mathbb{R}^d)$ be a nonzero measure such that $\mu(\mathbb{R}^d\backslash\mathcal{C}_1) = 0$, i.e., $\varnothing \subsetneqq \operatorname{supp}\mu \subseteq \mathcal{C}_1$. Let ξ be an $\mathbb{R}^d$-valued random variable uniformly distributed on $\mathcal{C}_1$. Then

$$\eta := \sum_{y\in\mathbb{Z}^d} \theta_{y+\xi}\ \mu$$

is a stationary non-Poisson SPP with the intensity measure $\mu(\mathcal{C}_1)|\cdot|_d$ (an exercise).

Proposition 7.1.
Let η be a stationary SPP on $\mathbb{R}^d$ with intensity measure μ such that

$$\gamma := \mu(\mathcal{C}_{1,\mathbb{R}^d}) = \mathbb{E}\ \eta([0,1)^d) < +\infty$$

Then

$$\mu = \gamma|\cdot|_d.$$

In this case, the quantity $\gamma \in [0, +\infty)$ is called the intensity of the stationary SPP η.

Proof. The proof follows from the following facts:

(a) $\theta_x\mu = \mu$ for all $x \in \mathbb{R}^d$ (translation invariance);

(b) the family of deterministic locally finite translation invariant measures on $\mathbb{R}^d$ is

$$\{\alpha|\cdot|_d\}_{\alpha\in\overline{\mathbb{R}}_+}.$$

□

Proposition 7.2.
Let η be a Poisson SPP on $\mathbb{R}^d$ with intensity measure μ such that

$$\gamma := \mu(\mathcal{C}_1) = \mathbb{E}\ \eta([0,1)^d) < +\infty.$$

Then η is stationary if and only if $\mu = \gamma|\cdot|_d$.

Proof. The proof follows immediately from combination of Example 7.2 and Proposition 7.1. □

7.2 Independent marking and Boolean model of stochastic geometry

We need a particular type of marked SPP, which are produced by a independent $p_{\mathbb{V}}$-marking of a proper SPP η on $\mathbb{X}$.

Definition 7.2 (independent marking)**.**
Let $p_{\mathbb{V}}$ be a certain probability measure on a certain measurable space $\mathbb{V}$, and let v_1, v_2, ... be i.i.d. $\mathbb{V}$-valued random variables with distribution $p_{\mathbb{V}}$. Let $\eta = \sum_{j=1}^{\varkappa} \delta_{y_j}$ be a proper SPP on a measure space $\mathbb{X}$ independent of $\{v_j\}_{j\in\mathbb{N}}$. Then the proper SPP ξ defined on the product measure space $\mathbb{X} \times \mathbb{V}$ by

$$\xi := \sum_{j=1}^{\varkappa} \delta_{(y_j,v_j)}$$

is called an independent $p_{\mathbb{V}}$-marking of η.

This definition is a simplified version of a general definition of K-marking, see [LP, Sections 5.2 and 16.1]. For existence of such independent $p_{\mathbb{V}}$-markings, we refer to [MR, Sections 1.4], and in more general settings in [LP, Section 5.2].

Theorem 7.1.
Let η be a proper SPP with intensity measure μ. Let ξ be an independent $p_{\mathbb{V}}$-marking of η. Then:

(a) The intensity measure of ξ is the product measure $\mu \otimes p_{\mathbb{V}}$.

(b) In the case where η is a proper Poisson SPP, ξ is also a proper Poisson SPP.

Proof of statement (a) of Theorem 7.1. It is enough to verify the formula for the intensity measure on the sets $A \times B$ with $A \in \mathcal{F}_{\mathbb{X}}$ and $B \in \mathcal{F}_{\mathbb{V}}$. Then

$$\begin{aligned}\mathbb{E}\,\xi(A\times B) &= \sum_{m\in\hat{\mathbb{N}}_0} \mathbb{E}\left[\chi_{\{\varkappa=m\}} \sum_{j=1}^{m} \chi_{\{(y_j,v_j)\in A\times B\}}\right] = \sum_{m\in\hat{\mathbb{N}}_0} \mathbb{E}\left[\chi_{\{\varkappa=m\}} \sum_{j=1}^{m} \chi_{\{y_j\in A\}}\chi_{\{v_j\in B\}}\right] \\ &= \mathbb{E}\chi_{\{v_j\in B\}} \sum_{m\in\hat{\mathbb{N}}_0} \mathbb{E}\left[\chi_{\{\varkappa=m\}} \sum_{j=1}^{m} \chi_{\{y_j\in A\}}\right] = p_{\mathbb{V}}(B)\ \mathbb{E}\,\eta(A) = \mu(A)p_{\mathbb{V}}(B).\end{aligned}$$

□

We do not give a complete proof of statement (b) of Theorem 7.1. One of the proofs can be obtained using the notion of Laplace functional.

Definition 7.3.
The Laplace functional of an SPP η on $\mathbb{X}$ is the map $L_\eta : M(\mathbb{X}, \overline{\mathbb{R}}_+) \to [0,1]$ defined by

$$L_\eta(u) := \mathbb{E}\ \mathrm{e}^{-\langle\eta,u\rangle} = \mathbb{E}\ \exp\left[-\int_{\mathbb{X}} u\mathrm{d}\eta\right], \quad u \in M(\mathbb{X}, \overline{\mathbb{R}}_+).$$

The Laplace functional plays for SPPs the role similar to that of characteristic functions and Laplace transforms of real-valued random variables.

Remark 7.1.

(a) Let η_1 and η_2 are two SPPs on $\mathbb{X}$. Then $\eta_1 \stackrel{\mathrm{d}}{=} \eta_2$ if and only if $L_{\eta_1} = L_{\eta_2}$.

(b) If η has is a binomial process on $\mathbb{X}$ with sample size m and sampling distribution p_{s}, then

$$L_\eta(u) = \left(\int_{\mathbb{X}} \mathrm{e}^{-u(x)} p_{\mathrm{s}}(\mathrm{d}x) \right)^m .$$

(c) An SPP η on $\mathbb{X}$ is a Poisson SPP with intensity measure μ if and only if

$$L_\eta(u) = \exp\left[\int (\mathrm{e}^{-u(x)} - 1) \ \mu(\mathrm{d}x) \right]$$

for all $u \in M(\mathbb{X}, \overline{\mathbb{R}}_+)$.

Our main example of the use of marked SPP is the following stochastic geometry construction.

Definition 7.4.
Let $\eta = \sum_{j\in\mathbb{N}} \delta_{y_j}$ be a homogeneous Poisson SPP on $\mathbb{R}^d$ with intensity $\gamma \in \mathbb{R}_+$. Let p_{r} be a probability measure on $[0, +\infty)$, and let

$$\xi := \sum_{j=1}^{\varkappa} \delta_{(y_j, r_j)}$$

be an independent p_{r}-marking of η (where r_j are i.i.d. $\overline{\mathbb{R}}_+$-valued random variables with distribution p_{r} like in Definition 7.2). Then the randomized set

$$\mathcal{Z} := \bigcup_{j\in\mathbb{N}} \overline{\mathbb{B}}_{r_j}(y_j) = \bigcup_{j\in\mathbb{N}} \{x \in \mathbb{R}^d : |x - y_j| \leqslant r_j\},$$

is called (Poisson spherical) Boolean model with intensity γ and radius distribution p_{r}.

This randomized set $\mathcal{Z} : \omega \mapsto \mathcal{Z}(\omega)$, $\omega \in \Omega$, is random in the sense of probabilistic measurability explained in the next theorem.

Let $\mathrm{Comp}(\mathbb{R}^d)$ be the family of all compact subsets of $\mathbb{R}^d$. Recall that Minkowski sum of sets $S_1, S_1 \subseteq \mathbb{R}^d$ is defined by

$$S_1 + S_2 = \{ \ y_1 + y_2 \ : \ y_1 \in S_1, \ y_2 \in S_2\}.$$

The sum $S_1 + \overline{\mathbb{B}}_r$ is called the r-parallel set of S_1.

Theorem 7.2.
Let $\mathcal{Z}$ be a Boolean model with intensity γ and radius distribution p_{r}. Then:

(a) $\mathcal{Z} : \omega \mapsto \mathcal{Z}(\omega)$ is random in the sense of the following probabilistic measurability statement: for every $S \in \mathrm{Comp}(\mathbb{R}^d)$,

$$\{\mathcal{Z} \cap S = \varnothing\} = \{\omega \in \Omega \ : \ \mathcal{Z}(\omega) \cap S = \varnothing\} \in \mathcal{F}_\Omega.$$

(b) For every $S \in \mathrm{Comp}(\mathbb{R}^d)$,

$$\mathbb{P}(\mathcal{Z} \cap S = \varnothing) = \exp\left[-\gamma \int |S + \overline{\mathbb{B}}_r|_d \ p_{\mathrm{r}}(\mathrm{d}r) \right].$$

The map $S \mapsto \mathbb{P}(\mathcal{Z} \cap S = \varnothing)$ is called the capacity functional of $\mathcal{Z}$.

Proof. Let $S \in \mathrm{Comp}(\mathbb{R}^d)$ be arbitrary.

(a) It is easy to see that the set

$$A := \{(x, r) \in \mathbb{R}^d \times \overline{\mathbb{R}}_+ : \ \overline{\mathbb{B}}_r(x) \cap S \neq \varnothing\}$$

is closed. Indeed, let $(x_n, r_n) \to (x_*, r_*)$ as $n \to \infty$ and $\{(x_n, r_n)\}_{n\in\mathbb{N}} \subseteq A$. Then there exist $y_n \in \overline{\mathbb{B}}_{r_n}(x_n) \cap S$ for all n. Since S is compact, the exists a converging subsequence $\{y_{n_j}\}_{j\in\mathbb{N}}$ with a limit $y_* \in S$. Besides, $|y_* - x_*| \leqslant r_*$. Hence, $(x_*, r_*) \in A$.

Since A is closed, we see that $A \in \mathcal{B}(\mathbb{R}^d \times \overline{\mathbb{R}}_+)$. On the other hand,

$$\{\omega \in \Omega : \ \mathcal{Z} \cap S = \varnothing\} = \{\omega \in \Omega : \ \xi(A) = 0\}. \tag{7.1}$$

Since ξ is an SPP on $\mathbb{R}^d \times \overline{\mathbb{R}}_+$, we see that $\{\mathcal{Z} \cap S = \varnothing\}$ is an event, i.e., $\{\mathcal{Z} \cap S = \varnothing\} \in \mathcal{F}_\Omega$.

(b) By Theorem 7.1, ξ is a Poisson SPP on $\mathbb{R}^d \times \overline{\mathbb{R}}_+$ with intensity measure $\mu = \gamma|\cdot|_d \otimes p_{\mathrm{r}}$. Using (7.1), we get

$$\begin{aligned}
\mathbb{P}(\mathcal{Z} \cap S = \varnothing) = \mathrm{e}^{-\mu(A)} &= \exp\left[-\gamma \int_{\overline{\mathbb{R}}_+} \int_{\mathbb{R}^d} \chi_A(x, r) \ \mathrm{d}x \ p_{\mathrm{r}}(\mathrm{d}r)\right] \\
&= \exp\left[-\gamma \int_{\overline{\mathbb{R}}_+} \int_{\mathbb{R}^d} \chi_{\{\overline{\mathbb{B}}_r(x) \cap S \neq \varnothing\}}(x, r) \ \mathrm{d}x \ p_{\mathrm{r}}(\mathrm{d}r)\right] \\
&= \exp\left[-\gamma \int_{\overline{\mathbb{R}}_+} |S + \overline{\mathbb{B}}_r|_d \ p_{\mathrm{r}}(\mathrm{d}r)\right].
\end{aligned}$$

□

7.3 Volume fraction and covering property for Boolean models

Let $\mathcal{Z}$ be a Boolean model with intensity $\gamma \in \mathbb{R}_+$ and radius distribution p_{r}. Let r_0 be an $\overline{\mathbb{R}}_+$-valued random variable with distribution p_{r}.

Applying Theorem 7.2 to the case $S = \{x\}$, we obtain the following result for arbitrary $x \in \mathbb{R}^d$, in particular, for $x = 0 = 0_{\mathbb{R}^d}$.

Corollary 7.1 (volume fraction)**.**
Let

$$\Pi_0 := \mathbb{P}(0 \in \mathcal{Z}).$$

Then

$$\Pi_0 = \mathbb{P}(x \in \mathcal{Z}) = 1 - \exp\left[-\gamma |\mathbb{B}_1|_d \ \mathbb{E}[r_0^d]\right] \quad \textit{for all } x \in \mathbb{R}^d.$$

The quantity Π_0 *is called the* volume fraction of $\mathcal{Z}$*.*

Note that $\Pi_0 > 0$ if and only if $\mathbb{P}(r_0 \neq 0) > 0$. In other words, $\Pi_0 = 0$ if and only if $r_0 = 0$ a.s..

Proposition 7.3.
The set

$$\{(\omega, x) \in \Omega \times \mathbb{R}^d : \ x \in \mathcal{Z}(\omega)\}$$

is measurable in the measurable product-space $\Omega \times \mathbb{R}^d$*.*

Theorem 7.3.

(a) Let $B \in \mathcal{B}(\mathbb{R}^d)$. If $\Pi_0 > 0$ or $|B|_d < +\infty$, then

$$\mathbb{E}\ |\mathcal{Z} \cap B|_d = \Pi_0 |B|_d$$

(this formula explains the name 'volume fraction' for Π_0).

(b) $\mathbb{P}(\mathcal{Z} = \mathbb{R}^d) = 1$ if and only if $\mathbb{E}[r_0^d] = +\infty$.

References for Lecture 7.

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[MR] Meester, R. and Roy, R., 1996. Continuum percolation. Cambridge University Press.

8 Continuum percolation.

Let us recall the definition of Boolean model of stochastic geometry driven by a homogeneous Poisson SPP. Let $\eta = \sum_{j\in\mathbb{N}} \delta_{y_j}$ be a homogeneous Poisson SPP on $\mathbb{R}^d$ with intensity $\gamma \in \mathbb{R}_+ = (0, +\infty)$. Let p_{r} be a probability measure on $\overline{\mathbb{R}}_+ = [0, +\infty)$, and let

$$\xi := \sum_{j\in\mathbb{N}} \delta_{(y_j, r_j)}$$

be an independent p_{r}-marking of η. Then the random set

$$\mathcal{Z} := \bigcup_{j\in\mathbb{N}} \overline{\mathbb{B}}_{r_j}(y_j) = \bigcup_{j\in\mathbb{N}} \{x \in \mathbb{R}^d : |x - y_j| \leqslant r_j\},$$

is called Boolean model with intensity γ and radius distribution p_{r}. The set-valued map $\mathcal{Z} : \omega \mapsto \mathcal{Z}(\omega)$ is random in the sense that, for every $S \in \mathrm{Comp}(\mathbb{R}^d)$,

$$\{\mathcal{Z} \cap S = \varnothing\} = \{\omega \in \Omega \ : \ \mathcal{Z}(\omega) \cap S = \varnothing\} \in \mathcal{F}_\Omega.$$

In the sequel, r_0 is an $\overline{\mathbb{R}}_+$-valued random variable with distribution p_{r}. Note that r_1, r_2, ..., are i.i.d. random variables with the distribution p_{r}.

8.1 Percolation functions for Boolean models on $\mathbb{R}^d$.

By Theorem 7.3, $\mathbb{P}(\mathcal{Z} = \mathbb{R}^d) = 1$ if and only if $\mathbb{E}[r_0^d] = +\infty$. The event $\{\mathcal{Z} = \mathbb{R}^d\}$ is an extreme example of the percolation. The (continuum) percolation for a random set is generally understood as an existence of unbounded connected component of this random set or the existence of the unbounded connected component of this random set that has a nonempty intersection with a given deterministic set S, see [MR]. It seems that the first appearance of continuum percolation models is attributed to Gilbert [G61].

Definition 8.1 (occupied components)**.**
Let $\mathcal{Z} : \omega \mapsto \mathcal{Z}(\omega)$ be a Boolean model on $\mathbb{R}^d$ with intensity $\gamma \in \mathbb{R}_+$ and radius distribution p_{r}.

(a) Connected components of $Z(\omega)$ are called occupied components.

(b) Let S be a deterministic subset of $\mathbb{R}^d$. The union of all occupied components that has a nonempty intersection with S is denoted by

$$W^S : \omega \mapsto W^S(\omega).$$

So, W^S is a randomized subset of $\mathbb{R}^d$.

(c) By $W := W^{\{0\}}$, the occupied component containing the origin $0 \in \mathbb{R}^d$ is denoted.

Note that

$$\{\omega \in \Omega : \ W = \varnothing\} = \{\omega \in \Omega : \ 0 \notin \mathcal{Z}\}.$$

Proposition 8.1 (percolation function)**.**
(a) The map $\eta(W) : \omega \mapsto \eta_\omega(W(\omega))$ is an $\widehat{\mathbb{N}}_0$-valued random variable.

(b) The following equality holds

$$\mathbb{P}(\ \eta(W) = +\infty\ ) = \mathbb{P}(\ W \text{ is unbounded }\ ).$$

The percolation function for the Boolean model $\mathcal{Z}$ is defined as

$$\Theta(\gamma, p_{\mathrm{r}}) := \mathbb{P}(\ W \text{ is unbounded }\ ).$$

We leave this proposition without a proof. One of the proofs can be based on Gilbert graphs and random-connection models, see [LP, Section 16.4] and [MR, Section 1.5]. Note that here $W : \omega \mapsto W(\omega)$ is a randomized set, and so, it is not obvious that

$$\eta(W) : \omega \mapsto \int_{W(\omega)} \mathrm{d}\eta_\omega$$

is a random variable. Statement (b) of this proposition follows from the fact that the homogeneous Poisson SPP η is locally finite.

Definition 8.2.

(a) For two real-valued random variables α_1 and α_2, one says that α_1 stochastically dominates α_2 and writes $\alpha_2 \preceq \alpha_1$ if for their cumulative distribution functions

$$F_{\alpha_j}(s) = \mathbb{P}(\alpha_j \leqslant s), \qquad s \in \mathbb{R},$$

the inequality

$$F_{\alpha_1}(s) \leqslant F_{\alpha_2}(s)$$

holds for all $s \in \mathbb{R}$.

(b) The property of stochastic dominance is actually property of distributions of random variables. Therefore, we will apply this definition also to the corresponding probability measures on $\mathbb{R}$ using the same notation '$\preceq$'.

Theorem 8.1 (monotonicity of percolation functions)**.**
Let $\mathcal{Z}_j$, $j = 1, 2$, are two Boolean models with intensities γ_j and radius distributions p_{r}^j, $j = 1, 2$. If $\gamma_1 \leqslant \gamma_2$ and $p_{\mathrm{r}}^1 \preceq p_{\mathrm{r}}^2$, then

$$\Theta(\gamma_1, p_{\mathrm{r}}^1) \leqslant \Theta(\gamma_2, p_{\mathrm{r}}^2).$$

The main question is to find sufficient and/or necessary conditions for $\Theta(\gamma, p_{\mathrm{r}}) > 0$. Percolation models are used, e.g., for modeling of dielectric breakdown and lightning leaders [DGB$^+$98] (see Figure 1).

By Theorem 7.3, the condition

$$\mathbb{E}[r_0^d] = \int_0^{+\infty} r^d \, \mathrm{d}p_{\mathrm{r}} = +\infty$$

ensures $\Theta(\gamma, p_{\mathrm{r}}) = 1$ for every $\gamma > 0$. In the case $d = 1$, this condition completely characterizes the percolation.

Figure 1: Lighting barrage [O].

8.2 Continuum percolation in 1-dimensional case.

Theorem 8.2.
Let $d = 1$ and let $\mathcal{Z}$ be a Boolean model on $\mathbb{R}$ with intensity $\gamma \in \mathbb{R}_+$ and radius distribution p_{r}. Then the following statements hold:

(a) Let $\mathbb{E}[r_0] < +\infty$. Then all occupied components are bounded with probability 1. In particular, $\Theta(\gamma, p_{\mathrm{r}}) = 0$ for all $\gamma \in \mathbb{R}_+$.

(b) Let $\mathbb{E}[r_0] = +\infty$. Then $\mathbb{P}(\mathcal{Z} = \mathbb{R}) = 1$, and so, $\Theta(\gamma, p_{\mathrm{r}}) = 1$ for all $\gamma \in \mathbb{R}_+$.

We leave this theorem without a proof (for the proof, see [MR, Sections 3.1-3.2]).

8.3 Continuum percolation in $\mathbb{R}^d$ with $d \geqslant 2$.

Let $d \geqslant 2$ and let $\mathcal{Z}$ be a Boolean model on $\mathbb{R}^d$ with intensity $\gamma \in \mathbb{R}_+$ and a radius distribution p_{r}. Let us assume the radius distribution p_{r} to be fixed and let us study the function

$$\Theta(\gamma) := \Theta(\gamma, p_{\mathrm{r}})$$

changing γ in the interval $(0, +\infty)$. Recall that

$$\Theta(\gamma) = \mathbb{P}(\ W \text{ is unbounded } \ ) = \mathbb{P}(\ \eta(W) = +\infty\ ).$$

Theorem 8.3 (critical intensity parameter, [MR, G08, GT19])**.**
Suppose $0 < \mathbb{E}[r_0^d] < +\infty$. Then there exists a critical intensity $\gamma_{\mathrm{c}} = \gamma_{\mathrm{c}}(p_{\mathrm{r}}) \in (0, +\infty)$ such that:

(a) $\Theta(\gamma) = 0$ for all $\gamma \in (0, \gamma_{\mathrm{c}})$,

(b) $\Theta(\gamma) > 0$ for all $\gamma > \gamma_{\mathrm{c}}$.

Remark 8.1.

(a) In other words, the phase transition *takes place when γ crosses the critical value γ_{c}.*

(b) From Theorems 8.3 and 7.3 (b), we see that (for $d \geqslant 2$) a nontrivial critical value $\gamma_{\mathrm{c}} \in (0, +\infty)$ exists if and only if $0 < \mathbb{E}[r_0^d] < +\infty$. Indeed, if $\mathbb{E}[r_0^d] = 0$, we have with probability 1 that all $r_j = 0$ and $\mathcal{Z} = \{y_j\}_{j=1}^{+\infty}$ consists of isolated points. So, $\mathbb{E}[r_0^d] = 0$ implies $\Theta(\gamma) = 0$ for all $\gamma \in (0, +\infty)$. If $\mathbb{E}[r_0^d] = +\infty$, Theorem 7.3 (b) implies that $\Theta(\gamma) = 1$ for all $\gamma \in (0, +\infty)$.

Theorem 8.4 (uniqueness of an unbounded component [MR, Theorem 3.6])**.**
With probability 1, $\mathcal{Z}$ has at most one unbounded occupied component.

Corollary 8.1.
Suppose $0 < \mathbb{E}[r_0^d] < +\infty$. Then

(a) For $\gamma \in (0, \gamma_c)$, all occupied components are bounded with probability 1.

(b) For $\gamma \in (\gamma_c, +\infty)$, $0 < \Theta(\gamma) < 1$ and

$$\Theta(\gamma) \leqslant \mathbb{P}(\exists \text{ exactly one unbounded occupied component}) = \mathbb{P}(\exists \text{ an unbounded occupied component}).$$

Proof. Step 1. Since η is a stationary SPP,

$$\mathbb{P}(\ W^{\{x\}} \text{ is unbounded }\ ) = \mathbb{P}(\ W^{\{0\}} \text{ is unbounded }\ ) = \Theta(\gamma). \tag{8.1}$$

for every deterministic $x \in \mathbb{R}^d$. (While this statement is intuitively clear, its rigorous proof is lengthy, and we take (8.1) without proof).

Step 2. The proof of (a). Let $\gamma \in (0, \gamma_c)$. By Theorem 8.3 and equality (8.1), for every $x \in \mathbb{R}^d$, we have

$$\mathbb{P}(\ W^{\{x\}} \text{ is unbounded }\ ) = 0.$$

Applying this equality to x belonging to the countable dense set $\mathbb{Q}^d$, we see from the σ-additivity of the measure $\mathbb{P}(\cdot)$ that

$$\mathbb{P}\left(\bigcup_{x \in \mathbb{Q}^d} \{\omega : W^{\{x\}} \text{ is unbounded } \} \right) = 0.$$

However, every unbounded occupied component contains a certain $x \in \mathbb{Q}^d$. Thus,

$$\mathbb{P}(\exists \text{ an unbounded occupied component}) \leqslant \mathbb{P}\left(\bigcup_{x \in \mathbb{Q}^d} \{\omega : W^{\{x\}} \text{ is unbounded } \} \right) = 0.$$

This proves (a).

Step 3. The proof of (b). Let $\gamma \in (\gamma_c, +\infty)$. Theorems 8.3 and 8.4 imply

$$0 < \Theta(\gamma) \leqslant \mathbb{P}(\exists \text{ an unbounded occupied component}) = \mathbb{P}(\exists \text{ exactly one unbounded occupied component}).$$

It remains to prove $\Theta(\gamma) < 1$. Assume that $\Theta(\gamma) = 1$. Then

$$\Theta(\gamma) = \mathbb{P}(\ W \text{ is unbounded }\ ) = 1,$$

and so, the volume fraction of $\mathcal{Z}$ is $\Pi_0 = \mathbb{P}(0 \in \mathcal{Z}) = 1$. This and Corollary 7.1 (about the value of the volume fraction) imply $\mathbb{E}[r_0^d] = +\infty$. This contradicts the assumption $\mathbb{E}[r_0^d] < +\infty$. □

Example 8.1 (the case of deterministic radius $r_0 \equiv R = 1$, see [MR, Theorem 3.10])**.**
Let $p_r = \delta_1$ (i.e., $r_0 = 1$ a.s.). Then for the Boolean model on $\mathbb{R}^2$,

$$0.174 < \gamma_c < 0.843.$$

8.4 Scaling of continuum percolation

Let $\mathcal{Z}$ be a Boolean model on $\mathbb{R}^d$ with $d \geqslant 2$ and

$$0 < \mathbb{E}[r_0^d] = \int_0^{+\infty} r^d \, \mathrm{d}p_{\mathrm{r}} < +\infty.$$

Then Theorem 8.3 defines the critical intensity $\gamma_{\mathrm{c}} = \gamma_{\mathrm{c}}(p_{\mathrm{r}})$ as a function depending on the radius distribution p_{r}.

Recall that, for $\alpha > 0$, the transformation $\mathrm{Dil}_{\alpha,n}$ of $\mathbb{R}^n$ defined by

$$\mathrm{Dil}_{\alpha,n}\, x = \alpha x, \quad x \in \mathbb{R}^n,$$

generates a transformation $\mathrm{Dil}_{\alpha,n}$ of measures μ on $\mathbb{R}^n$ by the formula

$$\mathrm{Dil}_{\alpha,n}\, \mu(B) = \mu(\mathrm{Dil}_{\alpha,n}^{-1}(B)) = \mu(\mathrm{Dil}_{\alpha^{-1},n}(B)), \qquad B \in \mathcal{B}(\mathbb{R}^n).$$

Corollary 8.2.
Let $0 < \alpha < +\infty$. Then

$$\gamma_{\mathrm{c}}(\mathrm{Dil}_{\alpha,1}\, p_{\mathrm{r}}) = \frac{\gamma_{\mathrm{c}}(p_{\mathrm{r}})}{\alpha^d}. \tag{8.2}$$

(Note that the probability measure $\mathrm{Dil}_{\alpha,1}\, p_{\mathrm{r}}$ supported on $[0, +\infty)$ is the distribution of the random variable αr_0.)

Proof. The corollary follows from Example 7.1. Indeed, the dilation $\mathrm{Dil}_{\alpha,d}$ applied to the Boolean model $\mathcal{Z}$ with intensity γ and a radius distribution p_{r} does not change the probability $\mathbb{P}(\ W \text{ is unbounded }\ )$. However, the underlying homogeneous Poisson SPP changes its intensity to $\alpha^{-d}\gamma$ and the radius distribution becomes $\mathrm{Dil}_{\alpha,1}\, p_{\mathrm{r}}$. □

8.5 Critical radius for random geometric graph

Let $R, R_1, R_2 \in (0, +\infty)$ be deterministic. Consider now the case where the radius distribution

$$p_{\mathrm{r}} = \delta_R$$

is the distribution of a deterministic random variable $r_0 \equiv R$. (The corresponding Gilbert graph of the random connection model is called *random geometric graph*, see [LP, Section 16.4]).

Then (8.2) becomes the scaling law for the associated critical intensities

$$\gamma_{\mathrm{c}}(\delta_{R_1}) R_1^d = \gamma_{\mathrm{c}}(\delta_{R_2}) R_2^d = \gamma_{\mathrm{c}}(\delta_1). \tag{8.3}$$

Corollary 8.3.
Let $d \geqslant 2$ and $\gamma \in \mathbb{R}_+$ be fixed. For a Boolean model $\mathcal{Z}$ on $\mathbb{R}^d$ with intensity γ and the deterministic radius distribution $p_{\mathrm{r}} = \delta_R$, we define

$$\widetilde{\Theta}(R) := \Theta(\gamma, \delta_R) = \mathbb{P}(\ W \text{ is unbounded }\ ).$$

Then there exists the critical radius value $R_{\mathrm{c}} = \left(\frac{\gamma_{\mathrm{c}}(\delta_1)}{\gamma}\right)^{1/d} \in (0, +\infty)$ such that the following statements hold:

(a) For all $R \in (0, R_c)$,

$$\widetilde{\Theta}(R) = 0$$

and all occupied components are bounded with probability 1.

(b) For all $R \in (R_c, +\infty)$,

$$0 < \widetilde{\Theta}(R) < 1$$

and

$$\begin{aligned}\widetilde{\Theta}(R) &\leqslant \mathbb{P}(\exists \textit{ exactly one unbounded occupied component}) \\ &= \mathbb{P}(\exists \textit{ an unbounded occupied component}).\end{aligned}$$

Proof. In order to obtain the proof from Corollary 8.1 and formula (8.3), it is enough to consider in $\mathbb{R}^2$ the graph of the function $R \mapsto \gamma_c(\delta_R) = \frac{\gamma_c(\delta_1)}{R^d}$ and its intersection with the horizontal level line $\{(R, \gamma) \in \mathbb{R}^2 : R \in \mathbb{R}_+\}$. □

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9 Closed and closable operators. Closures of operators and connections with adjoint operators.

9.1 Abstract closed operators. Closable operators and closure.

Let V, X, X_1, and X_2 be Hilbert spaces. Recall the operators in this course are linear (if it is not states explicitly otherwise). Recall that the graph of an operator

$$A : \operatorname{dom} A \subseteq X \to V$$

is defined as the following linear subspace of $X \times V$:

$$\operatorname{Gr} A = \Big\{ \{u, Au\} \ : \ u \in \operatorname{dom} A \Big\}.$$

The orthogonal sum of Hilbert spaces $X \oplus V$ is the Hilbert space produced by the linear space $X \times V$ and the inner product

$$(\{x_1, v_1\}|\{x_2, v_2\})_{X\oplus V} := (x_1|x_2)_X + (v_1|v_2)_V.$$

Recall that, for a subset of S of a metric (or topological space) space, we denote by $\overline{S}$ the closure of S.

Definition 9.1 (closed operator, e.g., [Kato])**.**
An operator $A : \operatorname{dom} A \subseteq X \to V$ is called closed if its graph $\operatorname{Gr} A$ is a closed subspace of the orthogonal sum $X \oplus V$.

Definition 9.2 (graph norm, e.g., [Kato])**.**

(a) The orthogonal sum $X \oplus V$ is a Hilbert space that induces an inner product and a norm on its linear subspace $\operatorname{Gr} A$. This norm is called the graph norm.

(b) The associated norm on $\operatorname{dom} A$

$$\|u\|_A := \big(\|u\|_X^2 + \|Au\|_V^2\big)^{1/2},$$

is also called the graph norm of A.

The normed spaces $(\operatorname{Gr} A, \|\cdot\|_{X\oplus V})$ and $(\operatorname{dom} A, \|\cdot\|_A)$ can be equipped with inner products compatible with the graph norms. An operator A is closed if and only if the normed space $(\operatorname{Gr} A, \|\cdot\|_{X\oplus V})$ is complete (or equivalently, if and only if the normed space $(\operatorname{dom} A, \|\cdot\|_A)$ is complete). If this is the case, than $(\operatorname{Gr} A, \|\cdot\|_{X\oplus V})$ and $(\operatorname{dom} A, \|\cdot\|_A)$ are Hilbert spaces with inner products associated with the corresponding graph norms.

Definition 9.3 (kernel, range, and inverse)**.**

(a) The kernel of an operator $A : \operatorname{dom} A \subseteq X \to V$ is defined as

$$\ker A := \{u \in \operatorname{dom} A \ : \ Au = 0_V\}.$$

(b) The operator A is injective if and only if $\ker A = \{0\}$. In this case, one says that A is invertible (we do not assume here that the inverse A^{-1} is bounded).

(c) The domain $\operatorname{dom} A^{-1}$ of the inverse operator A^{-1} is by the definition the range $\operatorname{ran} A$ of A.

(d) The range of A is defined by

$$\operatorname{ran} A = A \operatorname{dom} A = \{Au : u \in \operatorname{dom} A\}.$$

Recall that an operator $U : X \to V$ is called unitary if it is a bijection that preserves the value of the inner product in the sense

$$(Ux_1|Ux_2)_V = (x_1|x_2)_X \qquad \forall x_1, x_2 \in X.$$

Exercise 9.1.

(a) An operator $U : X \to V$ is unitary if and only if $U \in \mathcal{L}(X, V)$, $\ker U = \{0\}$, and $U^* = U^{-1}$.

(b) Let U be unitary. Then U is an isometry and maps orthogonal vectors to orthogonal vectors.

Proposition 9.1.
Let $A : \operatorname{dom} A \subseteq X_1 \to X_2$ be such that $\ker A = \{0\}$. Then A is closed if and only if A^{-1} is closed.

Proof. The statement follows from Definition 9.1 since $\operatorname{Gr}(A^{-1}) = \begin{pmatrix} 0 & I_{X_2} \\ I_{X_1} & 0 \end{pmatrix} \operatorname{Gr} A$. Indeed,

$$\begin{pmatrix} 0 & I_{X_2} \\ I_{X_1} & 0 \end{pmatrix} : X_1 \oplus X_2 \to X_2 \oplus X_1 \text{ is a unitary operator.}$$

Thus, $\operatorname{Gr} A$ and $\operatorname{Gr}(A^{-1})$ are closed or non-closed simultaneously. □

Exercise 9.2.
For $A : \operatorname{dom} A \subseteq X \to X$,

$$\rho(A) \neq \varnothing \qquad \Longrightarrow \qquad A \text{ is closed.}$$

Recall that the resolvent set $\rho(A)$ of A is the set of $\lambda \in \mathbb{C}$ such that $\ker(A - \lambda) = \{0\}$ and $(A - \lambda)^{-1} \in \mathcal{L}(X)$.

Definition 9.4 (closable operator, e.g., [AG, Kato])**.**
An operator $A : \operatorname{dom} A \subseteq X_1 \to X_2$ that has a closed extension $B : \operatorname{dom} B \subseteq X_1 \to X_2$ is called closable.

Clearly, every closed operator is closable.

Proposition 9.2 (closure of an operator)**.**
(a) Assume that $A : \operatorname{dom} A \subseteq X_1 \to X_2$ is closable. Then

$$\overline{\operatorname{Gr} A} = \operatorname{Gr} B \text{ for a certain closed operator } B : \operatorname{dom} B \subseteq X_1 \to X_2.$$

Besides, this operator B is a restriction of every closed extension of A, and therefore,

$$B \text{ is called } \text{the closure of } A \text{ and is denoted by } \overline{A}.$$

(b) An operator A is closable if and only if $\overline{\operatorname{Gr} A}$ does not contain elements of the form $\{0, x_2\} \in X_1 \oplus X_2$ with $x_2 \neq 0$.

Proof. The proof of (a) immediately follows from the definitions of closed and closable operators, and the definitions of restrictions and extensions.

Statement (b) follows from (a). □

Exercise 9.3 (see, e.g., [Leis])**.**
Let $G \subseteq \mathbb{R}^d$ be a nonempty open set.

(a) The operator $\mathbf{grad}_{\mathrm{comp}} : C_0^\infty(G) \subset L^2(G) \to L^2(G, \mathbb{C}^d)$ defined by $u \mapsto \nabla u$ is not closed.

(b) The operator $\mathbf{grad} : H^1(G) \subset L^2(G) \to L^2(G, \mathbb{C}^d)$ defined by $u \mapsto \nabla u$ is a closed operator and is an extension of $\mathbf{grad}_{\mathrm{comp}}$. Hence, $\mathbf{grad}_{\mathrm{comp}}$ is closable. The space with the graph norm $(\operatorname{dom} \mathbf{grad}, \|\cdot\|_{\mathbf{grad}})$ is the Hilbert space $H^1(G)$.

(c) The closure $\overline{\mathbf{grad}_{\mathrm{comp}}}$ is $\mathbf{grad}_0 : H_0^1(G) \subset L^2(G) \to L^2(G, \mathbb{C}^d)$, which is a restriction of $\mathbf{grad}$. The space $(\operatorname{dom} \mathbf{grad}_0, \|\cdot\|_{\mathbf{grad}_0})$ is the Hilbert space $H_0^1(G)$.

(d) If $G = \mathbb{R}^d$,then $\overline{\mathbf{grad}_{\mathrm{Comp}}} = \mathbf{grad}_0 = \mathbf{grad}$ and $H_0^1(G) = H^1(G)$, but this situation is atypical for general G. (For a typical G, $\mathbf{grad}_0(G) \neq \mathbf{grad}(G)$ and $H_0^1(G) \neq H^1(G)$).

(e) If G is a bounded domain with the boundary ∂G of the Lipschitz regularity, then

$$\overline{\mathbf{grad}_{\mathrm{Comp}}} = \mathbf{grad}_0 \neq \mathbf{grad} \quad \text{and} \quad H_0^1(G) \subsetneq H^1(G).$$

For an operator $A : X_1 \to X_2$ (with $\operatorname{dom} A = X_1$), the following statements are equivalent: (i) A is continuous (i.e., $A \in \mathcal{L}(X_1, X_2)$), (ii) A is bounded (in the sense $\|A\| := \sup\limits_{\|x\|_{X_1} \leqslant 1} \|Ax\|_{X_2} < \infty$), (iii) A is closed (Banach's closed graph theorem), (iv) A is closable.

We can use this equivalence to provide a pathological example of non-closable operator. If $A : X \to X$ with $\operatorname{dom} A = X$ is an unbounded operator, then A is not closable.

Examples of unbounded operators $A : X \to X$ can be constructed with the use of a Hamel basis, and sometimes are included into courses of Functional Analsysis (see [AG]).

An example of a non-closable operator that is not so pathological can be found in [AG, Section 43].

9.2 Connection between closures and adjoint operators. Essentially selfadjoint operators.

Recall that, for a densely defined operator $A : \operatorname{dom} A \subseteq X \to V$, the adjoint operator $A^* : \operatorname{dom} A^* \subseteq V \to X$ is defined in the following way:

(i) $\operatorname{dom} A^*$ consists of all $v \in V$ with the property that there exists $f_v \in X$ such that

$$(v|Au)_V = (f_v|u)_X \qquad \forall u \in \operatorname{dom} A$$

(note that such f_v is unique since $\overline{\operatorname{dom} A} = X$);

(ii) $A^* v = f_v$.

One can describe this definition in terms of the inner product of $X \oplus V$ and the graphs

of A and A^*. Indeed, the inclusion $\{v, A^*v\} \in \operatorname{Gr} A^*$ is equivalent to

$$0 = (v| - Au)_V + (A^*v|u)_X = (\{v, A^*v\}|\{-Au, u\})_{V\oplus X} \quad \forall u \in \operatorname{dom} A. \tag{9.1}$$

This implies the following statement.

Proposition 9.3.
Let $A : \operatorname{dom} A \subseteq X_1 \to X_2$ be a densely defined operator. Then:

(a) The graph $\operatorname{Gr} A^* \subset X_2 \oplus X_1$ is the orthogonal complement $S^\perp$ to the set

$$S = \begin{pmatrix} 0 & -I_{X_2} \\ I_{X_1} & 0 \end{pmatrix} \operatorname{Gr} A = \{\{-Au, u\} \in X_2 \oplus X_1 : \ u \in \operatorname{dom} A\}.$$

(b) The graph $\operatorname{Gr} A^*$ is closed.

(c) The operator A^* is closed.

(d) If $X_1 = X_2$ and $A = A^*$, then A is closed.

Proof. Equation (9.1) implies (a). Since orthogonal complemets are always closed, we obtain (b) and (c). Statement (d) follows from (c). □

Theorem 9.1.
Let $A : \operatorname{dom} A \subseteq X_1 \to X_2$ and $A^* :\subseteq X_2 \to X_1$ be densely defined operators. Then

$$\overline{A} = A^{**}.$$

Proof. Checking the definition of the unitary opertor, one sees that the operator

$$U_{X_1\oplus X_2} = \begin{pmatrix} 0 & -I_{X_2} \\ I_{X_1} & 0 \end{pmatrix}$$

is unitary from $X_1 \oplus X_2$ to $X_2 \oplus X_1$. Similarly one can define the unitary operator

$$U_{X_2\oplus X_1} = \begin{pmatrix} 0 & -I_{X_1} \\ I_{X_2} & 0 \end{pmatrix} : X_2 \oplus X_1 \to X_1 \oplus X_2.$$

We will use now the following obvious facts:

$$U_{X_2\oplus X_1} U_{X_1\oplus X_2} = -I_{X_1\oplus X_2}$$

and the fact that unitary operators maps orthogonal vectors into orthogonal vectors.

By Proposition 9.3, $\operatorname{Gr} A^* = S^\perp$, where $S = U_{X_1\oplus X_2} \operatorname{Gr} A$. Similarly, $\operatorname{Gr} A^{**} = \widetilde{S}^\perp$, where

$$\begin{aligned} \widetilde{S} &= U_{X_2\oplus X_1} \operatorname{Gr} A^* = U_{X_2\oplus X_1} S^\perp = (U_{X_2\oplus X_1} S)^\perp \\ &= (U_{X_2\oplus X_1} U_{X_1\oplus X_2} \operatorname{Gr} A)^\perp = (-I_{X_1\oplus X_2} \operatorname{Gr} A)^\perp = \operatorname{Gr} A^\perp. \end{aligned}$$

Summarizing,

$$\operatorname{Gr} A^{**} = \widetilde{S}^\perp = (\operatorname{Gr} A^\perp)^\perp = \overline{\operatorname{Gr} A}.$$

□

Exercise 9.4.
Let $A : \operatorname{dom} A \subseteq X_1 \to X_2$ be densely defined. Then A is closable if and only if A^* is densely defined.

Recall that, by Proposition 4.1, a densely defined operator $A : \operatorname{dom} A \subseteq X \to X$ is symmetric if and only if $\operatorname{Gr} A \subseteq \operatorname{Gr} A^*$.

Corollary 9.1.
Assume that a densely defined operator $A : \operatorname{dom} A \subseteq X \to X$ is symmetric. Then:

(a) The operator A^* is densely defined.

(b) The operator A is closable and $\overline{A} = A^{**}$.

Proof. (a) follows immediately from Proposition 4.1 and the assumption that $\overline{\operatorname{dom} A} = X$. Statement (b) follows from Exercise 9.4 and Theorem 9.1. □

Definition 9.5 (essential selfadjointness)**.**
A densely defined symmetric operator $A : \operatorname{dom} A \subseteq X \to X$ is called essentially selfadjoint if $\overline{A}$ is selfadjoint.

Exercise 9.5 (e.g., [Kato])**.**
Assume that $A : \operatorname{dom} A \subseteq X \to X$ is essentially selfadjoint. Then:

(a) the closure $\overline{A}$ is a unique selfadjoint extension of A,

(b) $A^* = \overline{A}$.

Theorem 9.2 (von Neumann)**.**
Let $T : \operatorname{dom} T \subseteq X_1 \to X_2$ be densely defined and closed. Then T^*T is a selfadjoint operator in X_1.

We take this theorem without proof (for the proof see [Kato, Theorem V.3.24]).

Example 9.1.
Assume that $G \subseteq \mathbb{R}^d$ be a nonempty open set. Let

$$A = -\Delta_G \restriction_{C_0^\infty(G)} \colon C_0^\infty(G) \subset L^2(G) \to L^2(G)$$

where $\Delta_G : u \mapsto \Delta u$ in the distributional sense of $(C_0^\infty(G))'$.

(a) The operator A is symmetric, which follows from the Green's formula, but is not a closed operator.

(b) The operator A is closable. Indeed, by Theorem 9.2 and Exercise 9.3, A has the following selfadjoint (and so closed) extensions

$$-\Delta^{\mathrm{D}} := \mathbf{grad}_0^* \, \mathbf{grad}_0, \qquad -\Delta^{\mathrm{N}} := \mathbf{grad}^* \, \mathbf{grad},$$

which are called Dirichlet and, respectively, Neumann nonnegative Laplace operators in G.

(c) Let $G = \mathbb{R}^d$. Then A is essentially selfadjoint and its closure is

$$\overline{A} = A^* = -\Delta = (-\Delta)^*,$$

where $\Delta : H^2(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is the standard nonpositive selfadjoint Laplacian. In this case,

$$-\Delta = -\Delta^{\mathrm{D}} = -\Delta^{\mathrm{N}}.$$

(c) Assume additionally that G is a bounded Lipschitz domain. Then $\operatorname{dom} \overline{A} = H_0^2(G)$. The operators $\overline{A}$ and A^* are not selfadjoint. Moreover, $\Delta^{\mathrm{D}} \neq \Delta^{\mathrm{N}}$ and

$$\operatorname{Gr} A \subsetneqq \operatorname{Gr} \overline{A} \subsetneqq \operatorname{Gr}(-\Delta^{\mathrm{D}}) \subsetneqq \operatorname{Gr} A^*, \qquad \operatorname{Gr} A \subsetneqq \operatorname{Gr} \overline{A} \subsetneqq \operatorname{Gr}(-\Delta^{\mathrm{N}}) \subsetneqq \operatorname{Gr} A^*.$$

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10 Schrödinger operators with Poisson-Anderson potentials and their spectra.

10.1 The scheme of the proof of Theorem 5.2.

We follow [KMN19] and start from the deterministic part.

Let $N \in \widehat{\mathbb{N}}$ and $a = \{a_j\}_{j=1}^N$. Let $Y = \{y_j\}_{j=1}^N \subset \mathbb{R}^3$ be a collection of distinct deterministic points such that the counting measure of Y is locally finite (i.e., Y has no finite accumulation points).

Recall that, in Lecture 5, we defined 3 operators associated with Y and a:

- the symmetric operator with "minimal reasonable" domain

$$\mathcal{H}_Y^{\min} := -\Delta \restriction_{C_0^\infty(\mathbb{R}^3 \setminus Y)};$$

- the Laplace-type operator

$$\mathcal{H}_Y^{\max} = -\Delta_{\mathbb{R}^3 \setminus Y} := (\mathcal{H}_Y^{\min})^*$$

with "maximal reasonable" domain in $G = \mathbb{R}^3 \setminus Y^3$;

- the deterministic operator with point interactions

$$\mathcal{H}_{Y,a} = -\Delta + \text{“} \sum_{j=1}^N \mathfrak{m}(a_j)\delta_{y_j} \text{“}, \quad \mathcal{H}_{Y,a} : \operatorname{dom} \mathcal{H}_{Y,a} \subset L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$$

defined as a restriction of $\mathcal{H}_Y^{\max}$ by means of special boundary conditions

$$u_{y_j}^1 - 4\pi a_j u_{y_j}^0 = 0, \qquad 1 \leqslant j \leqslant N, \tag{10.1}$$

where

$$u_{y_j}^0 = \lim_{x \to y_j} |x - y_j| u(x), \qquad u_{y_j}^1 = \lim_{x \to y_j} (u(x) - u_{y_j}^0 |x - y_j|^{-1}), \qquad u \in \operatorname{dom} \mathcal{H}_Y^{\max}$$

(see Proposition 5.1).

Lemma 10.1 (singular Green's formula)**.**
Let $u, v \in \operatorname{dom} \mathcal{H}_Y^{\max}$ be such that $\operatorname{supp} u, \operatorname{supp} v \in \operatorname{Comp}(\mathbb{R}^3)$. Then

$$(\mathcal{H}_Y^{\max} u | v)_{L^2} - (u | \mathcal{H}_Y^{\max} v)_{L^2} = -4\pi \sum_{y \in Y} \left(u_y^0 \overline{v_y^1} - u_y^1 \overline{v_y^0} \right).$$

(Note that the sum is finite due to the assumption $\operatorname{supp} u, \operatorname{supp} v \in \operatorname{Comp}(\mathbb{R}^3)$).

We take this lemma without proof. The proof uses the representation of Proposition 5.1 for u and v near singularities $y \in Y$, see [KMN19].

Let us define and additional operator

$$\mathcal{H}_{Y,a}^{\text{comp}} := \mathcal{H}_{Y,a} \restriction_{\operatorname{dom} \mathcal{H}_{Y,a}^{\text{comp}}}, \quad \operatorname{dom} \mathcal{H}_{Y,a}^{\text{comp}} = \{u \in \operatorname{dom} \mathcal{H}_{Y,a} : \ \operatorname{supp} u \in \operatorname{Comp}(\mathbb{R}^3)\}.$$

Lemma 10.2 (adjoint of $\mathcal{H}_{Y,a}^{\mathrm{comp}}$)**.**
The operator $\mathcal{H}_{Y,a}^{\mathrm{comp}}$ is a symmetric densely defined operator in $L^2(\mathbb{R}^3)$, and

$$(\mathcal{H}_{Y,a}^{\mathrm{comp}})^* = \mathcal{H}_{Y,a}.$$

We take this lemma without proof (for the proof, see [KMN19]).

By Theorem 5.1, $\mathcal{H}_{Y,a} = \mathcal{H}_{Y,a}^*$ if $d_* = \inf_{j\neq n} |y_j - y_n|$ is positive. Now, the aim is to relax the assumption $d_* > 0$.

Assumption 10.1.
There exists $R > 0$ such that, for the R-parallel set of Y

$$Y + \overline{\mathbb{B}}_R = \{x \in \mathbb{R}^d : \min_{1\leqslant j\leqslant N} |x - y_j| \leqslant R\},$$

all connected components are bounded.

Lemma 10.3.
Suppose that Assumption 10.1 holds. Then

$$\overline{\mathcal{H}_{Y,a}^{\mathrm{comp}}} = \mathcal{H}_{Y,a}.$$

We take this lemma without proof (for the proof, see [KMN19]).

Corollary 10.1.
Suppose that Assumption 10.1 holds. Then

$$\mathcal{H}_{Y,a} = \mathcal{H}_{Y,a}^*.$$

Proof. Combining Lemmata 10.3 and 10.2 with Theorem 9.1, we get

$$\mathcal{H}_{Y,a}^* = ((\mathcal{H}_{Y,a}^{\mathrm{comp}})^*)^* = \overline{\mathcal{H}_{Y,a}^{\mathrm{comp}}} = \mathcal{H}_{Y,a}.$$

□

In other words, Assumption 10.1 implies that $\mathcal{H}_{Y,a}^{\mathrm{comp}}$ is essentially selfadjoint and $\mathcal{H}_{Y,a}$ is selfadjoint.

Let us recall the formulation of Theorem 5.2.
Theorem 5.2 ([KMN19]). Let $Y = \{y_j\}_{j\in\mathbb{N}}$ be a collection of random points y_j in $\mathbb{R}^3$ associated with a certain homogeneous Poisson SPP $\eta = \sum_{j\in\mathbb{N}} \delta_{y_j}$ of positive intensity. Let $a = \{a_j\}_{j=1}^{+\infty}$ be an arbitrary sequence of $\mathbb{R}$-valued random variables a_j. Then $\mathcal{H}_{Y,a} = \mathcal{H}_{Y,a}^*$ with probability 1.

We see that Theorem 5.2 follows immediately from the combination of Corollary 10.1 with Corollary 8.3.

Indeed, for a Boolean model $\mathcal{Z}$ on $\mathbb{R}^3$ with an intensity γ and the deterministic radius distribution $p_{\mathrm{r}} = \delta_R$, there exists a critical radius $R_{\mathrm{c}} = R_{\mathrm{c}}(\gamma) > 0$ such that for $R < R_{\mathrm{c}}(\gamma)$ all occupied components are bounded with probability 1. This implies that Assumption 10.1 is fulfilled with the probability 1. Corollary 10.1 completes the proof of Theorem 5.2.

Remark 10.1.
Analogues of Corollary 10.3 and Theorem 5.2 are valid for Schrödinger operators with point interactions in $L^2(\mathbb{R})$ and $L^2(\mathbb{R}^3)$, see [KMN19].

10.2 Spectra of Poisson-Anderson-Schrödinger operators with i.i.d. strength-type parameters.

Let $\eta = \sum_{j\in\mathbb{N}} \delta_{y_j}$ be a homogeneous Poisson SPP of positive intensity on $\mathbb{R}^d$ with $d = 1$ or $d = 3$. (The case $d = 2$ is somewhat similar to the case $d = 3$, see [KMN19]).

Let α_0 be a certain $\mathbb{R}$-valued random variable with distribution p_{α_0}. Let

$$\xi = \sum_{j\in\mathbb{N}} \delta_{(y_j,\alpha_j)}$$

be an independent p_{α_0}-marking of η (see Definition 7.2). That is, $\alpha = \{\alpha_j\}_{j\in\mathbb{N}}$ be a sequence of i.i.d. $\mathbb{R}$-valued random variables with the distribution p_{α_0} and an additional property that $\{\alpha_j\}_{j\in\mathbb{N}}$ is independent of η.

Let us define in $L^2(\mathbb{R}^d)$ the Poisson-Anderson-Schrödinger operator $\mathcal{H}_{Y,\alpha}$ (in short, PAS operator) following [KMN19]. As before, $Y = \{y_j\}_{j\in\mathbb{N}}$ is a collection of random points y_j in $\mathbb{R}^d$ associated with with homogeneous Poisson SPP η.

Consider first the case $d = 3$. Then a PAS operator is defined by

$$\mathcal{H}_{Y,\alpha} := -\Delta + \text{“} \sum_{j=1}^{N} \mathfrak{m}(\alpha_j)\delta_{y_j} \text{“}$$

where the strength-type parameters $1/\alpha_j$ are generated by the i.i.d. random variables α_j of the independent marking.

Theorem 10.1 ([KMN19])**.**
Let $\mathcal{H}_{Y,\alpha}$ be a PAS operator in $L^2(\mathbb{R}^3)$. Then:

(a) $\mathcal{H}_{Y,\alpha}$ is selfadjoint with probability 1.

(b) $\sigma(\mathcal{H}_{Y,\alpha}) = \mathbb{R}$ with probability 1 (for any distribution p_{α_0} of the $\mathbb{R}$-valued i.i.d. random variables α_j).

The proof of statement (a) of Theorem 10.1. The proof of statement (a) can be obtained by the literal repetition of the proof of Theorem 5.2. □

We take statement (b) without proof (for the proof, see [KMN19, Section 3.3]).

Consider now the case $d = 1$. Then a PAS operator in $L^2(\mathbb{R})$ is defined by

$$\mathcal{H}_{Y,\alpha} := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{j\in\mathbb{N}} \alpha_j \delta_{y_j},$$

where i.i.d. random variables α_j play the role of the strength parameters.

Theorem 10.2 ([M88, KMN19])**.**
Let $\mathcal{H}_{Y,\alpha}$ be a PAS operator in $L^2(\mathbb{R})$. Then:

(a) $\mathcal{H}_{Y,\alpha}$ is selfadjoint with probability 1.

(b) If $\mathbb{P}(\alpha_0 \geqslant 0) = 1$ (i.e., if $\operatorname{supp} p_{\alpha_0} \subseteq [0, +\infty)$), then

$$\sigma(\mathcal{H}_{Y,\alpha}) = [0, +\infty) \text{ with probability 1.}$$

(c) If $\mathbb{P}(\alpha_0 \geqslant 0) < 1$ *(i.e., if* $\operatorname{supp} p_{\alpha_0} \cap (-\infty, 0) \neq \varnothing$*), then*

$$\sigma(\mathcal{H}_{Y,\alpha}) = \mathbb{R} \text{ with probability 1.}$$

We take this theorem without proof (for the proof, see [KMN19]). Statement (a) can be proved by an easy adaptation of the proof of Theorem 5.2 to the 1-dimensional case (an exercise).

Another proof of statement (a) can be obtained from the Kostenko-Malamud theorem [KM10] (Theorem 4.2 in this lecture series) combined with the interval theorem for 1-dimensional Poisson SPPs [LP, Section 7.1].

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11 Various types of spectra.

11.1 Eigenvalues, multiplicities, discrete and essential spectra.

Let $A : \operatorname{dom} A \subseteq X \to X$ be an operator $A : \operatorname{dom} A \subseteq X \to X$ in a Hilbert space X.

Definition 11.1.
The resolvent set $\rho(A)$ of A is the set of points $k \in \mathbb{C}$ such that the following three conditions hold true:

(a) $A - k$ is invertible, i.e., $\ker(A - kI) = \{0\}$;

(b) $\operatorname{ran}(A - k) = X$, i.e., the operator $A - k$ maps $\operatorname{dom}(A - k) := \operatorname{dom} A$ onto the whole X;

(c) $(A - k)^{-1} \in \mathcal{L}(X)$, i.e., $(A - k)^{-1}$ is a bounded operator in X.

Definition 11.2.
The set $\sigma(A) := \mathbb{C} \backslash \rho(A)$ is called the spectrum of A.

Exercise 11.1.

(a) Definition 11.1 implies the following equivalence: $k \in \rho(A)$ is equivalent to the statement that the equation $Au - ku = f$ has a unique solution u_f for every $f \in \mathbb{X}$ and

$$\|u_f\|_X \lesssim \|f\|_X, \qquad f \in X,$$

where the notation '$\lesssim$' means that there exists a constant $c > 0$ independent of f such that $\|u_f\|_X \leqslant c\|f\|_X$ for all $f \in X$.

(b) The set $\rho(A)$ is open. Consequently, $\sigma(A)$ is closed.

One can say that $k \in \rho(A)$ if and only if the equation $Au - ku = f$ is well posed in X in the sense of Hadamard.

Definition 11.3.

(a) A number $k \in \mathbb{C}$ is called an eigenvalue of A if $\ker(A - k) \neq \{0\}$. The elements of $\ker(A - k) \backslash \{0\}$ are called eigenvectors of A associated with the eigenvalue k.

(b) We denote the set of all eigenvalues of A by $\sigma_{\mathrm{p}}(A)$ Sometimes $\sigma_{\mathrm{p}}(A)$ is called the point spectrum of A (from Definition 11.1(a), one sees that $\sigma_{\mathrm{p}}(A) \subseteq \sigma(A)$).

(c) The geometric multiplicity $\operatorname{mult}_{\mathrm{g}}(k) \in \widehat{\mathbb{N}}$ of an eigenvalue k is by definition the dimensionality

$$\dim \ker(A - k)$$

of the linear subspace $\ker(A - k)$.

(d) The algebraic multiplicity $\operatorname{mult}_{\mathrm{a}}(k) \in \widehat{\mathbb{N}}$ of an eigenvalue k can be defined as

$$\operatorname{mult}_{\mathrm{a}} := \lim_{n \to +\infty} \dim(\ker(A - k)^n).$$

(e) An eigenvalue k_0 of A is called an isolated eigenvalue if k_0 is an isolated point of $\sigma(A)$.

An eigenvalue k of A is called simple if $\operatorname{mult}_{\mathrm{g}}(k) = 1$. An eigenvalue k of A is called semi-simple if $\operatorname{mult}_{\mathrm{g}}(k) = \operatorname{mult}_{\mathrm{a}}(k)$.

Definition 11.4 ([Kato, ?])**.**

(a) We define the discrete spectrum $\sigma_{\mathrm{disc}}(A)$ of A as the set of all isolated eigenvalues of finite algebraic multiplicity [Kato, **?**].

(b) The essential spectrum $\sigma_{\mathrm{ess}}(A)$ is defined by $\sigma_{\mathrm{ess}}(A) := \sigma(A)\backslash\sigma_{\mathrm{disc}}(A)$ [**?**].

(c) If $\sigma(A) = \sigma_{\mathrm{disc}}(A)$, we say that A is an operator with purely discrete spectrum (cf. [RS1]).

It should be emphasized that

- here the terminology concerning operators with purely discrete spectrum is related to the terminology of [RS1], but is not completely standard (in the more standard terminology of [Kato], if $\sigma(A) = \sigma_{\mathrm{disc}}(A)$, it is said that A has a discrete spectrum).
- there exists a variety of other non-equivalent definitions of the essential spectrum (see, e.g., the discussion in [Kato, Section IV.5.6]).

Exercise 11.2.
Let A be a symmetric operator in X. Then:

(a) $\sigma_{\mathrm{p}}(A) \subseteq \mathbb{R}$

(b) $\ker(A-k) = \ker(A-k)^n$ for all $n \in \mathbb{N}$. In particular, the geometric multiplicity of an eigenvalue k coincides with its algebraic multiplicity.

(c) Let $u_j \in \ker(A-k_j)$ for $j = 1, 2$. If $k_1 \neq k_2$, then $u_1 \perp u_2$.

(d) If X is separable, then $\sigma_{\mathrm{p}}(A) \subsetneqq \mathbb{R}$.

(e) Since selfadjoint operators form a subclass of the symmetric operators, statements (a)-(d) are valid for all selfadjoint operators.

11.2 Basic examples, unitary equivalence, and multiplication operators

The (topological) support $\operatorname{supp}\mu$ of a (Borel) measure μ on $\mathbb{R}^d$ is the complement $\mathbb{R}^d\backslash S$ of the maximal open set S such that $\mu(S) = 0$. A maximal set in a certain class is understood in the sense of the partial order '$\subseteq$'.

Exercise 11.3 (multiplication operators)**.**
Let $X = L^2(\mathbb{R}^d;\mu)$ be the L^2-space (of equivalence classes) corresponding to a certain locally finite measure μ on $\mathbb{R}^d$, i.e.,

$$(u|v)_X = \int_{\mathbb{R}^d} u\overline{v}\ \mu(\mathrm{d}x)$$

(note that the case of the trivial space $L^2(\mathbb{R}^d;\mu) = \{0\}$ is not excluded). Let $f \in M(\mathbb{R}^d, \mathbb{C})$ be a measurable function. The multiplication operator $\mathcal{M}_f : u(\cdot) \mapsto f(\cdot)u(\cdot)$ is defined on its natural domain

$$\operatorname{dom}\mathcal{M}_f = \{u \in L^2(\mathbb{R}^d;\mu) :\ fu \in L^2(\mathbb{R}^d;\mu)\}.$$

Then the following statements hold true:

(a) $\rho(\mathcal{M}_f) = \{k \in \mathbb{C} : \mu(f^{-1}(\mathbb{B}_\varepsilon(k))) = 0 \text{ for a certain } \varepsilon > 0\}$

(b) Let us denote by $\varphi = \mu \circ f^{-1}$ the image of μ under f (see Lecture 7), i.e.,

$$\varphi(S) = \mu(f^{-1}(S)), \qquad S \in \mathcal{B}(\mathbb{C}).$$

Then

$$\sigma(\mathcal{M}_f) = \operatorname{supp} \varphi.$$

(c) $k \in \sigma_p(\mathcal{M}_f)$ if and only if $\varphi(\{k\}) = \mu(f^{-1}(k)) > 0$. If this is the case, then the geometric and algebraic multiplicities of k are equal to $\dim L^2\left(f^{-1}(k); \mu\right)$.

(d) All multiplication operators $\mathcal{M}_f$ are densely defined.

(e) $\mathcal{M}_f^* = \mathcal{M}_{\overline{f}}$.

(f) All multiplication operators $\mathcal{M}_f$ are closed.

(g) A multiplication operator $\mathcal{M}_f$ is symmetric if and only if $f(x) \in \mathbb{R}$ for almost all x w.r.t. the measure μ.

(h) A multiplication operator $\mathcal{M}_f$ is selfadjoint if and only if $f(x) \in \mathbb{R}$ for almost all x w.r.t. the measure μ (note that $\mathcal{M}_f$ is symmetric if and only if $\mathcal{M}_f$ is selfadjoint).

(i) A multiplication operator $\mathcal{M}_f$ is unitary in $L^2(\mathbb{R}^d; \mu)$ if and only if $|f(x)| = 1$ for almost all x w.r.t. the measure μ.

Remark 11.1.
Let $U : X_1 \to X_2$ be an unitary operator between Hilbert spaces X_1 and X_2. Consider certain operators $A_j : \operatorname{dom} A_j \subseteq X_j \to X_j$ for $j = 1, 2$. Then:

(a) $\dim X_1 = \dim X_2$.

(b) Let

$$A_1 = U^{-1} A_2 U \text{ for a certain unitary } U : X_1 \to X_2.$$

Then one says that A_1 and A_2 are unitary equivalent. Note that $U^{-1} : X_2 \to X_1$ is also a unitary operator and that $A_2 = U A_1 U^{-1}$.

(c) Assume that A_1 and A_2 are unitary equivalent in the sense of (b). Then all "reasonable spectral properties" of A_1 and A_2 coincide, e.g., $\sigma(A_1) = \sigma(A_2)$ and $\sigma_\bullet(A_1) = \sigma_\bullet(A_2)$ for $\bullet = \mathrm{p}, \mathrm{disc}, \mathrm{ess}$. The same is true for the density of domains, closedness, and closability. If adjoint operators, closures, inverse operators or resolvents exist they are connected by similar equalities

$$A_1^* = U^{-1} A_2^* U, \qquad \overline{A}_1 = U^{-1} \overline{A}_2 U, \qquad (A_1 - k)^{-1} = U^{-1} (A_2 - k)^{-1} U.$$

Exercise 11.4 (multiplication operators with discrete spectra)**.**

(a) Let $n \in \mathbb{N}$. Let A be a selfadjoint operator in the standard Hilbert space $\mathbb{C}^n$ represented by a symmetric matrix $(a_{i,j})_{i,j=1}^n$ in the standard orthonormal basis. Find a measure μ on $\mathbb{R}$, a function $f \in M(\mathbb{R}, \mathbb{C})$, and a unitary operator $U : \mathbb{C}^n \to L^2(\mathbb{R}; \mu)$, such that

$$A = U^{-1} \mathcal{M}_f U.$$

(b) Characterize multiplication operators $\mathcal{M}_f$ with purely discrete spectra.

Multiplication operators $\mathcal{M}_f$ plays for infinite-dimensional spectral theory the same role as diagonal matrices play for finite-dimensional spectral theory.

Proposition 11.1 (operators without eigenvalues)**.**
Let $\mu = |\cdot|_d$ and $X = L^2(\mathbb{R}^d)$ with $d \in \mathbb{N}$.

(a) $\sigma_{\mathrm{p}}(\mathcal{M}_{|x|^2}) = \varnothing = \sigma_{\mathrm{disc}}(\mathcal{M}_{|x|^2})$ and $\sigma(\mathcal{M}_{|x|^2}) = \sigma_{\mathrm{ess}}(\mathcal{M}_{|x|^2}) = \overline{\mathbb{R}}_+ = [0, +\infty)$.

(b) For nonnegative selfadjoint Laplace operator $(-1)\Delta$ in $L^2(\mathbb{R}^d)$,

$$\sigma_{\mathrm{p}}(-\Delta) = \varnothing, \qquad \sigma(-\Delta) = \sigma_{\mathrm{ess}}(-\Delta) = \overline{\mathbb{R}}_+$$

(c) Consider the following symmetric restriction $A = -\Delta \restriction_{C_0^\infty(\mathbb{R}^n)}$. Then

$$\sigma_{\mathrm{p}}(A) = \varnothing, \qquad \sigma(A) = \sigma_{\mathrm{ess}}(A) = \mathbb{C}.$$

Proof. (a) can be obtained by direct application of Exercise 11.3.

(b) can be obtained from (a) with the use of the unitary Fourier transform $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ and the equality

$$-\Delta = \mathcal{F}^{-1}\mathcal{M}_{|x^2|}\mathcal{F}$$

combined with Remark 11.1.

(c) Since $(-1)\Delta$ has no eigenvalues, then its restriction A also has no eigenvalues. However, for arbitrary k,

$$\operatorname{ran}(A - k) \subseteq C_0^\infty(\mathbb{R}^d) \subsetneq L^2(\mathbb{R}^d).$$

Thus, every $k \in \mathbb{C}$ is a point of the spectrum of A. □

11.3 Extended spectrum and empty spectrum

Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the standard compactification of $\mathbb{C}$ via the stereographic projection.

Definition 11.5.
The extended spectrum $\widetilde{\sigma}(A) \subseteq \widehat{\mathbb{C}}$ of $A : \operatorname{dom} A \subseteq X \to X$ is defined in the following way.
(a) If $A \in \mathcal{L}(X)$, then $\widetilde{\sigma}(A) := \sigma(A)$.

If A is unbounded, then $\widetilde{\sigma}(A) := \sigma(A) \cup \{\infty\}$.

Theorem 11.1.
Assume that the Hilbert space X is not trivial, i.e., $X \neq \{0\}$. Then, for an arbitrary operator $A : \operatorname{dom} A \subseteq X \to X$, we have $\widetilde{\sigma}(A) \neq \varnothing$.

Example 11.1 (empty spectrum)**.**
Consider the operator $A : u \mapsto -u''$ in $L^2(0,1)$ with the domain

$$\operatorname{dom} A = \{u \in H^2[0,1] : \ u(0) = \partial_+ u(0) = 0\},$$

where $\partial_+ u(0)$ is the derivative from the right. Then $\sigma(A) = \varnothing$.

11.4 Operators with compact resolvent.

We denote by $\mathfrak{S}_\infty(X)$ the space of compact operators in a Hilbert space X.

Theorem 11.2.
Assume that

$$\text{there exists } k_0 \in \rho(A) \text{ such that } (A-k_0)^{-1} \in \mathfrak{S}_\infty(X). \tag{11.1}$$

Then:

(a) $(A-k)^{-1} \in \mathfrak{S}_\infty(X)$ *for every* $k \in \rho(A)$.

(b) $\sigma(A) = \sigma_{\text{disc}}(A)$, *i.e., the operator* A *has purely discrete spectrum.*

Definition 11.6.
If an operator A satisfies (11.1), one says that A is an operator with compact resolvent.

The proof of Theorem 11.2 uses the following particular case of the spectral mapping theorem.

Theorem 11.3.
Let A *be a closed invertible operator in* X. *Then*

$$\widetilde{\sigma}(A^{-1}) = \{k \in \widehat{\mathbb{C}} : \ \frac{1}{k} \in \widetilde{\sigma}(A)\},$$

where $0^{-1} = \frac{1}{0} := \infty$ *and* $\infty^{-1} = \frac{1}{\infty} := 0$.

The most well-known application of Theorem 11.2 is the following result.

Corollary 11.1.
Let G *be a bounded nonempty open set. Let* $(-1)\Delta^{\text{D}} = \mathbf{grad}_0^* \mathbf{grad}_0$ *be the selfadjoint Laplace operator in* $L^2(G)$ *associated with the Dirichlet boundary condition (see Lecture 9). Then:*

(a) $(-1)\Delta^{\text{D}}$ *has compact resolvent;*

(b) $\sigma(-\Delta^{\text{D}}) = \sigma_{\text{disc}}(-\Delta^{\text{D}})$;

(c) there exists an orthonormal basis $\{y_j\}_{j\in\mathbb{N}}$ *of eigenfunctions of* $(-1)\Delta^{\text{D}}$.

References for Lecture 11.

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12 Spectral theorem for selfadjoint operators.

12.1 Spectral theorem in the form of multiplication operators.

For $\mathbb{X} \in \mathcal{B}(\mathbb{C})$, we assume that a Borel measurable function $g \in M(\mathbb{X}, \mathbb{C})$ is defined on the whole $\mathbb{X}$. Then a composition of Borel measurable functions is a Borel measurable function.

Theorem 12.1 (e.g., [RS1])**.**
Let A be a selfadjoint operator in a separable Hilbert space X. Then there exists a measurable space $(\mathbb{Y}, \mu)$ with $\mu(\mathbb{Y}) < +\infty$, and a measurable function $f \in M(\mathbb{Y}, \mathbb{R})$ such that A is unitary equivalent to the multiplication operator $\mathcal{M}_f$. That is, there exists a unitary operator $U : X \to L^2(\mathbb{Y}; \mu)$ such that $A = U^{-1}\mathcal{M}_f U$.

Exercise 12.1.
Let A be a selfadjoint operator in a separable Hilbert space, which is unitary equivalent to a multiplication operator $\mathcal{M}_f$ in the way described in Theorem 12.1.

(a) For a multiplication operator $\mathcal{M}_f$ of Theorem 12.1 (or, more generally, for $\mathcal{M}_f$ of Lecture 11),
$$\|\mathcal{M}_f\| = \operatorname{ess\,sup} |f| = \|f\|_{L^\infty(\mathbb{Y};\mu)},$$
where the essential supremum is taken w.r.t. the measure μ.

(b) In particular, $A \in \mathcal{L}(X)$ if and only if $f \in L^\infty(\mathbb{Y}; \mu)$. If this is the case, then
$$\|A\| = \|f\|_{L^\infty(\mathbb{Y};\mu)}.$$

This formulation of the spectral theorem allows one to define functions $g(A)$ of selfadjoint operators, where $g : \mathbb{R} \to \mathbb{C}$ is a Borel measurable function (i.e., measurable in the sense of Borel σ-algebras $\mathcal{B}(\mathbb{R})$ and $\mathcal{B}(\mathbb{C})$)

Example 12.1.
Let A be a selfadjoint operator in a separable Hilbert space, which is unitary equivalent to a multiplication operator $\mathcal{M}_f$ in the way described in Theorem 12.1.

(a) Let $g(x) = x^2$, $x \in \mathbb{R}$. Then it is natural to define $g(A) := A^2$. In this case,
$$UA^2U^{-1} = (\mathcal{M}_f)^2 = \mathcal{M}_{f^2}.$$
This definition can be extended to any polynomial $p(x) = \sum_{j=0}^n c_j x^j$ with $n \in \mathbb{N}_0$, $c_j \in \mathbb{C}$, and $c_n \neq 0$. However, one need to define the domain of $p(A)$ appropriately, i.e.,
$$\operatorname{dom} p(A) = \operatorname{dom} A^n = \{u \in X : A^j u \in \operatorname{dom} A \text{ for all } 0 \leqslant j < n\}$$
Then
$$Up(A)U^{-1} = p(\mathcal{M}_f) = \mathcal{M}_{p\circ f}$$

(b) Note that for $g(x) = x$, $x \in \mathbb{R}$, we get $g(A) = A$.

(c) Let $g(x) = \frac{1}{x-k_0}$, $x \in \mathbb{R}$, $k_0 \in \mathbb{C}\backslash\mathbb{R}$. Then it is reasonable to define $g(A)$ as the resolvent at k_0, i.e., $g(a) = (A - k_0)^{-1}$. Then
$$U(A - k_0)^{-1}U^{-1} = (\mathcal{M}_f - k_0)^{-1} = \mathcal{M}_{(f-k_0)^{-1}} = \mathcal{M}_{g\circ f}.$$

Note that this definition can be extended to $k_0 \in \mathbb{R}\backslash\sigma_p(A)$.

These examples show that the following definition of $g(A)$ is resonable.

Definition 12.1.
Let A be a selfadjoint operator in a separable Hilbert space, which is unitary equivalent to a multiplication operator $\mathcal{M}_f$ in the way described in Theorem 12.1. Let $g \in M(\mathbb{R}, \mathbb{C})$ (or, more generally, $g \in M(\mathbb{R}, \widehat{\mathbb{C}})$ is such that $g(k) \neq \infty$ for all $k \in \sigma_p(A)$). Then the function $g(A)$ of A can be defined by

$$g(A) = U^{-1}\mathcal{M}_{g\circ f}U.$$

Remark 12.1.
In the unitary equivalence for A in Theorem 12.1, the choice of the measurable space $(\mathbb{Y}, \mu)$ and a function $f \in M(\mathbb{Y}, \mathbb{R})$ is not unique. However, the definition 12.1 of $g(A)$ does not depends on this choice and so $g(A)$ is well-defined (an exercise).

This definition allows to write the spectral theorem in the functional calculus form, see [RS1, Theorem VIII.5].

We need another form of the spectral theorem that is written in terms of (orthogonal)-projection-valued measures $S \mapsto P^S$, $S \in \mathcal{B}(\mathbb{R})$, where $P^S := \chi_S(A)$.

Recall that $\chi_S \in M(\mathbb{R}, \mathbb{R})$ is an indicator function of the set S, i.e., $\chi_S(x) = 1$ if $x \in S$, and $\chi_S(x) = 0$ if $x \notin S$.

12.2 Orthogonal projections

Let us recall the projection theorem (see, e.g., [RS1, Theorem II.3] and [Kato]).

Theorem 12.2 (orthogonal projection)**.**
Let X_1 be a closed subspace of a Hilbert space X, and let

$$X_1^\perp := \{u_2 \in X : (u_2|u_1)_X = 0 \text{ for all } u_1 \in X_1\}.$$

Then:

(a) $X_1^\perp$ is a closed subspace of X and $(X_1^\perp)^\perp = X_1$.

(b) Every $u \in X$ has a unique decomposition

$$u = u_1 + u_2, \qquad u_1 \in X_1, \quad u_2 \in X_1^\perp.$$

(c) The mapping $P : u \mapsto u_1$ acting according to the decomposition (b) defines a bounded operator $P \in \mathcal{L}(X)$ with $\|P\| \leqslant 1$. This operator P is called a projection on X_1 (or more precisely, the orthogonal projection on X_1).

Remark 12.2.
There is another more general definition of projections that include also non-orthogonal projections. Namely, an operator $Q \in \mathcal{L}(X)$ is called a projection if $Q^2 = Q$ (i.e., Q is an idempotent element of $\mathcal{L}(X)$ in the algebraic terminology). We will use only orthogonal projection defined by Theorem 12.2, and simply call them projections (skipping "orthogonal").

Exercise 12.2.

(a) An operator $P : \operatorname{dom} P \subseteq X \to X$ is a projection if and only if $P^2 = P = P^*$.

(b) If P is a projection, then its range $\operatorname{ran} P = PX = \{Pu : u \in X\}$ is the corresponding closed subspace on that P projects. (Note that the zero operator $0_{\mathcal{L}(X)}$ is also a projection on the trivial subspace $\{0\}$).

(c) Let P be a projection. Then $I - P$ is also a projection. (What is the corresponding closed subspace?)

(d) Let $U : X \to Y$ be a unitary operator from a Hilbert space X to a Hilbert space Y. Then, P is a projection in X if and only if UPU^{-1} is a projection in Y.

Lemma 12.1.
Let A be a selfadjoint operator in a separable Hilbert space X, and let $S \in \mathcal{B}(\mathbb{R})$. Then $\chi_S(A)$ is a projection.

Proof. Let A be unitary equivalent to a multiplication operator $\mathcal{M}_f$ in the way described in Theorem 12.1. Then $\chi_S(A) = U^{-1}\mathcal{M}_{\chi_S \circ f}U$. By Exercise 12.2, it is enough to prove that $\mathcal{M}_{\chi_S \circ f}$ is a projection in $L^2(\mathbb{Y}; \mu)$.

Since $\chi_S : \mathbb{R} \to \{0, 1\}$ is real-valued and bounded, we see that $\mathcal{M}^*_{\chi_S \circ f} = \mathcal{M}_{\chi_S \circ f}$ and that, by Exercise 12.1,

$$\|\mathcal{M}_{\chi_S \circ f}\| = \|\chi_S \circ f\|_{L^\infty(\mathbb{Y};\mu)} \leqslant 1.$$

The equality $\mathcal{M}^2_{\chi_S \circ f} = \mathcal{M}_{\chi_S \circ f}$ follows from the definition of an indicator-function.

$$\chi_S(f(y))\chi_S(f(y)) = \chi_{f^{-1}(S)}\chi_{f^{-1}(S)} = \chi_{f^{-1}(S)}.$$

Exercise 12.2 (a) implies that $\mathcal{M}_{\chi_S \circ f}$ is a projection. □

The projection-valued measure for a selfadjoint operator A is the mapping $S \mapsto \chi_S(A)$ defined for all $S \in \mathcal{B}(\mathbb{R})$.

12.3 Various types of convergence of sequences of operators.

Let us recall the main types of convergencies for sequences $\{T_n\}_{n\in\mathbb{N}} \subset \mathcal{L}(X)$ of bounded operators in a Hilbert space X.

The sequence $\{T_n\}_{n\in\mathbb{N}}$ is said to converge to $T \in \mathcal{L}(X)$ uniformly (or in the operator norm) if $\|T - T_n\|_{\mathcal{L}(X)} \to 0$ as $n \to \infty$.

The sequence T_n is said to converge to $T \in \mathcal{L}(X)$ strongly

$$\text{if, for every } u \in X,\ \|Tu - T_n u\|_X \to 0 \text{ as } n \to \infty.$$

In this case, one writes

$$T = \text{s-}\lim_{n\to\infty} T_n$$

and says that T is the strong (operator) limit of the sequence $\{T_n\}_{n\in\mathbb{N}} \subset \mathcal{L}(X)$. The logic behind the name of the strong convergence is that $T = \text{s-}\lim_{n\to\infty} T_n$ is equivalent to the statement that, after evaluation of $\{T_n\}_{n\in\mathbb{N}} \subset \mathcal{L}(X)$ on any vector $u \in X$, the sequence of vectors $\{T_n u\}_{n\in\mathbb{N}} \subset X$ converges to Tu strongly in X (i.e., $\{T_n u\}_{n\in\mathbb{N}}$ converges to Tu w.r.t. the norm of X).

The sequence $\{T_n\}_{n\in\mathbb{N}}$ is said to converge to $T \in \mathcal{L}(X)$ weakly

$$\text{if, for every pair } u, v \in X, \text{ one has } \lim_{n\to\infty} (T_n u|v)_X = (Tu|v)_X \text{ as } n \to \infty.$$

In this case, one writes

$$T = \text{w-}\lim_{n\to\infty} T_n$$

and says that T is the weak (operator) limit of the sequence $\{T_n\}_{n\in\mathbb{N}} \subset \mathcal{L}(X)$.

If the Hilbert space X is finite dimensional, then all three convergencies are equivalent.

If the Hilbert space X is infinite-dimensional, then these are three different (pairwise non-equivalent) types of convergencies (see [RS1, Section VI.1]). However,

$$\begin{aligned} \|T - T_n\|_{\mathcal{L}(X)} \to 0 \quad &\text{implies} \quad T = \text{s-}\lim_{n\to\infty} T_n, \\ \text{while} \quad T = \text{s-}\lim_{n\to\infty} T_n \quad &\text{implies} \quad T = \text{w-}\lim_{n\to\infty} T_n. \end{aligned}$$

We use the strong operator convergence in the following section.

12.4 Spectral theorem written via projection-valued measures

Let X be a Hilbert space.

Definition 12.2 (projection-valued measure, e.g., [RS1])**.**
The mapping $S \to P^S$ from $\mathcal{B}(\mathbb{R})$ to $\mathcal{L}(X)$ is called a projection-valued measure if the following conditions hold true:

(a) For each $S \in \mathcal{B}(\mathbb{R})$, the operator P^S is a projection.

(b) $P^{\varnothing} = 0_{\mathcal{L}(X)}$ and $P^{\mathbb{R}} = I$.

(c) If $S = \bigcup_{j\in\mathbb{N}} S_j$ is a union of disjunct sets S_j, then $P^S = \text{s-}\lim_{n\to\infty} \sum_{j=1}^{n} P^{S_j}$.

(d) $P^{S_1} P^{S_2} = P^{S_1 \cap S_2}$

Lemma 12.2.
Let $S \to P^S$ be a projection-valued measure, and let us denote $P_k := P^{(-\infty,k)}$. For every $u \in X$, the function $k \mapsto (P_k u|u)_X$ is a $\overline{\mathbb{R}}_+$-valued non-decreasing function defined for all $k \in \mathbb{R}$.

Proof. Since P_k is a projection, Exercise 12.2 implies $P_k = P_k^2 = P_k^*$ and

$$(P_k u|u)_X = (P_k^2 u|u)_X = (P_k u|P_k u)_X = \|P_k u\|^2 \in \overline{\mathbb{R}}_+.$$

Let $k_1 < k_2$ and put $S_1 = (-\infty, k_1)$ and $S_2 = [k_1, k_2)$. Using Definition 12.2, we see that the property (c) implies

$$P_{k_2} = P_{(-\infty,k_2)} = P^{S_1} + P^{S_2} = P_{k_1} + P^{S_2},$$

and properties (d) and (b) imply

$$0 = P^{\varnothing} = P^{S_1} P^{S_2} = P^{S_2} P^{S_1} = P^{S_2} P_{k_1} = P_{k_1} P^{S_2}.$$

Hence,

$$\|P_{k_2}u\|^2 = ([P_{k_1} + P^{S_2}]u \mid [P_{k_1} + P^{S_2}]u)_X = \|P_{k_1}u\|^2 + 2\,\mathrm{Re}(P_{k_1}u \mid P^{S_2}u)_X + \|P^{S_2}u\|^2$$
$$= \|P_{k_1}u\|^2 + 2\ \mathrm{Re}\ (P^{S_2}P_{k_1}u \mid u)_X + \|P^{S_2}u\|^2 = \|P_{k_1}u\|^2 + \|P^{S_2}u\|^2 \geqslant \|P_{k_1}u\|^2.$$

Thus, $k \mapsto (P_k u|u)_X$ is non-decreasing. □

Exercise 12.3.
The combination of the conditions of Definition 12.2 implies that the function

$$k \mapsto (P_k u|u)_X = \|P_k u\|^2, \qquad k \in \mathbb{R},$$

of Lemma 12.2 has the following additional properties:

(a)
$$\lim_{k\to-\infty} \|P_k u\|_X^2 = 0, \qquad \lim_{k\to+\infty} \|P_k u\|_X^2 = \|u\|_X^2.$$

(b) $\text{s-}\lim_{k\to-\infty} P_k = 0, \qquad \text{s-}\lim_{k\to+\infty} P_k = I.$

(c) $k \mapsto \|P_k u\|^2$ is left-continuous.

Definition 12.3.
For every $u \in X$, the Lebesgue–Stieltjes integral $\mu_u(S) := \int \chi_S \mathrm{d}(P_k u|u)_X$ taken first for intervals S, and then extended to all $S \in \mathcal{B}(\mathbb{R})$, defines

$$\text{a finite Borel measure } \mu_u = \mathrm{d}(P_k u|u)_X \text{ on } \mathbb{R},$$

which is called the spectral measure associated with the vector u (and with the projection-valued measure $S \to P^S$).

Let $u, v \in X$. The equality $P_k = P_k^2 = P_k^*$ and the polarization identity

$$(P_k u|v)_X = \frac{1}{4}\left(\|P_k(u+v)\|_X^2 - \|P_k(u-v)\|_X^2 - \mathrm{i}\|P_k(u+\mathrm{i}v)\|_X^2 + \mathrm{i}\|P_k(u-\mathrm{i}v)\|_X^2\right)$$

allows us to define a complex-valued measure $\mathrm{d}(P_k u|v)_X$ on $\mathbb{R}$ via the linear combination

$$\mathrm{d}(P_k u|v)_X = \frac{1}{4}\left(\mu_{u+v} - \mu_{u-v} - \mathrm{i}\mu_{u+\mathrm{i}v} + \mathrm{i}\mu_{u-\mathrm{i}v}\right).$$

Proposition 12.1.
Let $g \in M(\mathbb{R}, \mathbb{C})$ be a Borel measurable function.

(a) The set
$$D_g := \{u \in X : \int_{\mathbb{R}} |g(k)|^2 \mathrm{d}(P_k u|u)_X < \infty\}.$$
is a dense linear subspace of X.

(b) There exists a unique operator $T_g : D_g \subseteq X \to X$ such that
$$(T_g u|v)_X = \int_{\mathbb{R}} g(k)\mathrm{d}(P_k u|v)_X \qquad \forall \quad u \in D_g, \quad v \in X. \tag{12.1}$$

The most important application of this proposition is the case of the function $g(k) = k$, $k \in \mathbb{R}$.

Theorem 12.3 (spectral theorem, e.g., [AG, RS1])**.**

(a) Let A be a selfadjoint operator in X. There exists a unique projection-valued measure $S \to P^S$ such that

$$\operatorname{dom} A = \{u \in X : \int_{\mathbb{R}} k^2 \, \mathrm{d}(P_k u|u)_X < \infty\} \tag{12.2}$$

and

$$(Au|v)_X = \int_{\mathbb{R}} k \, \mathrm{d}(P_k u|v)_X \qquad \forall \quad u \in \operatorname{dom} A, \quad v \in X. \tag{12.3}$$

In this case, the projection-valued function $k \mapsto P_k$ is called the spectral function of A (or the spectral family, or a resolution of identity associated with A).

(b) Let $S \to P^S$ be a projection-valued measure. Then there exists a unique selfadjoint operator A such that (12.2) *and* (12.3) *hold true.*

Remark 12.3.
Let $k \mapsto P_k$ be the spectral function of a selfadjoint operator A like in Theorem 12.3.

(a) The functional calculus for the functions of the operator A can be defined via Proposition 12.1, i.e.,

$$g(A) := T_g \qquad \text{for Borel measurable functions } g.$$

This functional calculus is consistent with that of Definition 12.1.

(b) $P^S = \chi_S(A)$ for all $S \in \mathcal{B}(\mathbb{R})$.

The formulae (12.3) and (12.1) usually are written symbolically as

$$A = \int_{\mathbb{R}} k \, \mathrm{d}P_k, \qquad g(A) = \int_{\mathbb{R}} g(k) \mathrm{d}P_k.$$

However, it is possible to give a rigorous meaning to the formula

$$Au = \int_{\mathbb{R}} k \, \mathrm{d}P_k u$$

in the sense of an improper Riemann integral built with the use of the strong convergence of vectors in X. For continuous bounded functions $g : \mathbb{R} \to \mathbb{C}$ with compact support

$$g(A) = \int_{\mathbb{R}} g(k) \, \mathrm{d}P_k \qquad \forall u \in X$$

exists as a Riemann integral w.r.t. the convergence in the operator norm $\| \cdot \|_{\mathcal{L}(X)}$, see [AG].

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13 Classification of types of spectra of selfadjoint operators.

13.1 Norms of resolvents and spectra of selfadjoint operators.

Let $T : \operatorname{dom} T \subseteq X \to X$ be an operator in a Hilbert space X. Recall that T is closed if the resolvent set $\rho(T)$ is nonempty.

Theorem 13.1 (the 1st resolvent Neumann series, e.g., [Kato, Sect.III.6.1], [B, Thm.4.10])**.** *Let $\rho(T) \neq \varnothing$. Then:*

(a) For every $k_0 \in \rho(T)$,

$$(T-k)^{-1} = \sum_{j=0}^{+\infty} (k-k_0)^j ((T-k_0)^{-1})^{j+1} \quad \text{for all } |k-k_0| < \|(T-k_0)^{-1}\|. \quad (13.1)$$

(b) The resolvent set $\rho(T)$ is open and the resolvent $k \mapsto (T-k)^{-1}$ is an analytic $\mathcal{L}(X)$-valued function on $\rho(T)$.

(c) For every $k_0 \in \rho(T)$,

$$\frac{1}{\operatorname{dist}(k_0, \sigma(T))} \leqslant \|(T-k_0)^{-1}\|.$$

On formal level, formula (13.1) is obtained by straitforward verification, but on the rigorous level an additional step that uses the closedness of T is needed, see [Kato, B]. Statement (a) implies (b) (and so implies that $\sigma(T)$ is closed). Statement (c) also follows from (a) as an estimate on the radius of convergence in (13.1).

Theorem 13.2.
Let $A = A^*$ and let a function $g \in M(\mathbb{R}, \mathbb{C})$ be continuous on $\sigma(A)$. Then the norm of the operator $g(A) = \int_{\mathbb{R}} g(k) \mathrm{d}P_k$ is given by

$$\|g(A)\| = \|g\|_{L^\infty(\sigma(A))} = \operatorname*{ess\,sup}_{k \in \sigma(A)} |g(k)|.$$

We take this theorem without proof. For the proof of this theorem in the case of bounded selfadjoint A, we refer to [RS1, Section VII.1]. The case where A is unbounded can be reduced to the case of bounded selfadjoint operators by means of the decomposition

$$Au = A \sum_{n \in \mathbb{Z}} P^{[n,n+1)} u = \sum_{n \in \mathbb{Z}} A P^{[n,n+1)} u = \sum_{n \in \mathbb{Z}} P^{[n,n+1)} A P^{[n,n+1)} u, \quad u \in \operatorname{dom} A,$$

into a sum of bounded selfadjoint operators $A_n = AP^{[n,n+1)} = P^{[n,n+1)} A P^{[n,n+1)}$, where the summation of $A_n u$ is w.r.t. the norm $\|\cdot\|_X$. The sum here is essentially orthogonal, i.e., if A_n is perceived as an operator in the closed subspace $P^{[n,n+1)}X$, then this formula can be written as

$$A = \bigoplus_{n \in \mathbb{N}} A_n.$$

Corollary 13.1.
For a selfadjoint operator A and $k_0 \in \rho(A)$,

$$\|(A-k_0)^{-1}\| = \frac{1}{\operatorname{dist}(k_0, \sigma(T))}$$

Note that in the case of a separable X, this corollary and Theorem 13.2 can be easily obtained from Theorem 12.1 (the spectral theorem via a multiplication operator $\mathcal{M}_f$).

Theorem 13.3 (e.g., [AG, Section 82])**.**
Let $A = A^*$. Then

$$\sigma(A) := \{k \in \mathbb{R} : \quad P_{k+\varepsilon} - P_{k-\varepsilon} \neq 0_{\mathcal{L}(X)} \quad \forall \varepsilon > 0\},$$

i.e., $\sigma(A)$ consists of points of growth of the operator-valued measure $\{P^S\}_{S\in\mathcal{B}(\mathbb{R})}$ (note that $P_{k+\varepsilon} - P_{k-\varepsilon} = P^{[k-\varepsilon,k+\varepsilon)}$ and the function $k \mapsto P_k$ is nondecreasing in the sense of Lemma 12.2).

This theorem can be obtained from Corollary 13.1.

Theorem 13.4 (see [RS1, Section VII.2])**.**
Let $A = A^*$. Let S be the maximal open set S such that $\mu_u(S) = 0$ for all $u \in X$. Then $\sigma(A) = \mathbb{R}\backslash S$.

Corollary 13.2.
Let $X \neq \{0\}$ and $A = A^*$ in X. Then $\sigma(A) \neq \varnothing$.

13.2 Absolutely continuous, singular continuous, and pure point spectra

Let $\mathbb{Y}$ be a metric space equipped with the Borel σ-algebra, which makes it a measurable space. A measure μ on $\mathbb{Y}$ is called diffuse if $\mu(\{y\}) = 0$ for every $y \in \mathbb{Y}$.

Definition 13.1.
Let μ be a σ-finite Borel measure on $\mathbb{R}$.

(a) In this case, a diffuse measure μ is called also continuous.

(b) A measure μ is called discrete (or pure point) if there exists at most countable set $Y = \{y_j\}_{j=1}^N \subset \mathbb{R}$, $N \in \widehat{\mathbb{N}}$, such that $\mu(\mathbb{R}\backslash Y) = 0$.

(c) A continuous measure μ is called singular continuous if there exists a Borel set S with $|S|_1 = 0$ such that $\mu(\mathbb{R}\backslash S) = 0$.

Let μ be a discrete finite Borel measure. The set Y in Definition 13.1 can be chosen in a unique way such that $\mu(y_j) > 0$ for all $y_j \in Y$. Then one can express μ as

$$\mu(S) = \int_S \mathrm{d}\Psi_{\mathrm{pp}}, \qquad S \in \mathcal{B}(\mathbb{R}),$$

where

$$\Psi_{\mathrm{pp}}(x) = \sum_{y_j<x} \mu(y_j)$$

is a jump function, and the integral is understood as the Lebesgue–Stieltjes integral.

Theorem 13.5 (Lebesgue decomposition, e.g. [KF, RS1])**.**
Every σ-finite Borel measure μ on $\mathbb{R}$ can be decomposed in a unique way into a sum of Borel measures

$$\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{sc}} + \mu_{\mathrm{pp}}$$

such that the measure μ_{ac} is absolutely continuous (w.r.t. $|\cdot|_1$), the measure μ_{sc} is singular continuous, and μ_{pp} is discrete.

Definition 13.2.

(a) A closed subspace X_1 is called invariant subspace of an operator T if $Tu \in X_1$ for every $u \in \operatorname{dom} T \cap X_1$.

(b) Let X_j, $j = 1, 2$, be closed subspaces of X such that

$$X = X_1 \oplus X_2,$$

and let P_{X_j} be the associated projections. The subspace X_1 is called a reducing subspace for an operator T if, for every $u \in \operatorname{dom} T$, one has $P_{X_1} u \in \operatorname{dom} T$, $P_{X_2} u \in \operatorname{dom} T$, and each of the subspaces X_1 and X_2 is invariant for T.

Obviously a closed subspace X_1 is reducing for T if and only if $X_1^\perp$ is so. If this is the case and $X_2 = X_1^\perp$, then

$$T = T|_{X_1} \oplus T|_{X_2},$$

where $T|_{X_1}$ and $T|_{X_2}$ are the parts of T in the subspaces X_j having the domains $\operatorname{dom}(T|_{X_j}) := X_j \cap \operatorname{dom} T$. Note that $\{0\}$ and $\{0\}^\perp = X$ are reducing subspaces for every operator T.

Proposition 13.1 (e.g., [AG])**.**
Let $A = A^*$ and let X_1 be an invariant subspace for A. Then X_1 is a reducing subspace for A and $A|_{X_1}$ is a selfadjoint operator in X_1.

Theorem 13.6 (e.g., [RS1])**.**
Let $A = A^*$. Let us consider the sets

$$X_{\mathrm{ac}} := \{u \in X : \ \mu_u \text{ is absolutely continuous}\},$$
$$X_{\mathrm{sc}} := \{u \in X : \ \mu_u \text{ is singular continuous}\},$$
$$X_{\mathrm{pp}} := \{u \in X : \ \mu_u \text{ is discrete}\}.$$

Then:

(a) X_{ac}, X_{sc}, and X_{pp} are closed subspaces of X and each of them is a reducing subspace for A.

(b) $X = X_{\mathrm{ac}} \oplus X_{\mathrm{sc}} \oplus X_{\mathrm{pp}}$

(c) The part $A_{\mathrm{pp}} := A|_{X_{\mathrm{pp}}} : \operatorname{dom} A_{\mathrm{pp}} \subseteq X_{\mathrm{pp}} \to X_{\mathrm{pp}}$ has admits orthonormal basis $\{u_j\}_{j \in J}$ of X_{pp} consisting of eigenvectors of A_{pp} (where J is a certain index set, not necessarily countable).

Consequently,

$$A_{\mathrm{pp}} u = \sum_{\substack{j \in J \\ (u|u_k)_X \neq 0}} k_j (u|u_j)_X u_j, \qquad u \in \operatorname{dom} A_{\mathrm{pp}} = X_{\mathrm{pp}} \cap \operatorname{dom} A,$$

where k_j is an eigenvalue associated with an eigenvector u_j.

Note that it is possible that one or several of theses subspaces X_{ac}, X_{sc}, and X_{pp} are the zero spaces $\{0\}$ (or one of them is the the whole space X).

Definition 13.3 (ac-, sc-, and pp-spectra, e.g., [RS1])**.**
Let $A = A^*$.

(a) The set $\sigma_{\rm ac}(A) := \sigma(A|_{X_{\rm ac}})$ is called an absolutely continuous spectrum.

(b) The set $\sigma_{\rm sc}(A) := \sigma(A|_{X_{\rm sc}})$ is called a singular continuous spectrum.

(c) The set $\sigma_{\rm pp}(A) := \sigma(A|_{X_{\rm pp}})$ is called a pure point spectrum.

(d) If $X = X_{\rm pp}$ (and so $\{0\} = X_{\rm ac} = X_{\rm sc}$), one says that the spectrum of A is a pure point spectrum.

13.3 Characterization of eigenvalues of selfadjoint operators in terms of projector-valued measures

Theorem 13.7 (e.g., [AG])**.**
Let $A = A^*$. Then:

$k \in \sigma_{\rm p}(A)$ *if and only if* $P^{\{k_0\}} \neq 0$.

If $k_0 \in \sigma_{\rm p}(A)$, the corresponding to k_0 eigenspace is

$$\ker(A - k_0) = P^{\{k_0\}}X = \operatorname{ran} P^{\{k_0\}}.$$

In other words, eigenvalues are jumps of the function $k \mapsto P_k$.

Remark 13.1.
By comparison of Definitions 13.3, Theorem 13.6, and Theorem 13.7, it can be obtained that for a selfadjoint operator A,

$$\sigma_{\rm pp}(A) = \overline{\sigma}_p(A).$$

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14 Examples of various types of spectra.

14.1 Remarks and examples related to reducing subspaces and pure point spectra.

Let us recall that two closed subspaces X_j, $j = 1, 2$, of a Hilbert space X such that

$$X = X_1 \oplus X_2,$$

are called reducing for an operator $A : \operatorname{dom} A \subseteq X \to X$ if the following conditions are satisfied: for every $u \in \operatorname{dom} A$, one has $P_{X_1} u \in \operatorname{dom} T$, $P_{X_2} u \in \operatorname{dom} T$, and each of the subspaces X_1 and X_2 is invariant for A. In this lecture, P_{X_j} is an (orthogonal) projector to X_j.

The trivial examples of reducing subspaces for every operator are the subspaces $\{0\}$ and X. By Proposition 13.1, every invariant subspace of a selfadjoint operator A is reducing for A.

The following theorem characterizes (independently of the functional calculus) the projection-valued measure $\{P^S\}_{S \in \mathcal{B}(\mathbb{R})}$ associated with a selfadjoint operator A.

Theorem 14.1 ([AG, Section 75])**.**
Let $A = A^*$ in a Hilbert space X. Then the equality $A = \int_{\mathbb{R}} k \, \mathrm{d}P_k$ holds for a certain projection-valued measure $\{P^S\}_{S \in \mathcal{B}(\mathbb{R})}$ if and only if the following two conditions are satisfied:

(a) The space $P^{\mathcal{I}} X$ is a reducing subspace for A for every interval $\mathcal{I} \subseteq (-\infty, +\infty)$.

(b) For every $k_1, k_2 \in \mathbb{R}$ and $u \in \operatorname{dom} A$ such that $-\infty \leqslant k_1 < k_2 \leqslant +\infty$ and $u \in P^{(k_1, k_2]} X$, the following inequality holds

$$k_1 \|u\|_X^2 \leqslant (Au|u)_x \leqslant k_2 \|u\|_X^2.$$

Example 14.1 (eigenspaces as reducing subspaces)**.**
Let $A = A^*$ and $k_0 \in \sigma_{\mathrm{p}}(A)$. Then the eigenspace $\ker(A - k_0)$ of A corresponding to the eigenvalue k_0 is a reducing subspace of A. This follows from Theorem 14.1 (or from Proposition 13.1). Hence the orthogonal complement $X_2 := (\ker(A - k_0))^{\perp}$ is also a reducing subspace of A. Clearly, the part $A|_{X_2}$ of A in X_2 has no eigenvalue at k_0.

Proposition 14.1 ([AG, Section 75])**.**
Let $T : \operatorname{dom} T \subseteq X \to X$ be an operator in X. Let the family $\{X_j\}_{j \in J}$ of closed subspaces of X (indexed by a certain index set J, possibly uncountable) satisfy the following conditions:

(a) each X_j is a reducing subspace for T;

(b) X_j and X_n are mutually orthogonal if $j \neq n$.

Then the subspace

$$\widetilde{X} = \bigoplus_{j \in J} X_j := \left\{ \sum_{j \in J} u_j : \ u_j \in X_j \ \forall j \ \ \text{and} \ \sum_{j \in J} \|u_j\|^2 < +\infty \right\}$$

is a reducing subspace for T. The part $\widetilde{T} = T|_{\widetilde{X}}$ of T in $\widetilde{X}$ has the domain $\operatorname{dom} \widetilde{T}$ consisting

of all $u \in X$ such that

$$P_{X_j} u \in \operatorname{dom} T|_{X_j} \text{ for all } j \text{ and } \quad \sum_{j\in J} \|TP_{X_j} u\|_X^2 < +\infty.$$

Besides, the following formulae hold true for all $u \in \operatorname{dom} \widetilde{T}$,

$$Tu = \sum_{j\in J} T|_{X_j} P_{X_j} u = \sum_{j\in J} TP_{X_j} u = \sum_{j\in J} P_{X_j} TP_{X_j} u. \tag{14.1}$$

For a selfadjoint A, we considered in Lecture 13 the decomposition of X

$$X = X_{\mathrm{ac}} \oplus X_{\mathrm{sc}} \oplus X_{\mathrm{pp}}$$

into the reducing subspaces corresponding to $\sigma_{\mathrm{ac}}(A)$, $\sigma_{\mathrm{sc}}(A)$, and $\sigma_{\mathrm{pp}}(A)$.

Example 14.2 (X_{pp} as an orthogonal sum of eigenspaces)**.**
Let $A = A^*$ and $\{k_j\}_{j\in J} = \sigma_{\mathrm{p}}(A)$ be the family of all eigenvalues indexed by J without repetitions (i.e., $k_j \neq k_n$ if $j \neq n$) . The eigenspaces $X_j := \ker(A - k_j)$ are mutually orthogonal by Exercise 11.2, and satisfy Proposition 14.1 due to Example 14.1. Hence, the orthogonal sum of all eigenspaces

$$\widetilde{X} = \bigoplus_{j\in J} X_j = \bigoplus_{j\in J} \ker(A - k_j)$$

is a reducing subspace. It is easy to see that

$$\sigma(A|_{\widetilde{X}}) = \overline{\sigma_{\mathrm{p}}(A)} \text{ and } \sigma_{\mathrm{p}}(A|_{\widetilde{X}^\perp}) = \varnothing.$$

The space $\widetilde{X}$ is exactly the space X_{pp}, and this explains the equality

$$\sigma_{\mathrm{pp}} = \overline{\sigma_{\mathrm{p}}(A)}.$$

If additionally $X = X_{\mathrm{pp}}$, then A is an operator with purely point spectrum, A possess an orthonormal basis $\{u_{j'}\}_{j'\in J'}$ of eigenvectors (indexed by a certain index set J'), and the formula (14.1) becomes the eigenvector expansion

$$Au = \sum_{j'\in J'} \widetilde{k}_{j'} (u|u_{j'})_X u_{j'}, \qquad u \in \operatorname{dom} A,$$

where $\widetilde{k}_{j'}$ is an eigenvalue corresponding to $u_{j'}$.

14.2 Pure point spectra of differential operators

Theorem 14.2 (Minami [M89])**.**
Let $\eta = \sum_{j\in\mathbb{N}} \delta_{y_j}$ be a homogeneous proper Poisson SPP of positive intensity on $\mathbb{R}$. Let $a_{\max} > 0$ be a certain positive constant and α_0 be a certain $[0, a_{\max}]$-valued random variable with distribution p_{α_0}. Let

$$\xi = \sum_{j\in\mathbb{N}} \delta_{(y_j,\alpha_j)}$$

be an independent p_{α_0}-marking of η. Consider in $L^2(\mathbb{R})$ the selfadjoint Poisson-Anderson-Schrödinger operator

$$\mathcal{H}_{Y,\alpha} := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{j=1}^{N} \alpha_j \delta_{y_j}$$

associated with ξ as in Theorem 10.2. Then:

(a) The spectrum $\sigma(\mathcal{H}_{Y,\alpha})$ is deterministic and pure point in the following sense: there exists an event $\Omega_1 \in \mathcal{F}_\Omega$ of probability 1 such that

$$\sigma(\mathcal{H}_{Y,\alpha}) = \sigma_{\rm pp}(\mathcal{H}_{Y,\alpha}) = [0, +\infty) \qquad \forall \omega \in \Omega_1.$$

(b) Besides, the event $\Omega_1 \in \mathcal{F}_\Omega$ of probability 1 can be chosen such that for every $\omega \in \Omega_1$ and every eigenfunction u of $\mathcal{H}_{Y,\alpha}$ there exists a negative constant $c_{u,\omega}$ with the property

$$\limsup_{x \to \pm\infty} \frac{1}{|x|} \ln |u(x)| \leqslant c_{u,\omega} < 0 \quad \textit{(exponential decay).}$$

The effect described by the properties (a) and (b) of Theorem 14.2 is the spectral Anderson localization for the whole spectrum. It is proved for many 1-dimensional models with random potentials having reasonable ergodic properties.

The analogous question of the spectral Anderson localization in the multi-dimensional case is a long standing open problem. It is presently conjectured that there are multi-dimensional models where the spectral Anderson localization does not hold for the whole spectrum. The 3-dimensional Poisson-Anderson-Schrödinger operator of Kaminaga-Mine-Nakano (see Lecture 10) is possibly the least studied operator with good probabilistic properties.

Remark 14.1.
The space $L^2(\mathbb{R})$ is separable. Therefore the set of eigenvalues $\sigma_p(\mathcal{H}_{Y,\alpha})$ in Theorem 14.2 is at most countable, but

$$[0, +\infty) = \sigma_{\rm pp}(\mathcal{H}_{Y,\alpha}) = \overline{\sigma_{\rm p}(\mathcal{H}_{Y,\alpha})}.$$

Thus, for all $\omega \in \Omega_1$, the set $\sigma_p(\mathcal{H}_{Y,\alpha})$ is countable and dense in $[0, +\infty)$.

Proposition 14.2.
Let $A = A^*$ and $\sigma(A) = \sigma_{\rm disc}(A)$. Then $\sigma(A) = \sigma_{\rm pp}(A)$ and $X_{\rm pp} = X$.

This case takes place, for example, for the Laplace operator $(-1)\Delta^{\rm D} = \mathbf{grad}_0^* \mathbf{grad}_0$ with the Dirichlet boundary condition in $L^2(G)$, where $G \subset \mathbb{R}^d$ is a bounded open nonempty set. The operator $(-1)\Delta^{\rm D}$ is an invertible selfadjoint operator with compact resolvent. Hence, $(-1)(\Delta^{\rm D})^{-1} \in \mathcal{L}(X)$ is compact. Besides, $(-1)(\Delta^{\rm D})^{-1}$ is selfadjoint due to the next theorem.

Theorem 14.3.
Assume that $T : \operatorname{dom} T \subseteq X \to X$ be invertible and densely defined and that T^{-1} is densely defined. Then:

(a) $(T^*)^{-1} = (T^{-1})^*$

(b) If additionally T is selfadjoint, then T^{-1} is also selfadjoint.

Proposition 14.3 (spectral theorem for compact selfadjoint operators)**.**
Let $A = A^*$ and let A be compact. Then

$$X = X_{\rm pp}, \qquad \sigma_{\rm disc}(A) = \sigma(A) \backslash \{0\}, \qquad \sigma_{\rm ess}(A) = 0.$$

Moreover, the following statements hold:

(a) In the case $0 \in \sigma_{\rm p}(A)$*, we have*

$$\sigma(A) = \sigma_{\rm pp}(A) = \sigma_{\rm p}(A) \neq \sigma_{\rm disc}(A).$$

(b) In the case $0 \notin \sigma_{\rm p}(A)$*, we have*

$$\sigma(A) = \sigma_{\rm pp}(A) = \{0\} \cup \sigma_{\rm p}(A) \qquad \textit{and} \qquad \sigma_{\rm p}(A) = \sigma(A)\backslash\{0\} = \sigma_{\rm disc}(A).$$

14.3 The case of a finite number of δ-interactions in $\mathbb{R}^3$.

Let $Y = \{y_j\}_{j=1}^N$ be a finite collection of distinct points in $\mathbb{R}^3$. Let $a = \{a_j\}_{j=1}^N \subset \mathbb{R}$ be the corresponding deterministic "inverse strength" parameters. The corresponding deterministic operator with point interactions

$$\mathcal{H}_{Y,a} = -\Delta + \text{“} \sum_{j=1}^N \mathfrak{m}(a_j)\delta_{y_j} \text{“}, \quad \mathcal{H}_{Y,a} : \operatorname{dom} \mathcal{H}_{Y,a} \subset L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$$

was defined in Lecture 5 and, by Theorem 5.1,

$$\mathcal{H}_{Y,a} = (\mathcal{H}_{Y,a})^*.$$

Theorem 14.4.
(a) The set of eigenvalues of $\mathcal{H}_{Y,a}$ *is finite and is a subset of* $(-\infty, 0]$*, i.e.,*

$$\sigma_{\rm p}(\mathcal{H}_{Y,a}) = \sigma_{\rm pp}(\mathcal{H}_{Y,a}) = \{k_j\}_{j=1}^n \subset (-\infty, 0] \qquad \textit{with a certain } n \in \mathbb{N}_0.$$

(b) The absolutely continuous spectrum is

$$\sigma_{\rm ac}(\mathcal{H}_{Y,a}) = [0, +\infty).$$

(c) The singular continuous spectrum $\sigma_{\rm sc}(A)$ *is empty.*

(d) $\sigma(\mathcal{H}_{Y,a}) = \sigma_{\rm p}(\mathcal{H}_{Y,a}) \cup \sigma_{\rm ac}(\mathcal{H}_{Y,a}) = \{k_j\}_{j=1}^n \cup [0, +\infty)$.

The set of eigenvalues can be found using the following theorem.

Theorem 14.5 (Krein-type resolvent formula, e.g., [**?**])**.**
(a) Let $k \in \rho(\mathcal{H}_{Y,a})$ *and* $k = \lambda^2$ *for* $\lambda \in \mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$*. Then the integral kernel* $K_\lambda(x, x')$ *of the resolvent* $(\mathcal{H}_{Y,a} - \kappa)^{-1} = (\mathcal{H}_{Y,a} - \lambda^2)^{-1}$ *at* k *is given by the formula*

$$K_\lambda(x, x') = G_\lambda(x - x') + \sum_{j,j'=1}^N G_\lambda(x - Y_j)\,[\Gamma_{Y,a}]^{-1}_{j,j'}\,G_\lambda(x' - Y_{j'}), \tag{14.2}$$

where $x, x' \in \mathbb{R}^3\backslash Y$ *and* $x \neq x'$*. Here* $G_\lambda(x-x') := \frac{e^{\mathrm{i}\lambda|x-x'|}}{4\pi|x-x'|}$ *is the integral kernel associated with the resolvent* $(-\Delta - \lambda^2)^{-1}$ *of the nonnegative selfadjoint Laplacian* $(-1)\Delta$ *in* $L^2(\mathbb{R}^3)$*, whereas* $[\Gamma_{Y,a}]^{-1}_{j,j'}$ *denotes the* j, j'*-element of the inverse to the matrix*

$$\Gamma_{Y,a}(\lambda) = \Big[\big(a_j - \tfrac{\mathrm{i}\lambda}{4\pi}\big)\,\delta_{jj'} - \widetilde{G}_\lambda(Y_j - Y_{j'})\Big]_{j,j'=1}^N, \tag{14.3}$$

where $\widetilde{G}_\lambda(x) := \begin{cases} G_\lambda(x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ *and* $\delta_{jj'}$ *is the Kronecker delta.*

(b) The set of negative eigenvalues $\sigma_{\rm p}(\mathcal{H}_{Y,a})\backslash\{0\}$ *has the form* $\{\lambda_j^2\}_{j=1}^{\widetilde{n}}$*,* $\widetilde{n} \in \mathbb{N}_0$*, where* $\{\lambda_j\}_{j=1}^{\widetilde{n}}$ *is the set of solutions to the equations*

$$\det \Gamma_{Y,a}(\lambda) = 0 \quad \textit{ on the line } \mathrm{i}\mathbb{R}_+ := \{\mathrm{i}c : \ c \in (0, +\infty)\}.$$

14.4 Deterministic and stochastic resonances

The poles λ of generalized continuation of the resolvent $(\mathcal{H}_{Y,a} - \lambda^2)^{-1}$ are exactly the solutions $\det \Gamma_{Y,a}(\lambda) = 0$. They are the (continuation) resonances of Lectures 1-2.

If the positions y_j of point interactions become random, i.e., if Y is finite SPP, the set of eigenvalues associated with the operator $H_{Y,a}$ becomes a locally finite SPP on $(-\infty, 0]$. The corresponding set of resonances becomes a locally finite SPP on $\mathbb{C}$. The study of these point processes of random eigenvalues/resonances is presently in the initial stage, see [AK21, KMN25].

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