Script for the course

V5B6 - Selected Topics in Analysis and Calculus of Variations -

Homogenization-convergence and optimization of eigenvalues

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In this script the content of the lectures is given in a very compressed form and may include misprints. Corrections are welcome and may be sent to the email address of the lecturer:

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This script cannot substitute textbooks, monographs, and research papers that are the base of the course.

The main books and papers, which were used in the preparation of this course, are:

[A02] Allaire, G., Shape Optimization by the Homogenization Method. Springer, 2002.

[C00] Cherkaev, A., Variational methods for structural optimization. Springer Science & Business Media, New York, 2000.

[CK97] Cherkaev, A., and Kohn, R., Eds., Topics in the mathematical modelling of composite materials. Boston: Birkhäuser, 1997.

[CL96] Cox, S., and Lipton, R., 1996. Extremal eigenvalue problems for two-phase conductors. Archive for Rational Mechanics and Analysis 136(2), pp.101–118.

[H06] Henrot, A., Extremum problems for eigenvalues of elliptic operators. Springer Science & Business Media, 2006.

[JKO12] Jikov, V.V., Kozlov, S.M. and Oleinik, O.A., Homogenization of differential operators and integral functionals. Springer Science & Business Media, 2012.

[K79] Kesavan, S., 1979. Homogenization of elliptic eigenvalue problems: Part 1. Applied Mathematics and Optimization, 5, pp.153-167.

The complete list of references is given at the end of the script.

Contents

1 Overview: Two main topics in examples.

1.1 Example of optimization of termal conductivity via eigenvalues.

Consider the evolution of temperature $u(x,t)$ in a mixture of two materials with termal conductivities α_1 and α_2, $0 < \alpha_1 < \alpha_2$. The heat equation is

$$\partial_t u(x,t) = \nabla \cdot (a \nabla u(x,t)), \qquad x \in \Omega, \tag{1.1}$$

where $a = a(x) = \alpha_1 \chi_\omega(x) + \alpha_2 \chi_{\Omega \setminus \omega}(x)$ is the conductivity function, $\Omega \subset \mathbb{R}^d$ is a domain, $\omega \subset \Omega$ is a measurable set, and

$$\chi_E(x) = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}$$

is the indicator function of E.

Definition 1.1.
A set $\mathcal{D} \subset \mathbb{R}^d$ is a domain if $\mathcal{D}$ is nonempty, bounded, open, and connected.

The equation is equipped with the "ice bath" boundary condition and an initial boundary condition:

$$u(x,t) = 0 \qquad \text{for } x \in \partial\Omega, \quad t > 0, \tag{1.2}$$

$$u(x,0) = u^0(x) \qquad \text{for } x \in \partial\Omega \tag{1.3}$$

where $u_0 \geqslant 0$ almost everywhere. For simplicity we assume also

$$u^0 \in L^2(\Omega) = L^2(\Omega, \mathbb{R}).$$

Here $\partial\Omega$ is the boundary of Ω. With the differential expression

$$\ell(u) = \ell_a(u) = -\nabla \cdot (a \nabla u) = -\operatorname{div} a \operatorname{grad} u$$

and with the boundary condition (1.2), one can associate a selfadjoint operator $\mathcal{L} = \mathcal{L}_a$ in the complex Hilbert space $L^2(\Omega, \mathbb{C})$. The spectrum $\sigma(\mathcal{L})$ of $\mathcal{L}$ is a nondecreasing sequence of eigenvalues $\{\lambda_k\}_{k \in \mathbb{N}}$,

$$0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \dots . \tag{1.4}$$

Corresponding eigenfunctions $u_k(\cdot)$ can be chosen such that all u_k are real-valued and $\{u_k\}_{k \in \mathbb{N}}$ is an orthonormal basis in $L^2(\Omega)$. The latter means, in particular, that the eigenvalues are repeated in the sequence $\{\lambda_k\}_{k \in \mathbb{N}}$ according to their geometric multiplicities (since $\mathcal{L}$ is a selfadjoint operator, the geometric and algebraic multiplicities of eigenvalues coincide, see [K13] for details).

Remark 1.1.
This description of the spectrum of $\mathcal{L}$ follows from the spectral theorem [K13, Theorem III.6.29 and Sect. V.3.5] for selfadjoint operators $T = T^*$ with a compact resolvent

$(T-\lambda)^{-1}$, where $\lambda \in \rho(T)$ is in the resolvent set $\rho(T) := \mathbb{C}\backslash\sigma(T)$.
The unique strong solution in $L^2(\Omega)$ to the initial value problem (1.1)-(1.3) *is given by*

$$u(x,t) = \sum_{k=1}^{\infty} \mathrm{e}^{-\lambda_k t} \langle u^0 | u_k \rangle_{L^2} \; u_k(x)$$

where $\langle f|g\rangle_{L^2} = \int_\Omega f\overline{g}\mathrm{d}x$.

Remark 1.2.
Since we assumed that Ω is a domain, the 1st eigenvalue λ_1 is simple. That is, λ_1 has geometric (and algebraic) multiplicity 1 [GT77, Theorem 8.38]. This fact is included in (1.4) *in the form of the inequality $\lambda_1 < \lambda_2$. In a more general case, where a bounded open set Ω is not a domain because it is not connected, the inequality $\lambda_1 < \lambda_2$ does not necessary holds (as an example one can take as Ω a union of two disjoint unit balls).*

Proposition 1.1.

(i) $\|u(\cdot,t)\|_{L^2} \leqslant \mathrm{e}^{-\lambda_1 t}\|u^0\|_{L^2}$.

(ii) For generic $u_0 \in L^2(\Omega)$, we have $\langle u^0|u_1\rangle_{L^2} \neq 0$. In this case,

$$\|u(\cdot,t)\|_{L^2} \sim |\langle u_0|u_1\rangle_{L^2}| \; \mathrm{e}^{-\lambda_1 t} \quad \text{as } t \to +\infty.$$

The proof follows easily from Remark 1.1.

Remark 1.3.
Proposition 1.1 means that λ_1 is the (exponential) decay rate of $\|u(\cdot,t)\|_{L^2}$ in the generic situation $\langle u^0|u_1\rangle_{L^2} \neq 0$. In the exceptional case $\langle u^0|u_1\rangle_{L^2} = 0$, the decay rate of $\|u(\cdot,t)\|_{L^2}$ is faster (i.e., greater) than λ_1.

Conclusion. *One can use $\lambda_1 = \lambda_1(a)$ to measure the quality of conductivity (or opposite, of insulation) of the structure represented by the function $a(x)$, $x \in \Omega$.*

Let $\mathbb{F}$ be a feasible family/set of structures $a(\cdot)$, i.e., the family of composite structures which are feasible for fabrication.

Problem 1.1 (maximization of insulation, see [ATL89, CL96])**.**
We search for

$$\arg\min_{a\in\mathbb{F}} \lambda_1(a).$$

Here $\arg\min$ denotes the set of all optimizers $a_*(\cdot) \in \mathbb{F}$ (in this case, minimizers) and simultaneously is a short way to formulate the problem of finding this set. Similarly the notation $\arg\max$ is used.

Problem 1.2 (maximization of conductance, see [CL96])**.**
We search for

$$\arg\max_{a\in\mathbb{F}} \lambda_1(a).$$

1.2 Reasonable feasible families.

Example 1.1.
Let us consider the following set of L^∞-functions

$$\mathbb{F}^{\alpha_1,\alpha_2,\Omega} = \{a = \alpha_1 \chi_\omega + \alpha_2 \chi_{\Omega\setminus\omega} : \ \omega \subset \Omega \text{ is measurable}\},$$

that is, we consider all possible (measurable) mixtures of materials with α_1 and α_2.

This example is essentially trivial since

$$\operatorname*{arg\,min}_{a\in\mathbb{F}^{\alpha_1,\alpha_2,\Omega}} \lambda_1(a) = \{a_{\min}(\cdot)\} = \{\alpha_1\chi_\Omega(\cdot)\}. \tag{1.5}$$

Similarly,

$$\operatorname*{arg\,max}_{a\in\mathbb{F}^{\alpha_1,\alpha_2,\Omega}} \lambda_1(a) = \{a_{\max}(\cdot)\} = \{\alpha_2\chi_\Omega(\cdot)\}. \tag{1.6}$$

Remark 1.4.
Formulae (1.5) *and* (1.6) *follow from the formula*

$$\lambda_1(a) = \min_{\substack{v\in H_0^1(\Omega)\\ v\neq 0}} \mathcal{R}_{\ell_a}[v] = \min_{\substack{v\in \operatorname{dom}\mathcal{L}_a\\ v\neq 0}} \mathcal{R}_{\ell_a}[v],$$

where $0 = 0_{L^2}$ *and*

$$\mathcal{R}_{\ell_a}[v] := \frac{\int_\Omega a|\nabla v|^2 \mathrm{d}x}{\int_\Omega |v|^2 \mathrm{d}x}$$

is the Rayleigh quotient associated with the differential expression ℓ_a. Moreover, the minimal values of the Rayleigh quotient are achieved on the set $\{cu_1(\cdot)\}_{c\in\mathbb{C}\setminus\{0\}}$. These statements follow from the 1st Friedrichs representation theorem [K13, Th. VI.2.1] combined with the spectral theorem [K13, Theorem III.6.29 and Sect. V.3.5], which are applied to the selfadjoint operator $\mathcal{L}_a$.

Example 1.2 (prescribed ratio of materials)**.**
Let a constant $\gamma \in (0, \operatorname{meas}(\Omega))$ be fixed. Consider the feasible family

$$\mathbb{F}_\gamma = \mathbb{F}_\gamma^{\alpha_1,\alpha_2,\Omega} = \{a = \alpha_1 \chi_\omega + \alpha_2 \chi_{\Omega\setminus\omega} : \ \omega \subset \Omega \text{ and } \operatorname{meas}(\omega) = \gamma\},$$

where $\operatorname{meas}(\omega)$ denotes the Lebesgue measure in $\mathbb{R}^d$.

Theorem 1.1.
Let $d \geqslant 2$. Assume that a simply connected domain $\Omega \subset \mathbb{R}^d$ has a connected C^2-boundary $\partial\Omega$. Then:

(a) $\operatorname*{arg\,min}_{a\in\mathbb{F}_\gamma} \lambda_1(a) \neq \varnothing \iff \Omega$ *is a ball [C17].*

(b) If Ω is a ball in $\mathbb{R}^d$ centered at $0 = 0_{\mathbb{R}^d}$, then every optimal structure $a_*(\cdot)$ for the problem $\operatorname*{arg\,min}_{a\in\mathbb{F}_\gamma} \lambda_1(a)$ is a radial function and corresponding 1st eigenfunctions $u_{1,*}(\cdot)$ are also radial [ATL89].

1.3 H-convergence.

Let $\mathbb{R}^{d\times d}$ be the normed space of $d \times d$-matrices $M = (M^{i,j})_{i,j=1}^{d}$. The choice of the norm is not important, i.e., the operator norm for the linear operator $M : \mathbb{R}^d \to \mathbb{R}^d$ can be used.

Let $\alpha, \beta \in \mathbb{R}_+ = (0, +\infty)$ be constants such that $0 < \alpha < \beta^{-1}$. We consider the following subset of $\mathbb{R}^{d\times d}$:

$$\mathcal{M}_{\alpha,\beta} = \{M \in \mathbb{R}^{d\times d} : \ \alpha|y|^2 \leqslant \langle My|y\rangle_{\mathbb{R}^d}, \ \langle M^{-1}y|y\rangle_{\mathbb{R}^d} \geqslant \beta|y|^2 \quad \forall y \in \mathbb{R}^d\}.$$

The structures of composite materials in a domain $\Omega \subset \mathbb{R}^d$ will be represented in this lecture by measurable matrix-valued functions $A : \Omega \to \mathcal{M}_{\alpha,\beta}$. The family of all such matrix-valued functions functions (which also are called often material parameters) is denoted by $L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$.

Let α', β' be certain constants such that $0 < \alpha' \leqslant \alpha \leqslant \beta^{-1} \leqslant (\beta')^{-1}$.

Definition 1.2 (H-convergence [MT78], see also [A02])**.**
A sequence $\{A_n(\cdot)\}_{n\in\mathbb{N}} \subset L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$ is said to H-converge to an H-limit

$$A_*(\cdot) \in L^\infty(\Omega, \mathcal{M}_{\alpha',\beta'})$$

if, for any $f(\cdot) \in H^{-1}(\Omega)$, the weak solutions $v_n(\cdot)$ to

$$\begin{aligned} -\nabla \cdot (A_n \nabla v) &= f, \quad x \in \Omega \\ v(x) &= 0, \quad x \in \partial\Omega \end{aligned}$$

satisfy

$$v_n \rightharpoonup v \text{ in } H_0^1(\Omega), \qquad A_n \nabla v_n \rightharpoonup A_* \nabla v_* \text{ in } L^2(\Omega, \mathbb{R}^d),$$

where $v_*(\cdot)$ is the weak solution to the problem

$$\begin{aligned} -\nabla \cdot (A_* \nabla v) &= f, \quad x \in \Omega \\ v(x) &= 0, \quad x \in \partial\Omega \end{aligned}$$

Remark 1.5.

(a) Here an H-limit $A_*(\cdot)$ is also called homogenized limit (in the sense of Murat-Tartar). The corresponding notation is $A_n \xrightarrow{\text{H}} A_*$.

(b) If $A_n \xrightarrow{\text{H}} A_*$, then $A_* \in L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$.

(c) It is known that an H-limit is unique if it exists.

(d) The equation

$$-\nabla \cdot (A_* \nabla v) = f$$

is called homogenized equation. For any $f(\cdot) \in H^{-1}(\Omega)$ and $A(\cdot) \in L^\infty(\Omega, \mathcal{M}_{\alpha',\beta'})$, the problem

$$\begin{aligned} -\nabla \cdot (A \nabla v) &= f, \quad x \in \Omega \\ v(x) &= 0, \quad x \in \partial\Omega \end{aligned}$$

has a unique weak solution $v \in H_0^1(\Omega)$.

(e) *The Hilbert spaces* $H^1(\Omega)$, $H_0^1(\Omega)$, $H^{-1}(\Omega)$ *are Sobolev spaces "with* $p = 2$*", i.e.* $H^1(\Omega) = W^{1,2}(\Omega)$, $H_0^1(\Omega)$ *is the closure of of the space of test functions* $C_0^\infty(\Omega)$ *in* $H^1(\Omega)$, *the space* $H^{-1}(\Omega) = W^{-1,2}(\Omega)$ *can be defined as the dual space of* $H_0^1(\Omega)$ *(see [B68, GT77, B11]). Where necessary, we discuss in the course the basics of the Sobolev spaces and the theory of weak solutions.*

(f) *The notation* $g_n \rightharpoonup g$ *denotes the weak convergence in the corresponding space.*

1.4 H-closure and the existence of optimizers.

Consider a feasible family (of structures) $\mathbb{F} \subset L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$.

Definition 1.3 (H-closure)**.**
The H-closure $\overline{\mathbb{F}}^{\mathrm{H}}$ of $\mathbb{F}$ is the closure of $\mathbb{F}$ with respect to (w.r.t) the H-convergence.

Consider certain constants $\alpha_1, \alpha_2, \gamma > 0$ such that $0 < \alpha_1 < \alpha_2$ and $0 < \gamma < \mathrm{meas}(\Omega)$, and again consider the feasible family with a prescribed ratio of the materials

$$\mathbb{F}_\gamma = \mathbb{F}_\gamma^{\alpha_1,\alpha_2,\Omega} := \{\chi_\omega(\cdot)\alpha_1 I_{\mathbb{R}^d} + \chi_{\Omega\setminus\omega}(\cdot)\alpha_2 I_{\mathbb{R}^d} : \ \mathrm{meas}(\omega) = \gamma\}. \tag{1.7}$$

Here we identify the $\mathbb{R}$-valued structure-function $a(\cdot) = \chi_\omega(\cdot)\alpha_1 + \chi_{\Omega\setminus\omega}(\cdot)\alpha_2$ with the matrix-valued structure-function $A(\cdot) = a(\cdot)I_{\mathbb{R}^d} = \chi_\omega(\cdot)\alpha_1 I_{\mathbb{R}^d} + \chi_{\Omega\setminus\omega}(\cdot)\alpha_2 I_{\mathbb{R}^d}$, which takes only scalar matrices as its values.

In this course we will consider the important question of how to describe the H-closure $\overline{\mathbb{F}}_\gamma^{\mathrm{H}}$ of $\mathbb{F}_\gamma$. The answer is given by the G-closure theorem obtained independently by Lurie & Cherkaev and Murat & Tartar, see [MT85, C00, A02]. Our main application of $\overline{\mathbb{F}}_\gamma^{\mathrm{H}}$ is the following theorem.

Theorem 1.2 ([CL96])**.**
For all $k \in \mathbb{N}$,

$$\inf_{a\in\mathbb{F}_\gamma} \lambda_k(a) = \min_{A\in\overline{\mathbb{F}}_\gamma^{\mathrm{H}}} \lambda_k(A) \quad \text{and} \quad \sup_{a\in\mathbb{F}_\gamma} \lambda_k(a) = \max_{A\in\overline{\mathbb{F}}_\gamma^{\mathrm{H}}} \lambda_k(A).$$

Remark 1.6.
Theorem 1.2 implies in particular

$$\arg\min_{A\in\overline{\mathbb{F}}_\gamma^{\mathrm{H}}} \lambda_k(A) \neq \varnothing, \qquad \arg\max_{A\in\overline{\mathbb{F}}_\gamma^{\mathrm{H}}} \lambda_k(A) \neq \varnothing,$$

i.e., there exists at least one minimizer $A_k^{\min} \in \overline{\mathbb{F}}_\gamma^{\mathrm{H}}$ *and at least one maximizer* $A_k^{\max} \in \overline{\mathbb{F}}_\gamma^{\mathrm{H}}$ *for* $\lambda_k : L^\infty(\Omega, \mathcal{M}_{\alpha,\beta}) \to \mathbb{R}$. *Besides, for the infimum and the supremum over the family* $\mathbb{F}_\gamma$, *one has*

$$\inf_{a\in\mathbb{F}_\gamma} \lambda_k(a) = \lambda_k(A_k^{\min}), \qquad \sup_{a\in\mathbb{F}_\gamma} \lambda_k(a) = \lambda_k(A_k^{\max}).$$

1.5 Applications of eigenvalue optimization to wave equations.

A simple model of an optical cavity can be obtained if one takes two of the linear Maxwell(-Heaviside) equations

$$\partial_t \mathbf{E}(x,t) = \frac{1}{\varepsilon(x)} \nabla \times \mathbf{H}(x,t), \tag{1.8}$$

$$\partial_t \mathbf{H}(x,t) = -\frac{1}{\mu(x)} \nabla \times \mathbf{E}(x,t), \tag{1.9}$$

and makes the time-harmonic substitutions $\mathbf{E}(x,t) = e^{-i\varkappa t}\mathbf{E}(x)$, $\mathbf{H}(x,t) = e^{-i\varkappa t}\mathbf{H}(x)$. Together with the simplified perfect metal boundary condition

$$\boldsymbol{n}(x) \times \mathbf{E}(x) = 0, \qquad x \in \partial\Omega,$$

this leads to the eigenproblem

$$\begin{pmatrix} 0 & \frac{1}{\varepsilon(x)} \nabla \times \\ -\frac{1}{\mu(x)} \nabla \times & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \varkappa \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \qquad x \in \partial\Omega_3.$$

Here $\Omega_3 \subset \mathbb{R}^3$ is a domain with a sufficiently regular boundary $\partial\Omega_3$ (e.g., with a Lipschitz boundary) such that the outer normal vector-field $\boldsymbol{n} : \partial\Omega_3 \to \mathbb{R}^3$ is well-defined (e.g., as an $L^\infty(\partial\Omega, \mathbb{R}^3)$-vector field).

Under certain physically reasonable simplifying assumptions it is possible to connect this 3-D Maxwell eigenproblem with several 2-D and 1-D dimensionally reduced eigenproblems [BCNS12, ACL18]. One of such related equations after a 2-D reduction is the 2-nd order elliptic equation of section 1.1

$$-\nabla \cdot (a(x_1,x_2) \nabla u(x_1,x_2)) = \lambda u(x_1,x_2), \qquad x' = (x_1,x_2) \in \Omega_2 \subset \mathbb{R}^2, \tag{1.10}$$

where $u(x') = H_3(x')$, $\lambda = \varkappa^2$. The coefficient $a(\cdot)$ is constructed from $\mathbb{R}_+$-valued dielectric permittivity $\varepsilon(\cdot)$. The perfect metal boundary condition transforms into the Neumann boundary condition

$$(a \nabla u) \cdot \boldsymbol{n'} = 0, \quad x' \in \partial\Omega_2, \tag{1.11}$$

where $\boldsymbol{n'} : \partial\Omega_2 \to \mathbb{R}^2$ is the outer normal vector-field on $\partial\Omega_2$.

The corresponding evolution wave equation is

$$\partial_t^2 u(x',t) = \nabla \cdot (a(x') \nabla u(x',t)), \quad x' \in \Omega_2, t > 0.$$

The meaning of eigenvalues λ_k of (1.10)-(1.11) is that $\varkappa_k^\pm = \pm\sqrt{\lambda_k}$ are the frequencies of eigen-oscillations of the EM-field in the 2-D optical cavity Ω_2.

Consider now the case when a homogeneous cavity filled with one material has small impurities consisting of another material, and the corresponding structure is given by the coefficient $\widetilde{a}(\cdot)$.
The relevant feasible families $\mathbb{F}$ are

$$\mathbb{F}_{\gamma+} = \mathbb{F}_{\gamma+}^{\alpha_1,\alpha_2,\Omega_2} := \{a(\cdot) = \alpha_1 \chi_\omega(\cdot) + \alpha_2 \chi_{\Omega\setminus\omega}(\cdot) : \ \gamma \leqslant \text{meas}(\omega) \leqslant \text{meas}(\Omega)\}$$

with γ close to meas(Ω), and

$$\mathbb{F}_{\gamma-} = \mathbb{F}_{\gamma-}^{\alpha_1,\alpha_2,\Omega_2} = \{a(\cdot) = \alpha_1 \chi_\omega(\cdot) + \alpha_2 \chi_{\Omega\backslash\omega}(\cdot) : \ 0 \leqslant \mathrm{meas}(\omega) \leqslant \gamma\}$$

with small $\gamma > 0$.

The values of $\inf\limits_{a \in \mathbb{F}} \varkappa_k^+(a)$ and $\sup\limits_{a \in \mathbb{F}} \varkappa_k^+(a)$ have now the physical meaning of the bounds on the k-th eigen-frequency $\varkappa_k^+(\widetilde{a}(\cdot))$. Hence, the optimization problems

$$\arg\min_{a \in \overline{\mathbb{F}}^{\mathrm{H}}} \lambda_k(a), \qquad \arg\max_{a \in \overline{\mathbb{F}}^{\mathrm{H}}} \lambda_k(a)$$

for $\mathbb{F} = \mathbb{F}_{\gamma\pm}$ become meaningful for all $k \geqslant 2$. (What happens with $k = 1$ in the case of Neumann boundary condition?) These optimization problems are closely related to the problems

$$\arg\min_{a \in \overline{\mathbb{F}}_\gamma^{\mathrm{H}}} \lambda_k(a) \ \text{ and } \ \arg\max_{a \in \overline{\mathbb{F}}_\gamma^{\mathrm{H}}} \lambda_k(a).$$

2 Convergence of eigenvalues and existence of optimizers.

2.1 Dirichlet eigenvalues of 2-nd order elliptic operators.

Let $\Omega \subset \mathbb{R}^d$ be a domain. Let $\alpha, \beta \in \mathbb{R}_+ = (0, +\infty)$ be constants such that $0 < \alpha < \beta^{-1}$. Let

$$\mathbb{R}^{d\times d}_{\mathrm{sym}} = \{M \in \mathbb{R}^{d\times d} : M^{i,j} = M^{j,i} \quad \forall i, j\}$$

be the real linear space of symmetric matrices $M = M^\top$. By

$$\mathcal{M}^{\mathrm{sym}}_{\alpha,\beta} := \{M \in \mathbb{R}^{d\times d}_{\mathrm{sym}} : \ \alpha |y|^2 \leqslant \langle My|y\rangle_{\mathbb{R}^d}, \ \langle M^{-1}y|y\rangle_{\mathbb{R}^d} \geqslant \beta |y|^2 \quad \forall y \in \mathbb{R}^d\}$$

we denote the set of all symmetric real $d \times d$-matrices in

$$\mathcal{M}_{\alpha,\beta} := \{M \in \mathbb{R}^{d\times d} : \ \alpha |y|^2 \leqslant \langle My|y\rangle_{\mathbb{R}^d}, \ \langle M^{-1}y|y\rangle_{\mathbb{R}^d} \geqslant \beta |y|^2 \quad \forall y \in \mathbb{R}^d\}.$$

Then $L^\infty(\Omega, \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta})$ is the family of all measurable matrix-functions $A : \Omega \to \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta}$.

Let $\rho_-, \rho_+ \in \mathbb{R}_+$ be certain constants such that $0 < \rho_- \leqslant \rho_+$. We take $\rho(\cdot) \in L^\infty(\Omega, [\rho_-, \rho_+])$, i.e., the function $\rho : \Omega \to [\rho_-, \rho_+]$ is measurable.

Let $C_0^\infty(\Omega)$ be the space of test functions in Ω, i.e., the space of C^∞-functions $f : \Omega \to \mathbb{R}$ such that $\operatorname{supp} f := \overline{\{x \in \Omega : f(x) \neq 0\}}$ is a compact subset of Ω. The Hilbert space $H_0^1(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in the Sobolev space $H^1(\Omega) = W^{1,2}(\Omega)$.

Let $A \in L^\infty(\Omega, \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta})$ and $\rho \in L^\infty(\Omega, [\rho_-, \rho_+])$. The eigenproblem

$$-\nabla \cdot (A \nabla u) = \lambda \rho u, \qquad x \in \Omega, \tag{2.1}$$

$$u(x) = 0, \qquad x \in \partial\Omega, \tag{2.2}$$

is understood in the following sense: Find all pairs $\{\lambda, u(\cdot)\}$ with $\lambda \in \mathbb{R}$ (called eigenvalue) and $u \in H_0^1(\Omega)$ (called eigenfunction) such that (2.1) is valid in the sense of distributions and $u(\cdot) \not\equiv \mathbb{0}$ in $H_0^1(\Omega)$ (or equivalently $u(\cdot) \not\equiv \mathbb{0}$ in $L^2(\Omega)$), where $\mathbb{0}$ is the constant function equal to 0 everywhere.

Theorem 2.1 (spectral theorem for symmetric 2nd order Dirichlet problems).
Let $A \in L^\infty(\Omega, \mathcal{M}^{\text{sym}}_{\alpha,\beta})$ and $\rho \in L^\infty(\Omega, [\rho_-, \rho_+])$.

(a) The set $\Sigma = \Sigma(A, \rho)$ of eigenvalues of (2.1)-(2.2) can be represented as

$$\Sigma = \{\lambda_k\}_{k\in\mathbb{N}} = \{\lambda_k(A, \rho)\}_{k\in\mathbb{N}}$$

with a nondecreasing sequence

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \dots, \qquad \textit{such that } \lim_{k\to\infty} \lambda_k = +\infty.$$

(b) The eigenvalues can be numbered such that every eigenvalue is repeated according to its geometric multiplicity.

(c) Under convention (b), the corresponding eigenfunctions $u_k(\cdot)$ can be chosen such that $\{u_k\}_{k\in\mathbb{N}}$ is an orthonormal basis in the Hilbert space $L^2_\rho(\Omega, \mathbb{C})$.

The space $L^2_\rho(\Omega, \mathbb{C})$ is the Hilbert space of measurable $\mathbb{C}$-valued functions f such that $\langle f|f\rangle_{L^2_\rho} < +\infty$, where $\langle f|g\rangle_{L^2_\rho} := \int\limits_\Omega f(x)\overline{g(x)}\rho(x)\mathrm{d}x$ is the inner product in $L^2_\rho(\Omega, \mathbb{C})$. The real Hilbert space $L^2_\rho(\Omega)$ is the space of real-valued $L^2_\rho(\Omega, \mathbb{C})$-functions.

In what follows, we work under the conventions (a)–(b) on the numbering of λ_k.

Remark 2.1.
We do not give the proof of Theorem 2.1 in the course (for the proof, see [GT77]).

2.2 Spectral convergence under the homogenization.

Definition 2.1.
A vector v in a normed space V is called V-normalized if $\|v\|_V = 1$.

Definition 2.2 (reminder, dual space and weak-* convergence, e.g., [RS12, K13]).
Let $\mathbb{K} = \mathbb{R}$ oder $\mathbb{K} = \mathbb{C}$. Let V be a normed space over $\mathbb{K}$.

(a) The dual space V' is the linear space of continuous linear functionals $\varphi : V \to \mathbb{K}$ equipped with the operator norm.

(b) A sequence of functionals $\{\varphi_n\}_{n\in\mathbb{N}} \subset V'$ is weak-* convergent to $\varphi \in V'$ (the notation is $\varphi_n \xrightarrow{*} \varphi$) if $\forall v \in V$ $\langle v, \varphi_n\rangle := \varphi_n(v)$ converges to $\langle v, \varphi\rangle := \varphi(v)$.

Remark 2.2.

(a) The dual space V' is always complete, i.e., it is always a Banach space. For example, $(L^1(\Omega))'$ can be naturally identified with $L^\infty(\Omega)$. So, a sequence $\{\rho_n\} \subset L^\infty(\Omega)$ is weak- convergent exactly when there exists $\rho \in L^\infty(\Omega)$ such that*

$$\int\limits_\Omega v\rho_n \mathrm{d}x \to \int\limits_\Omega v\rho \mathrm{d}x \ \forall v \in L^1(\Omega).$$

(b) In a Hilbert space the weak convergence and weak- convergence coincide. (Why?)*

Lemma 2.1 ([A02]).

(a) Let $\{\rho_n\}_{n\in\mathbb{N}} \subset L^\infty(\Omega, [\rho_-, \rho_+])$ *be such that* $\rho_n \xrightarrow{*} \rho$ *in* $L^\infty(\Omega)$. *Then* $\rho \in L^\infty(\Omega, [\rho_-, \rho_+])$.

(b) Let $\{A_n\}_{n\in\mathbb{N}} \subset L^\infty(\Omega, \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta})$ *be such that* $A_n \xrightarrow{\mathrm{H}} A$. *Then* $A \in L^\infty(\Omega, \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta})$.

The proof will be given later.

Theorem 2.2 ([A02]).
Let $\{A_n\}_{n\in\mathbb{N}} \subset L^\infty(\Omega, \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta})$ *and* $\{\rho_n\}_{n\in\mathbb{N}} \subset L^\infty(\Omega, [\rho_-, \rho_+])$ *be such that* $A_n \xrightarrow{\mathrm{H}} A$ *and* $\rho_n \xrightarrow{*} \rho$ *in* $L^\infty(\Omega)$. *Then:*

(a) $\lambda_k(A_n, \rho_n) \to \lambda_k(A, \rho)$ *as* $n \to \infty \quad \forall k \in \mathbb{N}$.

(b) Let $k \in \mathbb{N}$. *For each* n, *let* u_k^n *be a certain* L^2*-normalized eigenfunctions that solve*

$$-\nabla \cdot (A_n \nabla u_k^n) = \lambda_k(A_n, \rho_n)\rho_n u_k^n, \qquad x \in \Omega,$$
$$u_k^n(x) = 0, \qquad x \in \partial\Omega.$$

Then there exists a subsequence $\{u_k^{n_j}\}_{j\in\mathbb{N}}$ *of* $\{u_k^n\}_{n\in\mathbb{N}}$ *and a* L^2*-normalized eigenfunction* u_k *solving*

$$-\nabla \cdot (A \nabla u_k) = \lambda_k(A, \rho)\rho u_k, \qquad x \in \Omega,$$
$$u_k(x) = 0, \qquad x \in \partial\Omega,$$

such that $u_k^{n_j} \rightharpoonup u_k$ *in* $H_0^1(\Omega)$ *as* $j \to \infty$.

The proof will be given later in Section 3

2.3 H-compactness, weak-*-compactness, and metrizability.

Theorem 2.3 (sequential H-compactness of $L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$, [MT78], see also [A02]).
For every sequence $\{A_n\}_{n\in\mathbb{N}} \subset L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$, *there exist a subsequence* $\{A_{n_j}\}_{j\in\mathbb{N}}$ *and a matrix-valued function* $A \in L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$ *such that* $A_{n_j} \xrightarrow{\mathrm{H}} A$ *as* $j \to \infty$.

Theorem 2.4 (reminder, sequential Banach-Alaoglu Theorem, e.g., [RS12]).
Let V *be a separable normed space. Then the closed unit ball* $\overline{\mathbb{B}_1(0_{V'}; V')}$ *in* V' *is sequentially compact w.r.t. the weak-* convergence.*

This is a standard theorem in Functional Analisys courses, so we use it without proof.

Corollary 2.1.

(a) Every closed ball in $L^\infty(\Omega)$ *is sequentially compact w.r.t. weak-* convergence.*

(b) In particular, for every sequence $\{\rho_n\}_{n\in\mathbb{N}} \subset L^\infty(\Omega, [\rho_-, \rho_+])$, *there exist a subsequence* $\{\rho_{n_j}\}_{j\in\mathbb{N}}$ *and a function* $\rho \in L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$ *such that* $\rho_{n_j} \xrightarrow{*} \rho$ *as* $j \to \infty$.

Proof. (a) Since $L^\infty(\Omega) = (L^1(\Omega))'$, $\overline{\mathbb{B}_1(\mathbb{0}; L^\infty(\Omega))}$ is sequentially weak-* compact. Since every closed ball $\overline{\mathbb{B}_r(\rho_0(\cdot); L^\infty(\Omega))}$ in L^∞ can be obtained from $\overline{\mathbb{B}_1(\mathbb{0}; L^\infty(\Omega))}$ by a homothety (homogeneous dilation) and shift, $\overline{\mathbb{B}_r(\rho_0(\cdot); L^\infty(\Omega))}$ is also sequentially weak-* compact.

(b) follows from (a) and the fact that $L^\infty(\Omega, [\rho_-, \rho_+]$ is a closed ball in L^∞. □

Theorem 2.5 (metrizability)**.**

(a) For every closed ball $\overline{B}$ in $L^\infty(\Omega)$ there exists a metric (distance-function) $\mu_{w*}(\cdot,\cdot)$ on $\overline{B}$ such that the weak- convergence in $\overline{B}$ coincides with the convergence of the metric space $(\overline{B}, \mu_{w*})$.*

(b) There exists a metric $\mu_{\mathrm{H}}(\cdot,\cdot)$ on $L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$ such that the H-convergence in $L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$ coincides with the convergence of the metric space $(L^\infty(\Omega, \mathcal{M}_{\alpha,\beta}), \mu_{\mathrm{H}})$ (see [A02]).

Part (a) is sometimes covered by Functional Analysis courses, we give the sketch of the proof later in Section 3. We also will prove there part (b) of this theorem.

Corollary 2.2.

(a) If $A_n \xrightarrow{\mathrm{H}} A$ and $A_{n_j} \xrightarrow{\mathrm{H}} B$ for a certain subsequence $\{A_{h_j}\}_{j\in\mathbb{N}}$, then $A = B$ almost everywhere in Ω.

(b) In particular, an H-limit is unique if it exists.

(c) The metric space $(L^\infty(\Omega, \mathcal{M}_{\alpha,\beta}), \mu_{\mathrm{H}})$ is compact.

Proof. Since the H-convergence is actually a convergence in a metric space, usual properties of general metric spaces imply (a) and (b). Statement (c) is a direct combination of Theorems 2.3 and 2.5. □

Remark 2.3.
Consider on $L^\infty(\Omega, \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta}) \times L^\infty(\Omega, [\rho_-, \rho_+]$ the metric

$$\mu_{\mathrm{H},w*}(\{A_1, \rho_1\}, \{A_2, \rho_2\}) := \mu_{\mathrm{H}}(A_1, A_2) + \mu_{w*}(\rho_1, \rho_2).$$

Then Theorem 2.2 implies that the nonlinear functionals $\lambda_k : \{A, \rho\} \mapsto \lambda_k(A, \rho)$ are continuous on the metric space

$$\left(L^\infty(\Omega, \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta}) \times L^\infty(\Omega, [\rho_-, \rho_+]),\ \mu_{\mathrm{H},w*}\right).$$

2.4 Existence of optimizers for functionals $\lambda_k(\cdot)$.

Let $\mathbb{F} \subseteq L^\infty(\Omega, \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta})$ be a feasible family of structures $A(\cdot)$. For $A \in L^\infty(\Omega, \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta})$, let $\lambda_k(A)$ be the k-th eigenvalue of

$$-\nabla \cdot (A \nabla u) = \lambda u, \qquad x \in \Omega,$$
$$u(x) = 0, \qquad x \in \partial\Omega$$

(here $\rho \equiv \mathbb{1}$). Recall that $\overline{\mathbb{F}}^{\mathrm{H}}$ is the H-closure of $\mathbb{F}$.

Theorem 2.6.
For all $k \in \mathbb{N}$,

$$\inf_{A\in\mathbb{F}} \lambda_k(A) = \min_{A\in\overline{\mathbb{F}}^{\mathrm{H}}} \lambda_k(A), \qquad \sup_{A\in\mathbb{F}} \lambda_k(A) = \max_{A\in\overline{\mathbb{F}}^{\mathrm{H}}} \lambda_k(A).$$

Proof. Step 1. As $\mathbb{F} \subseteq L^\infty(\Omega, \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta})$, we see from Lemma 2.1 (b), that

$$\overline{\mathbb{F}}^{\mathrm{H}} \subset L^\infty(\Omega, \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta}).$$

By Theorem 2.1, $\lambda_k(\cdot)$ is well-defined on $\overline{\mathbb{F}}^{\mathrm{H}}$.

Step 2. Lemma 2.1 (b) implies that $\overline{\mathbb{F}}^{\mathrm{H}}$ and $L^\infty(\Omega, \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta})$ are closed subsets of the compact metric space $(L^\infty(\Omega, \mathcal{M}_{\alpha,\beta}), \mu_{\mathrm{H}})$. Consequently, with the metric μ_{H}, $\overline{\mathbb{F}}$ and $L^\infty(\Omega, \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta})$ are compact metric space themselves.

Step 3. Since $\lambda_k : L^\infty(\Omega, \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta}) \to \mathbb{R}$ are continuous (nonlinear) functionals (see Remark 2.3), they are bounded and achieve their maxima and minima on compact sets, in particular, on $\overline{\mathbb{F}}$. Since $0 < \min_{A\in\overline{\mathbb{F}}^{\mathrm{H}}} \lambda_k(A)$ by Theorem 2.1, we conclude that

$$0 < \min_{A\in\overline{\mathbb{F}}^{\mathrm{H}}} \lambda_k(A) \leqslant \inf_{A\in\mathbb{F}} \lambda_k(A) \leqslant \sup_{A\in\mathbb{F}} \lambda_k(a) = \max_{A\in\overline{\mathbb{F}}^{\mathrm{H}}} \lambda_k(A) < +\infty.$$

Step 4. Let $A^{\min}$ be a minimizer for λ_k on $\overline{\mathbb{F}}^{\mathrm{H}}$. Then there exists $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathbb{F}$ such that $A_n \to A^{\min}$. From the continuity of $\lambda_k(\cdot)$, we have $\lambda_k(A_n) = \lambda_k(A^{\min}) = \min_{A\in\overline{\mathbb{F}}^{\mathrm{H}}} \lambda_k(A)$ and so $\min_{A\in\overline{\mathbb{F}}^{\mathrm{H}}} \lambda_k(A) = \inf_{A\in\mathbb{F}} \lambda_k(A)$. The proof of $\sup_{A\in\mathbb{F}} \lambda_k(a) = \max_{A\in\overline{\mathbb{F}}^{\mathrm{H}}} \lambda_k(A)$ is similar. □

Remark 2.4.
In the case of $\mathbb{F} = \mathbb{F}^{\alpha_1,\alpha_2,\Omega}_\gamma$ of the first chapter, Theorem 2.6 becomes Theorem 1.2, which is the starting point of [CL96].

2.5 General theorem on the existence of optimizers for $\lambda_k(A, \rho)$.

Now let the structure of a composite material be described by a pair $\{A(\cdot), \rho(\cdot)\}$ with $A \in L^\infty(\Omega, \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta})$ and $\rho \in L^\infty(\Omega, [\rho_-, \rho_+])$, where $0 < \rho_- \leqslant \rho_+$ as before. Let $\lambda_k(A, \rho)$ be the k-th eigenvalue of

$$\begin{aligned} -\nabla\cdot(A(x)\nabla u(x)) &= \lambda\rho(x)u(x), \qquad x \in \Omega, \\ u(x) &= 0, \qquad x \in \partial\Omega. \end{aligned}$$

Let μ_{w*} be a certain metric on $L^\infty(\Omega, [\rho_-, \rho_+])$ such that the corresponding convergence is the weak-* convergence.

By Theorem 2.2 and Remark 2.3, $L^\infty(\Omega, \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta}) \times L^\infty(\Omega, [\rho_-, \rho_+])$ becomes a metric space with the metric

$$\mu_{\mathrm{H},w*}(\{A_1, \rho_1\}, \{A_2, \rho_2\}) := \mu_{\mathrm{H}}(A_1, A_2) + \mu_{w*}(\rho_1, \rho_2).$$

Moreover, the functions $\lambda_k : L^\infty(\Omega, \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta}) \times L^\infty(\Omega, [\rho_-, \rho_+]) \to \mathbb{R}$ are continuous on this metric space.

Proposition 2.1.
The metric space $\left(L^\infty(\Omega, \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta}) \times L^\infty(\Omega, [\rho_-, \rho_+]), \mu_{\mathrm{H},w*}\right)$ *is compact.*

The proof follows easily from Theorems 2.3, 2.4, and 2.5 (exercise).

Theorem 2.7.
Let $\mathbb{F} \subseteq L^\infty(\Omega, \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta}) \times L^\infty(\Omega, [\rho_-, \rho_+])$ *be a feasible family of structures* $\{A, \rho\}$. *Let* $\overline{\mathbb{F}}$ *be the closure of* $\mathbb{F}$ *w.r.t. the metric* $\mu_{\mathrm{H},w*}$. *Then for all* $k \in \mathbb{N}$,

$$0 < \min_{\{A,\rho\}\in\overline{\mathbb{F}}} \lambda_k(A, \rho) = \inf_{\{A,\rho\}\in\mathbb{F}} \lambda_k(A, \rho) \leqslant \sup_{\{A,\rho\}\in\mathbb{F}} \lambda_k(A, \rho) = \max_{\{A,\rho\}\in\overline{\mathbb{F}}} \lambda_k(A, \rho).$$

The proof uses Proposition 2.1. In other points the proof is the same as the proof of Theorem 2.6.

2.6 Existence of optimal non-homogeneous membranes.

Consider another important particular case. One can fix $A \equiv I_{\mathbb{R}^d}$ and take a certain family $\mathcal{F} \subseteq L^\infty(\Omega, [\rho_-, \rho_+])$ of of weight-functions $\rho(\cdot)$.

Then the application of Theorem 2.7 to $\mathbb{F} = \{\{I_{\mathbb{R}^d}, \rho\} : \rho \in \mathcal{F}\}$ gives the existence of optimizers for the weighted Laplacian eigenproblem

$$\begin{aligned} -\Delta u(x) &= \lambda \rho(x) u(x), & x &\in \Omega, \\ u(x) &= 0, & x &\in \partial\Omega. \end{aligned}$$

This eigenvalue problem has a special applied name: vibrations of a non-homogeneous membrane. In the case $d = 2$, the domain Ω represents a membrane with possibly non-homogeneous (mass) density $\rho(x)$, $x \in \Omega$. Non-homogeneous means in this context non-constant. The membrane is firmly fixed at the boundary $\partial\Omega$, which corresponds to the Dirichlet boundary condition $u(x) = 0$, $x \in \partial\Omega$.

The elastic properties of the membrane are assumed to be isotropic and homogeneous, i.e., A is a constant scalar matrix $cI_{\mathbb{R}^d}$, $c \in \mathbb{R}_+$. By scaling we can reduce to the case $c = 1$. The eigenvalues $\lambda_k = \lambda_k(\rho)$ are squares ω^2 of frequencies ω of eigenoscillations.

In the case $d = 1$, this eigenproblem describes the eigenoscillations of a string.

Theorem 2.8.
Let $\mathcal{F} \subseteq L^\infty(\Omega, [\rho_-, \rho_+])$ *be a feasible family of densities* $\rho(\cdot)$. *Let* $\overline{\mathcal{F}}^{w*}$ *be the closure of* $\mathcal{F}$ *w.r.t. weak-* convergence. Then for all* $k \in \mathbb{N}$,

$$0 < \min_{\rho\in\overline{\mathcal{F}}^{w*}} \lambda_k(\rho) = \inf_{\rho\in\mathcal{F}} \lambda_k(\rho) \leqslant \sup_{\rho\in\mathcal{F}} \lambda_k(\rho) = \max_{\rho\in\overline{\mathcal{F}}^{w*}} \lambda_k(\rho).$$

Proof. The theorem is a direct corollary of Theorem 2.7 applied to $\mathbb{F} = \{\{I_{\mathbb{R}^d}, \rho\} : \rho \in \mathcal{F}\}$. □

Remark 2.5.
For the problems of type

$$-\Delta u(x) = \lambda \rho(x) u(x), \qquad x \in \Omega,$$

the appropriate homogenization convergence is the weak- convergence for* $\rho(\cdot)$. *Generally, different types of equations require different types of homogenization convergiencies.*

3 Properties of homogenization-convergencies.

3.1 Metrizability of weak-* convergences.

For more detailed theory of weak-* and weak convergences we refer to [KF67, DS88] (see also less detailed expositions in [RS12, K13]).

Let V be a normed space. Let V' be the space of continuous linear functionals on V. For $\varphi \in V'$ and $v \in V$, we use the notation $\langle v, \varphi \rangle = \varphi(v)$, where the bilinear form $\langle \cdot, \star \rangle$ on $V \times V'$ is called the pairing of V and V'.

Exercise 3.1.
A linear functional φ on V is continuous if and only if it is bounded, i.e., if and only if

$$\|\varphi\|_{V'} := \sup_{\|v\|_V \leqslant 1} |\langle v, \varphi \rangle| < +\infty.$$

The nonlinear functional $\| \cdot \|_{V'}$ is the norm in V', which makes V' a Banach space. The definition of $\| \cdot \|_{V'}$ implies

$$|\langle v, \varphi \rangle| \leqslant \|v\|_V \; \|\varphi\|_{V'}, \qquad \forall v \in V, \quad \varphi \in V'.$$

Exercise 3.2.

$$\|\varphi\|_{V'} = 0 \quad \Leftrightarrow \quad \langle v, \varphi \rangle = 0 \quad \forall v \in V \tag{3.1}$$

The linear functional with the property (3.1) is called zero-functional and is denoted by $0 = 0_{V'}$.

Exercise 3.3.
Let S be dense in V. Then

$$\varphi = 0_{V'} \quad \Leftrightarrow \quad \langle v, \varphi \rangle = 0 \quad \forall v \in S.$$

Exercise 3.4.

(a) $\mu_0(z_1, z_2) = \frac{|z_1 - z_2|}{1 + |z_1 - z_2|}$ is a metric on $\mathbb{C}$.

(b) Let $\mathbb{C}^N$ with $N \in \mathbb{N} \cup \{\infty\}$ be the linear space of all complex sequences $z = \{z_j\}_{n=1}^N$. In the case $N = \infty$ the standard notation for $\mathbb{C}^\infty$ is $\mathbb{C}^\mathbb{N}$. Then

$$\mu_N(y, z) = \sum_{n=1}^{N} 2^{-n} \frac{|y_n - z_n|}{1 + |y_n - z_n|}$$

is a metric on $\mathbb{C}^N$.

(c) The convergence in $(\mathbb{C}^N, \mu_N)$ is the componentwise convergense.

Assume now that the normed space V is separable, i.e., there exists a countable subset $S = \{v_n\}_{n\in\mathbb{N}} \subset V$ such that $\overline{S} = V$.

Theorem 3.1.
Let V be a separable normed space with a dense countable subset $S = \{v_n\}_{n\in\mathbb{N}}$. Then:

(a) The function

$$\mu_{w*}(\varphi_1, \varphi_2) = \sum_{n=1}^{\infty} 2^{-n} \frac{|\langle v_n, \varphi_1 - \varphi_2\rangle|}{1 + |\langle v_n, \varphi_1 - \varphi_2\rangle|}$$

is a metric on every closed ball $\overline{\mathbb{B}_r(\varphi_0)} = \overline{\mathbb{B}_r(\varphi_0; V')}$ in V'.

(b) The convergence of functionals $\varphi \in \overline{\mathbb{B}_r(\varphi_0)}$ w.r.t. the metric μ_{w*} is weak-* convergence.

(c) $(\overline{\mathbb{B}_r(\varphi_0)}, \mu_{w*})$ is a compact metric space.

Remark 3.1.
For metric spaces sequential compactness and their compactness (as topological spaces) are equivalent.

Proof of Theorem 3.1. (a) We only need to prove that

$$\mu_{w*}(\varphi_1, \varphi_2) = 0 \quad \Leftrightarrow \quad \varphi_1 = \varphi_2.$$

The other properties of a metric follow from Exercise 3.4 (b).

The implication “$\Leftarrow$” is obvious.

Assume that $\mu_{w*}(\varphi_1, \varphi_2) = 0$. Then $\langle v_n, \varphi_1 - \varphi_2\rangle = 0 \quad \forall v_n \in S$. Exercise 3.3 implies that $\varphi_1 - \varphi_2 = 0_{V'}$.

(b) By scaling, it is possible to reduce the statement to the case of the closed unit ball $\overline{\mathbb{B}_1(0_{V'})}$. Let us prove (b) for $\overline{\mathbb{B}_1(0_{V'})}$.

Assume $\varphi_k \xrightarrow{*} \varphi$ as $k \to \infty$, where $\varphi, \varphi_k \in \overline{\mathbb{B}_1(0_{V'})}$. The sequence $\{z^k\}_{k\in\mathbb{N}} \subset \mathbb{C}^\infty = \mathbb{C}^\mathbb{N}$ of $\mathbb{C}^\mathbb{N}$-sequences

$$z^k = (z^k_n)_{n\in\mathbb{N}} = (\langle v_n, \varphi_k - \varphi\rangle)_{n\in\mathbb{N}}$$

converges to the zero-sequence $0_{\mathbb{C}^\mathbb{N}}$ componentwise. Thus, Exercise 3.4 (c) implies

$$\mu_{w*}(\varphi_k, \varphi) \to 0.$$

Assume $\mu_{w*}(\varphi_k, \varphi) \to 0$, where $\varphi, \varphi_k \in \overline{\mathbb{B}_1(0_{V'})}$. By Exercise 3.4 (c), for all $v_n \in S$ we have $\lim\limits_{k\to\infty} \langle v_n, \varphi_k - \varphi\rangle = 0$.

Let $v \in V\backslash S$. Then, for every $\varepsilon > 0$, there exists $v_n \in S$ such that $\|v_n - v\|_V < \varepsilon/4$. Let us take $k_1 \in \mathbb{N}$ such that $|\langle v_n, \varphi_k - \varphi\rangle| < \epsilon/2$ for all $k \geqslant k_1$. Then,

$$|\langle v, \varphi_k - \varphi\rangle| \leqslant |\langle v - v_n, \varphi_k - \varphi\rangle| + |\langle v_n, \varphi_k - \varphi\rangle| < \frac{\varepsilon}{4}\|\varphi_k - \varphi\|_{V'} + \frac{\varepsilon}{2} < \varepsilon$$

for all $k \geqslant k_1$. Thus, $\langle v, \phi_k - \varphi \rangle \to 0$ for all $v \in V$.

(c) follows from the Banach-Alaouglu Theorem (Theorem 2.4). $\square$

3.2 Additional properties of weak-* and weak convergencies.

We always assume that a field $\mathbb{K}$ is $\mathbb{K} = \mathbb{R}$, or $\mathbb{K} = \mathbb{C}$.

Definition 3.1.
Let E be a subset of a linear space V over $\mathbb{K}$. The (linear) span of E, which is denoted by $\mathrm{span}(E)$, is the set of all finite linear combinations of vectors from E.

Let again V be a normed space.

Proposition 3.1.
Let $\overline{\mathrm{span}(E)} = V$. A sequence $\{\varphi_k\}_{k\in\mathbb{N}} \subset V'$ is weakly- convergent if and only if the two following conditions hold:*

(a) $\{\varphi_k\}_{k\in\mathbb{N}}$ is bounded,

(b) for a certain $\varphi \in V'$,

$$\lim_{k\to\infty} \langle v, \varphi_k - \varphi \rangle = 0 \qquad \text{for all } v \in E. \tag{3.2}$$

Remark 3.2.
A statement similar to Proposition 3.1 is valid also for the weak convergence in V with the analogous proof.

Proof of Proposition 3.1. Step 1. Proof of "only if". Let $\varphi_k \xrightarrow{*} \varphi$. Then the uniform boundedness principle implies (a). Statement (b) holds in a stronger form for all $v \in V$.

Step 2. Proof of "if". Assume that (a) and (b) holds. Then (3.2) for all $v \in E$ implies (3.2) for all $v \in \mathrm{span}(E)$. Let $\{\varphi_k\}_{k\in\mathbb{N}}$ be a bounded sequence in V'. Using $\overline{\mathrm{span}(E)} = V$ and the ε-type estimate in the proof of Theorem 3.1 (b), we obtain (3.2) for all $v \in V$, i.e., $\varphi_k \xrightarrow{*} \varphi$. $\square$

Let $\Omega \subset \mathbb{R}^d$ be a domain.

Remark 3.3.
Let $1 \leqslant q \leqslant +\infty$. Let E be the set of all indicator-functions χ_ω for measurable subsets ω of Ω. Then $E \subset L^q(\Omega)$ and $\overline{\mathrm{span}(E)} = L^q(\Omega)$ (where the closure is taken w.r.t. the strong convergence of the corresponding space $L^q(\Omega)$).

We see from Proposition 3.1 and Remark 3.3 that the weak-* convergence of sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ in $L^p(\Omega)$ with $1 < p \leqslant \infty$ is the combination of $\|\cdot\|_{L^p}$-boundedness with the convergence "in average". The convergence "in average" can be understood as

$$\int_\omega \varphi_k(x)\mathrm{d}x \to \int_\omega \varphi(x)\mathrm{d}x \qquad \forall \text{ measurable } \omega \subseteq \Omega.$$

Example 3.1.
The weak-* convergence of $\{\rho_k\}_{k\in\mathbb{N}} \subseteq L^\infty(\Omega, [\rho_-, \rho_+])$ is exactly convergence "in average" since $L^\infty(\Omega, [\rho_-, \rho_+])$ is bounded in $L^\infty(\Omega)$.

Remark 3.4.
Let $1 \leqslant q < \infty$. *Let* E *be the set of all indicator-functions* χ_ω *for open (or closed)* d*-dimensional cubes* $\omega \subset \Omega$ *with sides parallel to coordinate axes. Then* $\overline{\operatorname{span}(E)} = L^q(\Omega)$.

Exercise 3.5 ([A02]).
Let $u_n(x) = \sin(nx_1)$, where $x = (x_1, \ldots, x_d) \in \Omega$.

(a) Find the weak limit $\text{w-}\lim_{n\to\infty} u_n$ in $L^2(\Omega)$.

(b) Find $\lim_{n\to\infty} \|u_n\|_{L^2(\Omega)}$.

Remark 3.5.
In reflexive Banach spaces weak- topology (convergence) coincides with weak topology (resp., convergence). This is the case, e.g., for the spaces* $L^p(\Omega)$ *with* $1 < p < \infty$.

Example 3.2.
Any Hilbert space X over $\mathbb{K}$, e.g., $H^{-1}(\Omega)$, $L^2(\Omega)$, $H^1(\Omega)$, or $H^1_0(\Omega)$, is a reflexive Banach space.

Indeed, by the Riesz theorem, the paring $\langle v, \varphi\rangle$ of $v \in X$ with $\varphi \in X'$ can be identified with the inner product $\langle x|w_\varphi\rangle_X$, where the map $\varphi \mapsto w_\varphi$ is a bijective isometry from X' onto X (moreover, in the case $\mathbb{K} = \mathbb{R}$, this map is a isometric isomorphism).

3.3 The space of distributions $H^{-1}(\Omega)$.

The identification of $(H^1_0(\Omega))'$ with $H^1_0(\Omega)$ by Riesz's theorem, is not a unique reasonable way to produce all bounded linear functionals on $H^1_0(\Omega)$. We consider another useful approach to the space $(H^1_0(\Omega))'$.

Since Ω is a bounded open set, the following compact embedding holds

$$H^1_0(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega).$$

Presently, we need only a weaker statement that

$$\text{the continuous embedding} \quad H^1_0(\Omega) \hookrightarrow L^2(\Omega) \quad \text{holds.}$$

We need also the fact that $H^1_0(\Omega)$ is dense in $L^2(\Omega)$. The combination of such a density property and a continuous embedding will be called a dense continuous embedding with the notation $V_1 \overset{\mathrm{d}}{\hookrightarrow} V_2$. So,

$$H^1_0(\Omega) \overset{\mathrm{d}}{\hookrightarrow} L^2(\Omega).$$

The continuous embedding implies that for every $u \in L^2(\Omega)$,

$$\varphi_u(v) = \langle v, \varphi_u\rangle = \langle v|u\rangle_{L^2(\Omega)}$$

defines a bounded functional φ_u on $H_0^1(\Omega)$, i.e., $\varphi_u \in (H_0^1(\Omega))'$.

The set $\{\varphi_u : u \in L^2(\Omega)\}$ does not contain the whole $(H_0^1(\Omega))'$. Actually the space $L^2(\Omega)$ with the norm

$$\|u\|_{(H_0^1)'} = \|\varphi_u\|_{(H_0^1)'}$$

is not complete.

Definition 3.2 (space $H^{-1}(\Omega)$)**.**
The space $H^{-1}(\Omega, \mathbb{K})$ can be defined as a completion of $\Big(L^2(\Omega, \mathbb{K}), \|\cdot\|_{(H_0^1(\Omega,\mathbb{K}))'}\Big)$.

Remark 3.6.
Let us provide a rigorous interpretation to the statement $(H_0^1(\Omega))' = H^{-1}(\Omega)$.

(a) *The bounded bilinear form* $\langle v|u\rangle_{L^2}$ *on* $H_0^1(\Omega) \times L^2(\Omega)$ *can be extended by continuity to the pairing* $\langle\cdot,\cdot\rangle = {}_{H_0^1}\langle\cdot,\cdot\rangle_{H^{-1}}$ *of* $H_0^1(\Omega)$ *and* $H^{-1}(\Omega)$. *This pairing is a bounded bilinear form on* $H_0^1(\Omega) \times H^{-1}(\Omega)$. *Now every* $\varphi \in (H_0^1(\Omega))'$ *has a unique isometric representation* $\varphi(v) = {}_{H_0^1}\langle v, u_\varphi\rangle_{H^{-1}}$ *with a certain distribution* $u_\varphi \in H^{-1}(\Omega)$.

(b) *A similar statement is valid for the complex spaces* $H_0^1(\Omega, \mathbb{C})$ *and* $H^{-1}(\Omega, \mathbb{C})$ *if one replaces everywhere "bilinear form" with "sesquilinear form".*

The triple $H_0^1(\Omega)$, $L^2(\Omega)$, $H^{-1}(\Omega)$ united by the two dense continuous embeddings

$$H_0^1(\Omega) \overset{\mathrm{d}}{\hookrightarrow} L^2(\Omega) \overset{\mathrm{d}}{\hookrightarrow} H^{-1}(\Omega)$$

is an example of a rigged Hilbert space. The space $L^2(\Omega)$ is called a pivot space of this triple because its inner product is used to build the pairing ${}_{H_0^1}\langle\cdot,\cdot\rangle_{H^{-1}}$ between the two other spaces. It is said that $H^{-1}(\Omega)$ is dual to $H_0^1(\Omega)$ w.r.t. the pivot space $L^2(\Omega)$ (or, in short, w.r.t. the L^2-paring).

3.4 Exercises for the construction of the space $H^{-1}(\Omega)$.

More detailed constructions for rigged Hilbert spaces, for the spaces with the norm of negative order of regularity, and, in particular, for the space $H^{-1}(\Omega)$, can be found in the monograph of Berezanskiĭ [B68].

Let $\Omega \subset \mathbb{R}^d$ again be a domain. For every $u \in L^2(\Omega)$, we consider $\varphi_u \in (H_0^1(\Omega))'$ defined by $\varphi_u = \langle v|u\rangle_{L^2}$.

Exercise 3.6.
Consider the map $F : L^2(\Omega) \to (H_0^1(\Omega))'$ defined by $F : u \mapsto \varphi_u$. Prove the following statements.

(a) F is injective.

(b) F is a bounded linear operator. Note that we are in the case $\mathbb{K} = \mathbb{R}$. (How to modify this statement for the similar map $F : L^2(\Omega, \mathbb{C}) \to (H_0^1(\Omega, \mathbb{C}))'$ in the complex spaces?)

(c) The operator F is not surjective.
Hints: The proof is by contradiction. Assume the surjectivity. Then (a), (b), and the

bounded inverse theorem imply that F is homeomorphism and $\|\cdot\|_{L^2}$ and $\|\cdot\|_{(H_0^1(\Omega))'}$ are equivalent norms in $L^2(\Omega)$. Here, $\|\cdot\|_{(H_0^1)'}$ is perceived as the norm induced by F in $L^2(\Omega)$, i.e.,

$$\|u\|_{(H_0^1)'} = \sup_{v \neq 0} \frac{\langle v|u\rangle_{L^2}}{\|v\|_{H_0^1}}. \tag{3.3}$$

Show that this equivalence of norms leads to a contradiction.

(d) The image $F(L^2(\Omega))$ is dense in $(H_0^1(\Omega,\mathbb{C}))'$.
Hints: The proof is by contradiction. Assume that $F(L^2(\Omega))$ is not dense. Then $\exists \varphi \in (H_0^1(\Omega,\mathbb{C}))' \backslash \{0\}$ such that $\varphi \perp F(L^2(\Omega))$ (i.e., φ is orthogonal to $F(L^2(\Omega))$). This leads to a contradiction (one can use, e.g., the identification of $(H_0^1(\Omega))'$ and $H_0^1(\Omega)$ via the Riesz theorem in $H_0^1(\Omega)$).

(e) The completion $H^{-1}(\Omega)$ of $(L^2(\Omega), \|\cdot\|_{(H_0^1)'})$ can be isometrically identified with $(H_0^1(\Omega))'$ via the pairing ${}_{H_0^1}\langle\cdot|\cdot\rangle_{H^{-1}}$ constructed in the end of section 3.3.
Hint: Use (d).

3.5 Metrizability of the H-convergence.

We have seen that for a dual V' of a separable normed space V we can metrize weak-* convergence in bounded subsets of V' with the use of an arbitrary countable dense subset $\{v_n\}_{n\in\mathbb{N}}$ of V. Namely, the construction of the corresponding metric on bounded subsets of V' was

$$\mu_{w*}(\varphi_1, \varphi_2) = \mu_{w*,V'}(\varphi_1, \varphi_2) = \sum_{n=1}^{\infty} 2^{-n} \frac{|\langle v_n, \varphi_1 - \varphi_2\rangle|}{1 + |\langle v_n, \varphi_1 - \varphi_2\rangle|}. \tag{3.4}$$

We did this construction on closed balls. However it obviously generates weak-* convergence on every bounded subset. Concerning the metric μ_{w*} on the whole V', see the following exercises.

Exercise 3.7.
Note that μ_{w*} is a metric on the the whole V'. However, generally, the convergence w.r.t. μ_{w*} on the whole V' is not equivalent to weak-* convergence in V'. Namely, weak-* convergence always implies μ_{w*}-convergence. However, there are separable normed spaces V such that there exists a μ_{w*}-convergent sequence in V' that is not weak-* convergent. The understanding how and where this happens is the aim of this exercise.

(a) Let $V = V' = \ell^2$. Find a countable dense subset $\{v_n\}_{n\in\mathbb{N}} \subset \ell^2$ and a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset \ell^2$ such that $\mu_{w,\ell^2}(\varphi_n, 0_{\ell^2}) \to 0$ for the metric defined similarly to (3.4) and

$$\|\varphi_n\|_{\ell^2} \to \infty \qquad \text{as } n \to \infty. \tag{3.5}$$

So, the sequence $\{v_n\}_{n\in\mathbb{N}}$ converges to 0_{ℓ^2} in the metric space (ℓ^2, μ_{w,ℓ^2}). However, due to (3.5), this sequence cannot be weakly convergent.

(b) For an arbitrary countable dense $\{v_n\}_{n\in\mathbb{N}} \subset \ell^2$, show that there exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ with properties as in (a).

Exercise 3.8.
Characterize all separable normed spaces V such that the convergence in (V', μ_{w*}) is equivalent to weak-* convergence in V'.

If V is a reflexive Banach space with a separable dual V', then $V = V''$, and we denote by $\mu_w = \mu_{w,V}$ a certain metric on bounded subsets of V that is defined by a formula similar to (3.4) in such a way that μ_w-convergence is weak-convergence in every bounded set of V. That is, for $\mu_{w,V}$, we take a countable dense subset $\{v_n\}_{n\in\mathbb{N}}$ of V', and define by (3.4) the metric $\mu_{w,V}(\varphi_1, \varphi_2)$ for $\varphi_1, \varphi_2 \in V$.

Remark 3.7.
The Hilbert spaces $H^{-1}(\Omega)$, $L^2(\Omega)$, $H^1(\Omega)$, and $H_0^1(\Omega)$ are separable. Hence, we can define similarly to (3.4) metrics that generate weak convergence on their bounded subsets. In particular we can fix two such metrics μ_{w,H_0^1} and μ_{w,L^2} for the spaces $H_0^1(\Omega)$ and $L^2(\Omega)$.

Theorem 3.2 (metrizability of H-convergence)**.**
Let $\{f_n\}_{n\in\mathbb{N}}$ be a countable dense subset of $H^{-1}(\Omega)$. For $A \in L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$, let us define $u_n^A(\cdot)$ as the weak solution to

$$\begin{aligned} -\nabla\cdot(A\nabla u) &= f_n, \quad x \in \Omega \\ u(x) &= 0, \quad x \in \partial\Omega. \end{aligned}$$

Let us consider the function defined for $\{A, B\} \in L^\infty(\Omega, \mathcal{M}_{\alpha,\beta}) \times L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$ by

$$\mu_{\mathrm{H}}(A, B) = \sum_{n=1}^{\infty} 2^{-n} \frac{\mu_{w,H_0^1}(u_n^A, u_u^B) + \mu_{w,L^2}(A\nabla u_n^A, B\nabla u_n^B)}{1 + \mu_{w,H_0^1}(u_n^A, u_u^B) + \mu_{w,L^2}(A\nabla u_n^A, B\nabla u_n^B)}.$$

Then:

(a) $(L^\infty(\Omega, \mathcal{M}_{\alpha,\beta}), \mu_{\mathrm{H}})$ is a metric space.

(b) The convergence defined on $L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$ by the metric μ_{H} is the H-convergence.

We need some preparational results in order to prove this theorem. The metric μ_{H} defined here is different from the metric introduced in [A02] in order to metrize the H-convergence, but the idea is essentially the same.

3.6 Solvability of the Dirichlet problem.

Again we have $\alpha, \beta \in \mathbb{R}_+ = (0, +\infty)$ constants such that $0 < \alpha < \beta^{-1}$ and

$$\mathcal{M}_{\alpha,\beta} := \{M \in \mathbb{R}^{d\times d} : \ \alpha|y|^2 \leqslant \langle My|y\rangle_{\mathbb{R}^d}, \ \langle M^{-1}y|y\rangle_{\mathbb{R}^d} \geqslant \beta|y|^2 \quad \forall y \in \mathbb{R}^d\}.$$

Let $A \in L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$ and $f \in H^{-1}(\Omega)$.

Recall that a function $u \in H_0^1(\Omega)$ is called a weak solution to the problem

$$-\nabla\cdot(A\nabla u) = f, \quad x \in \Omega \tag{3.6}$$
$$u(x) = 0, \quad x \in \partial\Omega \tag{3.7}$$

if
$$\int_\Omega (A\nabla u)\cdot(\nabla v)\mathrm{d}x = {}_{H^{-1}}\langle f, v\rangle_{H_0^1} \qquad \forall v \in H_0^1(\Omega).$$
We use here ${}_{H^{-1}}\langle f, v\rangle_{H_0^1} = {}_{H_0^1}\langle v, f\rangle_{H^{-1}}$.

Definition 3.3.

(a) A bilinear form $b(\cdot,\cdot)$ on a normed space V is called bounded if

$$|b(u,v)| \lesssim \|u\|_V\|v\|_V \qquad \forall u, v \in V,$$

i.e., if

$$|b(u,v)| \leqslant C\|u\|_V\|v\|_V \qquad \forall u, v \in V \tag{3.8}$$

with a certain constant $C > 0$.

(b) A bilinear form $b(\cdot,\cdot)$ on a normed space V is called coercive if

$$\|v\|_V^2 \lesssim |b(v,v)| \qquad \forall v \in V,$$

i.e., if

$$\gamma\|v\|_V^2 \leqslant |b(u,v)| \qquad \forall v \in V \tag{3.9}$$

with a certain constant $\gamma > 0$.

Example 3.3.
Let the bilinear form $b : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ be defined by $b(u,v) = \int_\Omega (\nabla u)\cdot(\nabla v)\mathrm{d}x$.

(a) The Poincare inequality on $H_0^1(\Omega)$ states that b is coercive on $H_0^1(\Omega)$.

(b) Obviously, b is not coercive on $H^1(\Omega)$.

(c) It is easy to see that b is a bounded bilinear form on $H^1(\Omega)$.

(d) The formula

$$|v|_{H_0^1} = \left(\int_\Omega |\nabla v|^2\mathrm{d}x\right)^{1/2}$$

defines an equivalent norm in $H_0^1(\Omega)$. This follows from (a).

Theorem 3.3 (Lax-Milgram lemma)**.**
Let $b : V \times V$ be a bounded coercive bilinear form on a Hilbert space V. Then for every $\varphi \in V'$ the variational problem

$$b(u,v) = \varphi(v) \quad \forall v \in V$$

has a unique solution $u \in V$. Moreover,

$$\|u\|_V \leqslant \frac{1}{\gamma}\|\varphi\|_{V'},$$

where γ is the constant from (3.9).

This theorem we do not prove. The proof can be obtained, e.g., by a very minor modification from the proof of the Lax-Milgram lemma for $\mathbb{K} = \mathbb{R}$ in [GT77].

Theorem 3.4 (solvability of the Dirichlet boundary value problem (BVP))**.**
Let $A \in L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$. Then:

(a) The bilinear form

$$b_A(u, v) = \int_\Omega (A \nabla u) \cdot (\nabla v) \mathrm{d}x,$$

is bounded and coercive on $H_0^1(\Omega)$ with the estimate

$$\alpha |v|_{H_0^1} \leqslant b_A(v, v) \quad \forall v \in H_0^1(\Omega).$$

(b) For every $f \in H^{-1}(\Omega)$, the Dirichlet problem (3.6)-(3.7) *has a unique weak solution $u \in H_0^1(\Omega)$. Moreover,*

$$|u|_{H_0^1} \leqslant \frac{1}{\alpha} |f|_{H^{-1}},$$

where

$$|f|_{H^{-1}} := \sup_{v \neq \mathbb{0}} \frac{|\ {}_{H^{-1}}\langle f, v \rangle_{H_0^1}\ |}{|v|_{H_0^1}}.$$

In other words, Theorem 3.4 considers $H_0^1(\Omega)$ as a Hilbert space with the inner product $(u|v)_{H_0^1} = \int_\Omega (\nabla u) \cdot (\nabla \overline{v}) \mathrm{d}x$ instead of the standard inner product. This inner product generates the norm of Example 3.3 (d).

Proof of Theorem 3.4. The coercivity in statement (a) follows from $A \in L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$, Example 6.1 (d), and the pointwise inequality

$$\alpha |y|^2 \leqslant \langle A(x) y | y \rangle_{\mathbb{R}^d}, \qquad y \in \mathbb{R}^d$$

which is valid for almost all $x \in \Omega$ (see the definition of the family $\mathcal{M}_{\alpha,\beta}$).

The boundedness in statement (a) follows from the 2nd inequality in the definition of $\mathcal{M}_{\alpha,\beta}$

$$\beta |y|^2 \leqslant \langle (A(x))^{-1} y | y \rangle_{\mathbb{R}^d}, \qquad y \in \mathbb{R}^d, \tag{3.10}$$

which is also valid for almost all $x \in \Omega$. Indeed, let us take $z = (A(x))^{-1} y$. Then (3.10) implies

$$\beta |A(x) z|^2 \leqslant \langle z | A(x) z \rangle_{\mathbb{R}^d} \leqslant |z|\ |A(x) z|,$$

where the Cauchy–Bunyakovsky–Schwarz inequality is used. So $|A(x)z| \leqslant \frac{1}{\beta}|z|$ and using the Cauchy–Bunyakovsky–Schwarz inequality again we get

$$\langle A(x) z | z \rangle_{\mathbb{R}^d} \leqslant |A(x) z|\ |z| \leqslant \frac{1}{\beta} |z|^2 \tag{3.11}$$

for all $z \in \mathbb{R}^d$ and for almost all $x \in \Omega$. This estimate implies the boundedness of $b_A(\cdot, \cdot)$.

Statement (b) follows from (a) and the Lax-Milgram lemma. □

Exercise 3.9.
Let $\varphi \in (H_0^1(\Omega))'$ be defined by $\varphi(v) = \langle v|w\rangle_{L^2}$ with $w \in L^2(\Omega)$ or, more generally, by $\varphi(v) = {}_{H_0^1}\langle v, w\rangle_{H^{-1}}$ with $w \in H^{-1}(\Omega)$.

(a) By Riesz's theorem there exist a unique $u \in H_0^1(\Omega)$ such that $\varphi(v) = (v|u)_{H_0^1}$ for all $v \in H_0^1(\Omega)$. How to find this u?

(b) Show that, for u from (a), we always have $(-\Delta)u = w$ in the sense of distributions. How the norms $|u|_{H_0^1}$ and $|w|_{H^{-1}}$ are connected?

(c) Prove that $f \in H^{-1}(\Omega)$ if and only if f is a distribution having a representation $f = \Delta v$ with a certain $v \in H_0^1(\Omega)$. Prove that, for this representation, $|v|_{H_0^1} = |f|_{H^{-1}}$.

(d) Show that $f \in H^{-1}(\Omega)$ if and only if f is a distribution with the property that there exists $g_j \in L^2(\Omega)$, $j = 1, \dots, d$, such that $f = \sum_{j=1}^{d} \frac{\partial g_j}{\partial x_j}$ in the sense of distributions. Let $g \in L^2(\Omega, \mathbb{R}^d)$ be defined by $g = (g_1, \dots, g_d)$. How are the norms $\|g\|_{L^2}$ and f connected?

Corollary 3.1.
The linear operator

$$L_A : H_0^1(\Omega) \to H^{-1}(\Omega)$$

defined by $L_A(v) = -\nabla \cdot (A \nabla v)$ is a homeomorphism. (Note that the differentiations here are understood in the sense of distributions).

Proof. Theorem 3.4 implies that the operator $M_A = L_A^{-1}$ is continuous from $H^{-1}(\Omega)$ to $H_0^1(\Omega)$. For every $v \in H_0^1(\Omega)$, we have $g = A \nabla v \in L^2(\Omega, \mathbb{R}^d)$. By Exercise 3.9 (d), the distribution $L_A v = \nabla \cdot g$ belongs to $H^{-1}(\Omega)$. Hence, $M_A = L_A^{-1}$ is surjective, and so bijective. The bounded inverse theorem completes the proof. ☐

3.7 Proof of Theorem 3.2 (on the metrizability of H-convergence).

We split the proof into several steps and start from the verification of the properties of a metric for μ_{H}.

Step 1. The symmetry property for μ_{H} is obvious. Let us prove the triangle inequality for μ_{H}. For each n and $A, B \in L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$, consider the function

$$\mu_n(A, B) = \mu_{w,H_0^1}(u_n^A, u_n^B) + \mu_{w,L^2}(A \nabla u_n^A, B \nabla u_n^B).$$

As we discussed before, $\mu_{w,V}$ is a metric on a normed space V (which induces weak convergence on bounded subsets of V). Hence,

$$0 \leqslant \mu_n(A, C) \leqslant \mu_n(A, B) + \mu_n(B, C), \qquad A, B, C \in L^\infty(\Omega, \mathcal{M}_{\alpha,\beta}).$$

Now the triangle inequality for μ_H follows from the following statement: if $a, b, c \in [0, +\infty)$ satisfy $0 \leqslant a \leqslant b + c$, then

$$\frac{a}{1+a} \leqslant \frac{b}{1+b} + \frac{c}{1+c}.$$

Step 2. It is obvious that $A = B$ almost everywhere in $\Omega \quad \Rightarrow \quad \mu_{\mathrm{H}}(A,B) = 0$. Let us prove that $\mu_{\mathrm{H}}(A,B) = 0 \quad \Rightarrow \quad A = B$ almost everywhere in Ω. Note that

$$\begin{aligned}\mu_{\mathrm{H}}(A,B) = 0 \quad &\Rightarrow \quad 0 = \mu_{w,H_0^1}(u_n^A, u_n^B) = \mu_{w,L^2}(A \nabla u_n^A, B \nabla u_n^B) \\ &\Rightarrow \quad u_n^A = u_n^B \text{ in } H_0^1(\Omega) \text{ and } A \nabla u_n^A = B \nabla u_n^B \text{ in } L^2(\Omega, \mathbb{R}^d) \text{ for all } n.\end{aligned}$$

Using the operators $M_A = L_A^{-1}$ from Corollary 3.1, we write these equalities as $M_A f_n = M_B f_n$ in $H_0^1(\Omega)$ and $A \nabla (M_A f_n) = B \nabla (M_B f_n)$ in $L^2(\Omega, \mathbb{R}^d)$ for the dense in $H^{-1}(\Omega)$ set $\{f_n\}_{n \in \mathbb{N}}$.

By Theorem 3.4, M_A and M_B are bounded and continuous as operators from $H^{-1}(\Omega)$ to $H_0^1(\Omega)$. Hence,

$$M_A f = M_B f \qquad \forall f \in H^{-1}(\Omega). \tag{3.12}$$

Similarly, since the operator grad : $u \to \nabla u$ is bounded and continuous from $H^1(\Omega)$ to $L^2(\Omega, \mathbb{R}^d)$, we get

$$A \operatorname{grad} M_A f = B \operatorname{grad} M_B f \qquad \forall f \in H^{-1}(\Omega). \tag{3.13}$$

Let us show that (3.12)-(3.13) implies $A = B$ almost everywhere in Ω. Let a closed set ω be such that

$$\omega \subset\subset \Omega, \text{i.e., } \omega \text{ is compactly embedded into } \Omega.$$

In our settings, this means that ω is compact and $\omega \subset \Omega$. There exists $\varphi \in C_0^\infty(\Omega)$ such that $\varphi(x) = 1$ for $x \in \omega$.

Take an arbitrary $y \in \mathbb{R}^d$ and

$$f = -\nabla \cdot (A \nabla (\varphi(x) y \cdot x)),$$

we get from $\varphi(x) y \cdot x \in H_0^1(\Omega)$ and (3.12) that

$$u = M_B f = M_A f = \varphi(x) y \cdot x, \qquad x \in \Omega.$$

From $\nabla u = y$ in ω and (3.13) we get

$$Ay = A \nabla u = B \nabla u = By$$

in the sense of $L^2(\Omega, \mathbb{R}^d)$ for any $y \in \mathbb{R}^d$. This implies that $A = B$ almost everywhere in any compact subset of Ω, and in turn, almost everywhere in Ω.

We proved that $(L^\infty(\Omega, \mathcal{M}_{\alpha,\beta}), \mu_{\mathrm{H}})$ is a metric space. Now we proof part (b).

Step 3. Let us prove that the convergence in the metric space $(L^\infty(\Omega, \mathcal{M}_{\alpha,\beta}), \mu_{\mathrm{H}})$ is the H-convergence.

First we show that the H-convergence implies the μ_{H}-convergence. Indeed, for every $n \in \mathbb{N}$, $A_k \xrightarrow{\mathrm{H}} A_*$ implies that $u_n^{A_k} \rightharpoonup u_n^{A_*}$ in $H_0^1(\Omega)$ and $A_k \nabla u_n^{A_k} \rightharpoonup A_* \nabla u_n^{A_*}$ in $L^2(\Omega, \mathbb{R}^d)$ as $k \to \infty$. So $\mu_n(A_k, A_*) = \mu_{w,H_0^1}(u_n^{A_k}, u_n^{A_*}) + \mu_{w,L^2}(A_k \nabla u_n^{A_k}, A_* \nabla u_n^{A_*})$ goes to 0 as $k \to \infty$ for all n, and in turn, $\mu_{\mathrm{H}}(A_k, A_*) \to 0$.

Let us prove the inverse implication. Let $\mu_{\mathrm{H}}(A_k, A_*) \to 0$. Then for all n

$$\mu_{w,H_0^1}(u_n^{A_k}, u_n^{A_*}) \to 0, \tag{3.14}$$

$$\mu_{w,L^2}(A_k \nabla u_n^{A_k}, A_* \nabla u_n^{A_*}) \to 0, \tag{3.15}$$

as $k \to \infty$.

From Theorem 3.4, it follows that

$$|u_n^{A_k}|_{H_0^1} \leqslant \frac{1}{\alpha}|f_n|_{H^{-1}} \qquad \forall k, n \in \mathbb{N}. \tag{3.16}$$

So (3.14) and (3.16) imply $u_n^{A_k} \rightharpoonup u_n^{A_*}$ in $H_0^1(\Omega)$ as $k \to \infty$. Similarly, (3.16) implies $\|A_k \nabla u_n^{A_k}\|_{L^2} \leqslant \frac{1}{\alpha\beta}|f_n|_{H^{-1}}$ for all k and n. Together with (3.15), this implies $A_k \nabla u_n^{A_k} \rightharpoonup A_* \nabla u_n^{A_*}$ in $L^2(\Omega, \mathbb{R}^d)$ as $k \to \infty$ for all n.

We proved the desired weak convergencies from the definition of H-convergence for all f in the dense subset $\{f_n\}_{n\in\mathbb{N}}$ of $H^{-1}(\Omega)$. In order to extend these weak converegencies to all $f \in H^{-1}(\Omega)$, one uses Proposition 3.2 below. To formulate Proposition 3.2, we need some preparations.

Let V and U be Banach spaces. Following [A02], let us denote by $LC(V, U)$ the space of linear continuous operators $T : V \to U$ equipped with the operator norm $\|\cdot\|$. Recall that $LC(V, U)$ is a Banach space.

Remark 3.8.
We use the standard terminology of [DS88, § II.3.25] and [KF67] for weak convergence. That is, let V be a normed space. A sequence $\{v_n\}_{n\in\mathbb{N}} \subset V$ is called weak convergent in V if there exists $v \in V$ such that $\langle v_n, \varphi\rangle \to \langle v, \varphi\rangle \quad \forall \varphi \in V'$. Note that, the monograph of Kato [K13] uses slightly different and not completely equivalent terminology.

Definition 3.4.
A sequence $\{T_n\}_{n\in\mathbb{N}}$ of bounded (linear) operators $T_n : V \to U$ from a Banach space V to a Banach space U is said to converge weakly if $T_n v$ converges weakly in U for every $v \in V$.

Exercise 3.10 (see [K13, Section III.3.1], [RS12, Theorem VI.1])**.**
Let $\{T_n\}_{n\in\mathbb{N}} \subset LC(V, U)$ be a sequence of bounded operators from a Banach space V to a Banach space U.

(a) If $\{T_n\}_{n\in\mathbb{N}}$ converges weakly, then there exists a unique $T \in LC(V, U)$ such that $T_n v \rightharpoonup Tv$ in U for all $v \in V$. In this case, one says that $\{T_n\}_{n\in\mathbb{N}}$ converges weakly to T and writes $T_n \xrightarrow{w} T$.

(b) If $\{T_n\}_{n\in\mathbb{N}}$ converges weakly, then $\{T_n\}_{n\in\mathbb{N}}$ is bounded, i.e., there exists $C > 0$ such that $\|T_n\| \leqslant C$ for all n. In this case, for the weak limit $T = \operatorname{w-lim} T_n$, one has $\|T\| \leqslant C$.

Proposition 3.2.
Let $\{T_n\}_{n\in\mathbb{N}}$ be a bounded sequence of bounded operators from a Banach space V to a

Banach space U. Let S_1 be dense subset of V and S_2 be a dense subset of U'. Then T_n is weakly convergent if $\langle T_n v, \varphi\rangle \to \langle u_v, \varphi\rangle$ with a certain $u_v \in U$ (depending on v) for all $v \in S_1$ and $\varphi \in S_2$.

We give the proof of this proposition later.

Remark 3.9.
Note that special weak convergence of operators $T_n \in LC(V,U)$, $n \in \mathbb{N}$, defined above in Definition 3.4 does not necessarily coincide with the weak convergence in the Banach space $LC(V,U)$. Generally, weak convergence in the Banach space $LC(V,U)$ is stronger property than the weak convergence of operators (for the case, where $V = U$ is a Hilbert space, see the detailed explanations in [RS12, Sections VI.1 and VI.6]).

Proposition 3.2 applied to operators $M_{A_k} = (\operatorname{div} A_k \operatorname{grad})^{-1} \in LC(H^{-1}(\Omega), H_0^1(\Omega))$ and to operators $A_k \operatorname{grad} M_{A_k} \in LC(H^{-1}(\Omega), L^2(\Omega, \mathbb{R}^d))$ completes the proof of Theorem 3.2.

4 Abstract G-convergence and weak operator convergence.

4.1 Weak convergence and compactness for operators.

The next big aim is to give the proof of the H-compactness of $L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$, which requires more results on weak convergence and weak compactness for operators. That is why our next small aim is to prove Proposition 3.2 and consider weak convergence of operators in more detail.

Let V and U be Banach spaces. Recall that $LC(V,U)$ is the Banach space of linear continuous operators $T : V \to U$ equipped with the operator norm $\|\cdot\|$.

Exercise 4.1.

(a) For every $v \in V$, there exists $\varphi \in V'$ such that $\|\varphi\|_{V'} = 1$ and $\langle v, \varphi\rangle = \|v\|_V$ (see [K13, Sect. III.1.4]).

(b) The norm $\|\cdot\|$ is weakly lower semicontinuous, i.e., if $v_n \rightharpoonup v$ in V, then $\|v\|_V \leqslant \liminf_{n\to\infty} \|v_n\|$.
Hint: use (a).

(c) If $T_n \xrightarrow{w} T_0$ for $\{T_n\}_{n\in\mathbb{N}} \subset LC(U,V)$, then $\|T_0\| \leqslant \liminf_{n\to\infty} \|T_n\|$.
Hint: use (b).

Proof of Proposition 3.2. Since $\|T_n\| \leqslant C$ for all n, the sequence $\{T_n v\}_{n\in\mathbb{N}}$ is bounded for every $v \in V$. From the assumption of the proposition and the criterion of weak convergence in U (see Remark 3.2), we see that $T_n v \rightharpoonup u_v$ for every $v \in S_1$. We denote this limit $Tv := u_v$. This weak convergence $T_n v$ can be extended to $v \in \operatorname{span}(S_1)$ and defines on the linear (possibly nonclosed) subspace $\operatorname{span}(S_1)$ the linear operator Tv in such a way that

$$T_n v \rightharpoonup Tv \text{ in } U \qquad \forall v \in \operatorname{span}(S_1).$$

The subspace span(S_1) is equipped with the norm of V. Then Exercise 4.1 (a) implies $\|T\| \leqslant C$. So T can be extended by continuity to the whole V with $\|T\| \leqslant C$ for this extended operator.

Let now $v \in V$ be arbitrary. Then $\forall \varepsilon > 0$, there exists $v_1 \in \text{span}(S_1)$ such that $\|v - v_1\|_V < \varepsilon$ (or there exists $v_1 \in S_1$ with the same property; the use of dense subset S_1 or dense subset span(S_1) plays here no difference). Then for every $\varphi \in U'$,

$$\begin{aligned}|\langle T_n v, \varphi\rangle - \langle Tv, \varphi\rangle| &\leqslant |\langle (T_n - T)(v - v_1), \varphi\rangle| + |\langle (T_n - T)v_1, \varphi\rangle| \\ &\leqslant 2C\varepsilon\|\varphi\|_{U'} + |\langle (T_n - T)v_1, \varphi\rangle|.\end{aligned}$$

Similarly to the proof of Theorem 3.1 (b), this implies $\langle T_n v, \varphi\rangle \to \langle Tv, \varphi\rangle$. That is, $T_n \xrightarrow{w} T$. □

Theorem 4.1 (see e.g. [DS88, Theorem II.3.28]).
Let U be a reflexive Banach space. Then every bounded sequence in U contains a weakly convergent subsequence.

Theorem 4.1 we take without a proof. (Note that there is no the assumption that U' is separable, and so its is not immediate corollary of the Banach-Alaoglu theorem).

Proposition 4.1 (on the compactness for weak operator convergence, see e.g. [A02]).
Let V be a separable Banach space, and let U be a reflexive Banach space. Let $\{T_n\}_{n\in\mathbb{N}} \subset LC(V, U)$ be a bounded sequence of operators. Then there exists $T_0 \in LC(V, U)$ and a subsequence $\{T_{n_k}\}_{k\in\mathbb{N}}$ such that $T_{n_k} \xrightarrow{w} T_0$.

Proof. Let C be such that $\|T_n\| \leqslant C$ for all n. Let $S = \{f_j\}_{j\in\mathbb{N}}$ be a dense countable subset of V. For every $f \in S$, the sequence $\{T_n f\}$ is bounded in U. Applying Theorem 4.1, we see that there exists a subsequence $\{T_{n_k^f} f\}_{k\in\mathbb{N}}$ such that $T_{n_k^f} f \rightharpoonup u_f$ in U for a certain $u_f \in U$. Here the increasing sequence $\{n_k^f\}_{k\in\mathbb{N}} \subset \mathbb{N}$ depends on the choice of f, and u_f depends on the choice of f and $\{n_k^f\}_{k\in\mathbb{N}}$.

Consider now the procedure of the extraction of a diagonal subsequence. For $f_1 \in S$, there exists a subsequence $\{n_k^{f_1}\}_{k\in\mathbb{N}}$ such that $T_{n_k^{f_1}} f_1 \rightharpoonup u_{f_1}$ in U. From this subsequence we choose an increasing subsequence $\{n_k^{f_2}\}_{k\in\mathbb{N}} \subseteq \{n_k^{f_1}\}_{k\in\mathbb{N}}$ such that $T_{n_k^{f_2}} f_2 \rightharpoonup u_{f_2}$, and so on. Then the diagonal subsequence $\{n_k^{f_k}\}$ satisfies

$$T_{n_k^{f_k}} f \rightharpoonup u_f \qquad \forall f \in S = \{f_j\}_{j\in\mathbb{N}}.$$

This defines a linear operator T, for a time being on span(S), by

$$T\Big(\sum_{j=1}^{N} c_j f_j\Big) = \sum_{j=1}^{N} c_j u_{f_j}.$$

From Exercise 4.1 (b), $\|T\| \leqslant C$. So we can extend this operator by continuity to the whole V saving the inequality $\|T\| \leqslant C$ for the extended operator.

Summarizing, there exists $T \in LC(V,U)$ and a subsequence $\{n_k\}_{k\in\mathbb{N}} = \{n_k^{f_k}\}_{k\in\mathbb{N}}$ such that

$$T_{n_k} f \rightharpoonup Tf \qquad \forall f \in \operatorname{span}(S). \tag{4.1}$$

It remains to extend (4.1) to all $f \in V$ using the density $\overline{S} = V$. This can be done via the estimates with approximation of arbitrary $v \in V$ by $f \in S$ similar to that of the proofs of Proposition 3.2 and Theorem 3.1 (b). □

Remark 4.1 ([DS88, § II.3.25]).
Let V be a normed space.

(a) A sequence $\{v_n\}_{n\in\mathbb{N}} \subset V$ is called a weak Cauchy sequence (in V) if $\{\langle v_n, \varphi\rangle\}_{n\in\mathbb{N}}$ is a Cauchy sequence $\forall \varphi \in V'$.

(b) Equivalently, $\{v_n\}_{n\in\mathbb{N}} \subset V$ is a weak Cauchy sequence if and only if $\{\langle v_n, \varphi\rangle\}_{n\in\mathbb{N}}$ converges (in $\mathbb{R}$ or $\mathbb{C}$) $\quad \forall \varphi \in V'$.

(One has to be careful with the terminology for weak convergence in the book [K13], because in [K13] weak Cauchy sequences are called weak convergent, and weak convergent sequences in the standard sense are called weak convergent to a certain element $v \in V$. That is, the definition of weak convergent sequences in [K13] is generally not equivalent to the standard definition. These definitions are equivalent in the special situation of weak complete spaces.)

(c) If every weak Cauchy sequence in V is weak convergent in V, the space V is called weakly complete.

Exercise 4.2.(a) If a normed space V is weakly complete, it is complete.

(b) There exist Banach spaces that are not weakly complete.
Hint: try $V = C[0,1]$ or $V = c_0$ (see [DS88, Table IV.A]).

(c) Every reflexive Banach space is weakly complete.
Hint: use Theorem 4.1.

Remark 4.2.
There exist Banach spaces V having a separable pre-dual and simultaneously not weakly complete, e.g., $V = \ell^\infty$ [DS88, IV.13.5].

4.2 Abstract G-convergence of operators.

Let V be a separable reflexive Banach space. Assume that $T \in LC(V,V')$ satisfies for a certain $\alpha > 0$ the coercivity estimate

$$\alpha\|v\|_V \leqslant \langle Tv, v\rangle \qquad \forall v \in V. \tag{4.2}$$

Then the Lax-Milgram lemma implies that T is invertible and

$$\|T^{-1}\| \leqslant \frac{1}{\alpha}.$$

Assume now that T^{-1} satisfy the coercivity estimate for a certain $\beta > 0$

$$\beta \|f\|_{V'} \leqslant \langle T^{-1}f, f\rangle \qquad \forall f \in V'. \tag{4.3}$$

Then

$$\beta \|f\|_{V'} \leqslant \|T^{-1}f\|_V \|f\|_{V'} \leqslant \frac{1}{\alpha}\|f\|_{V'}^2 \qquad \forall f \in V',$$

and so $\alpha \leqslant \frac{1}{\beta}$. Recall that we use the notation $\mathbb{R}_{\pm} = \{\alpha \in \mathbb{R} : \pm\alpha > 0\}$.

Definition 4.1.
We define

(a) $\mathcal{E}(V) := \{T \in LC(V, V') :$ (4.2) is satisfied for a certain $\alpha > 0\}$

(b) for $\alpha, \beta \in \mathbb{R}_+$ satisfying $\alpha \leqslant \beta^{-1}$

$$\mathcal{E}_{\alpha,\beta}(V) := \{T \in LC(V, V') : \ T \text{ satisfies (4.2)}, \ T^{-1} \text{ satisfies (4.3)}\}.$$

Definition 4.2 (Spagnolo [S76]).
Let $\{T_n\}_{n\in\mathbb{N}} \subset \mathcal{E}(V)$ and $T \in \mathcal{E}(V)$. The sequence $\{T_n\}$ is said to G-converge to T if $T_n^{-1} \xrightarrow{w} T^{-1}$. In this case, we write $T_n \xrightarrow{G} T$.

Theorem 4.2 (G-compactness, [S76], see also [A02, JKO12]).
Let $\alpha, \beta \in \mathbb{R}_+$ satisfy $\alpha \leqslant \beta^{-1}$. Let V be a separable Banach space. Then for every sequence $\{T_n\}_{n\in\mathbb{N}} \subset \mathcal{E}_{\alpha,\beta}(V)$, there exists a subsequence $\{T_{n_k}\}$ such that $T_{n_k} \xrightarrow{G} T$ for a certain $T \in \mathcal{E}(V)$. In this case, $T \in \mathcal{E}_{\alpha,\beta}(V)$.

Exercise 4.3 ([B11, Corollaries 3.21 and 3.27]).
(a) A Banach space V is reflexive if and only if V' is reflexive.

(b) Banach space V is reflexive and separable if an only if V' is reflexive and separable.

Proof of Theorem 4.2. Using Exercise 4.3 and Proposition 4.1, we may pass to a weak limit $S = \text{w-}\lim_{k\to\infty} T_{n_k}^{-1}$ on a certain subsequence $\{T_{n_k}^{-1}\}$. Then the appropriate coercivity estimates for S and $T := S^{-1}$ can be obtained passing to limits for appropriately written coercive estimates for T_{n_k} and $T_{n_k}^{-1}$. □

Corollary 4.1 (application of the abstract G-convergence to div A grad-operators).
Let $\{A_n\}_{n\in\mathbb{N}} \subset L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$. Let $L_{A_n} \in LC(H_0^1(\Omega), H^{-1}(\Omega))$ be defined by

$$L_{A_n} = -\operatorname{div} A_n \operatorname{grad},$$

and let $Q_n \in LC(H^{-1}(\Omega), L^2(\Omega, \mathbb{R}^d))$ be defined by

$$Q_n = A_n \operatorname{grad} L_{A_n}^{-1}.$$

Then there exists a subsequence $\{n_k\} \subset \mathbb{N}$, an operator $T_0 \in \mathcal{E}_{\alpha,\beta}(H_0^1(\Omega))$, and an operator $Q_0 \in LC(H^{-1}(\Omega), L^2(\Omega, \mathbb{R}^d))$ such that

$$L_{A_{n_k}}^{-1} \xrightarrow{w} T_0^{-1} \quad \textit{and} \quad Q_{n_k} \xrightarrow{w} Q_0.$$

Here $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ are percieved as mutually dual spaces equipped with norms $|\cdot|_{H_0^1}$ and $|\cdot|_{H^{-1}}$.

Proof. The proof is based on Theorem 4.2 and Proposition 8.2. In the proof, we denote $L_n := L_{A_n}$ for brevity.

Step 1. We show that $\{L_n\} \subset \mathcal{E}_{\alpha,\beta}(H_0^1(\Omega), H^{-1}(\Omega))$. The estimate (4.2) for L_n follows from the coercivity estimates for A_n. In order to get (4.3), we write using the coercivity estimates for A_n^{-1}.

$$\langle L_n v, v\rangle = \int_\omega \langle A_n \nabla v, A_n^{-1} A_n \nabla v\rangle \mathrm{d}x \geqslant \beta \|A_n \nabla v\|_{L^2}.$$

For any $f \in H^{-1}(\Omega)$, we put $v = L_n^{-1} f$. This gives

$$\beta \|Q_n f\|_{L^2} \leqslant \langle f, L_n^{-1} f\rangle. \tag{4.4}$$

On the other hand,

$$|f|_{H^{-1}} = \sup_{w \neq \mathbb{0}} \frac{|\langle A_n \nabla v| \nabla w\rangle_{L^2}|}{|w|_{H^{-1}}} \leqslant \|A_n \nabla v\|_{L^2} = \|Q_n f\|_{L^2}. \tag{4.5}$$

Estimates (4.4)-(4.5) imply (4.3) for L_n^{-1}.

Step 2. We apply Theorem 4.2 to $\{L_n\}$ and get, after passing to a subsequence, $L_{n_k}^{-1} \xrightarrow{w} T_0^{-1}$ for a certain $T_0 \in \mathcal{E}_{\alpha,\beta}(H_0^1(\Omega))$.

Step 3. The sequence $\{Q_n\}$ is bounded in $LC(H^{-1}(\Omega), L^2(\Omega, \mathbb{R}^d))$. Indeed, using the coercivity estimate for A_n^{-1} and then $\|L_n^{-1}\| \leqslant 1/\alpha$, we get

$$\|Q_n f\|_{L^2} = \|A_n \operatorname{grad} L_n^{-1} f\|_{L^2} \leqslant \beta^{-1} \|\nabla (L_n^{-1} f)\|_{L^2} = \beta^{-1} |L_n^{-1} f|_{H_0^1} \leqslant \frac{1}{\beta\alpha} |f|_{H^{-1}}.$$

Step 4. We apply Proposition 8.2 to the bounded subsequence $\{Q_{n_k}\}$. After passing to a suitable subsequence one more time, Proposition 8.2 produces the subsubsequence, which with some abuse of notation we keep indexing by n_k, such that $Q_{n_k} \xrightarrow{w} Q_0$ for a certain $Q_0 \in LC(H^{-1}(\Omega), L^2(\Omega, \mathbb{R}^d))$. Thus, $Q_{n_k} \xrightarrow{w} Q_0$ and $L_{n_k}^{-1} \xrightarrow{w} T_0^{-1}$ simultaneously. □

5 The space $(H^1(\Omega))'$ and oscillatory test functions.

5.1 Oscillating test functions.

The proof of the H-compactness of $L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$ is based on two methods of Murat and Tartar [MT78, A02], namely, oscillating test functions and the compensated compactness.

Lemma 5.1 (oscillating test functions [MT78])**.**
Let $\{A_n\}_{n\in\mathbb{N}} \subset L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$. Let $1 \leqslant j \leqslant d$. Then there exists a subsequence $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ and a sequence of test functions $\{w_k^j\}_{k\in\mathbb{N}} = \{w_k\}_{k\in\mathbb{N}} \subset H^1(\Omega)$ with the properties:

(a) $w_k \rightharpoonup x_j$ in $H^1(\Omega)$, where by x_j we denote a $C^\infty(\Omega)$-function $x \mapsto x_j$,

(b) For all $k \in \mathbb{N}$,

$$-\nabla \cdot (A_{n_k} \nabla w_k) = g \qquad \text{in the sense of distributions for a certain } g \in H^{-1}(\Omega).$$

(c) $A_{n_k} \nabla w_k \rightharpoonup a^j$ in $L^2(\Omega, \mathbb{R}^d)$ for a certain vector-field $a^j \in L^2(\Omega, \mathbb{R}^d)$.

This lemma is proved in Section 5.1 after some preparations are done in Section 5.2.

Remark 5.1.
The role of the collection of the vector-fields $a^1(\cdot), \ldots, a^d(\cdot)$ is that, taken as vector-columns, they together produce a matrix-function $A_*(x) = (a^1(x), \ldots, a^d(x))$. Later it will be shown that A_* is a homogenized matrix-function in the sense that A_* is the H-limit of the subsequence $\{A_{n_k}\}_{k\in\mathbb{N}}$. As soon as this fact is proved, this proves Theorem 2.3 about the compactness of $L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$.

5.2 The space $(H^1(\Omega))'$ and weak convergence in $H^1(\Omega)$.

We identify pairs of $w_0 \in L^2(\Omega)$ and $w_1 \in L^2(\Omega, \mathbb{R}^d)$ with vector-fields $w = (w_0, w_1) \in L^2(\Omega, \mathbb{R}^{d+1})$. For the description of the spaces $(W^{k,p}(\Omega))'$ see [A75].

Theorem 5.1 (description of $(H^1(\Omega))'$).

(a) For every $w = (w_0, w_1) \in L^2(\Omega, \mathbb{R}^{d+1})$, the formula

$$\langle v, \psi_w \rangle = \langle v | w_0 \rangle_{L^2} + \langle \nabla v | w_1 \rangle_{L^2}$$

defines $\varphi_w \in (H^1(\Omega))'$.

(b) The mapping

$$F : w \mapsto \varphi_w, \qquad F : L^2(\Omega, \mathbb{R}^{d+1}) \to (H^1(\Omega))'$$

is surjective.

(c) For every $\varphi \in (H^1(\Omega))'$, the norm of linear functional φ equals

$$\|\varphi\|_{(H^1)'} = \min\{\|w\|_{L^2} \ : \ \varphi = \varphi_w\}.$$

Proof. (a) It follows from the definition of the norm in $H^1(\Omega)$ and the Cauchy-Bunyakovsky-Schwarz inequality that

$$|\langle v, \phi_w \rangle| \leqslant \|v\|_{L^2} \|w_0\|_{L^2} + \|\nabla v\|_{L^2} \|w_1\|_{L^2} \leqslant \|v\|_{H^1} \|w\|_{L^2}. \tag{5.1}$$

This implies (a).

(b) follows from the Riesz representation theorem in Hilbert spaces. That is, for every $\varphi \in (H^1(\Omega))'$ there exists a unique $u \in H^1(\Omega)$ such that

$$\langle v | \varphi \rangle = \langle v, u \rangle_{H^1} = \langle v | u \rangle_{L^2} + \langle \nabla v | \nabla u \rangle_{L^2} = \langle v | \varphi_w \rangle$$

with $w = (u, \nabla u)$. Moreover, in this case,

$$\|\varphi\|_{(H^1)'} = \|u\|_{H^1} = \|w\|_{L^2}. \tag{5.2}$$

(c) follows from (5.1) and (5.2). □

Corollary 5.1.
The weak convergence $v_n \rightharpoonup v_0$ in $H^1(\Omega)$ holds if and only if

$$\langle v_n|w_0\rangle_{L^2} + \langle \nabla v_n|w_1\rangle_{L^2} \to \langle v_0|w_0\rangle_{L^2} + \langle \nabla v_0|w_1\rangle_{L^2} \quad \forall w \in L^2(\Omega, \mathbb{R}^{d+1}).$$

Proof. Corollary 5.1 follows from Theorem 5.1. □

Corollary 5.2.
Let Ω and $\widehat{\Omega}$ be domains in $\mathbb{R}^d$ such that $\Omega \subset \widehat{\Omega}$. Then, for every $\varphi \in (H^1(\Omega))'$, there exists $\widehat{\varphi} \in (H^1(\widehat{\Omega}))'$ such that

$$\|\widehat{\varphi}\|_{(H^1(\widehat{\Omega}))'} \leqslant \|\varphi\|_{(H^1(\Omega))'}$$

and, for the restriction $v|_\Omega$ of an arbitrary $v \in H^1(\widehat{\Omega})$, the following equality holds

$$\varphi(v|_\Omega) = \widehat{\varphi}(v).$$

Proof. Let $\varphi \in (H^1(\Omega))'$. As in the proof of Corollary 5.1, there exists a unique $u \in H^1(\Omega)$ such that

$$\langle v|\varphi\rangle = \langle v|u\rangle_{L^2} + \langle \nabla v| \nabla u\rangle_{L^2} = \langle v|\varphi_w\rangle$$

with $w = (w_0, w_1) = (u, \nabla u)$. Let us extend w to a function $\widehat{w} \in L^2(\widehat{\Omega}, \mathbb{R}^{d+1})$ by zero, i.e., $\widehat{w}(x) = 0$ for $x \in \widehat{\Omega}\backslash\Omega$ and $\widehat{w}(x) = w(x)$ for $x \in \Omega$. Then $\widehat{\varphi} = \varphi_{\widehat{w}}$ satisfies the desired properties. □

Corollary 5.3.
Let Ω and $\widehat{\Omega}$ be domains in $\mathbb{R}^d$ such that $\Omega \subset \widehat{\Omega}$. Let $v_n \rightharpoonup v_0$ in $H^1(\widehat{\Omega})$ (or in $H^1_0(\widehat{\Omega})$). Then $v_n|_\Omega \rightharpoonup v_0|_\Omega$ in $H^1(\Omega)$.

Proof. For every $\varphi \in (H^1(\Omega))'$, we can use the extension of Corollary 5.2. This and Corollary 5.1 show that $v_n \rightharpoonup v_0$ in $H^1(\widehat{\Omega})$ implies $v_n|_\Omega \rightharpoonup v_0|_\Omega$ in $H^1(\Omega)$. If $v_n \rightharpoonup v_0$ in $H^1_0(\widehat{\Omega})$, we also have $v_n \rightharpoonup v_0$ in $H^1(\widehat{\Omega})$. Thus, $v_n|_\Omega \rightharpoonup v_0|_\Omega$ in $H^1(\Omega)$ follows as above. □

5.3 The proof of Lemma 5.1 (on oscillating test functions).

Exercise 5.1.
Let Ω and $\widehat{\Omega}$ be domains in $\mathbb{R}^d$ such that $\Omega \subset \widehat{\Omega}$. Let $u \in H^1(\widehat{\Omega})$. Then the restriction $u|_\Omega$ belongs to $H^1(\Omega)$ and $\nabla(u|_\Omega) = (\nabla u)|_\Omega$ in the L^2-sense.

The proof of Lemma 5.1 is given in several steps.

Step 1. Let us consider a certain domain $\widehat{\Omega}$ such that $\overline{\Omega} \subset \widehat{\Omega}$. We extend A_n to $\widehat{\Omega}$ such that the extended A_n belongs to $L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$, for example, such an extension can be done by $A_n(x) = \alpha I_{\mathbb{R}^3}$ for $x \in \widehat{\Omega}\backslash\Omega$.

We apply Corollary 4.1 to the operators $\widehat{L}_{A_n} = -\operatorname{div} A_n \operatorname{grad}$ and $\widehat{Q}_{A_n} = A_n \operatorname{grad} \widehat{L}^{-1}_{A_n}$ corresponding to the domain $\widehat{\Omega}$. This produces a subsequence $\{n_k\} \subset \mathbb{N}$, an operator

$\widehat{T}_0 \in \mathcal{E}_{\alpha,\beta}(H_0^1(\widehat{\Omega}))$, and an operator $\widehat{Q}_0 \in LC(H^{-1}(\widehat{\Omega}), L^2(\widehat{\Omega}, \mathbb{R}^d))$ such that

$$\widehat{L}_{A_{n_k}}^{-1} \xrightarrow{w} \widehat{T}_0^{-1} \quad \text{and} \quad \widehat{Q}_{n_k} \xrightarrow{w} \widehat{Q}_0.$$

Step 2. Let us take a smooth cut-off function $\psi \in C_0^\infty(\widehat{\Omega})$ such that $\psi(x) = 1$ for $x \in \Omega$. Denote by ψx_j the $C_0^\infty(\widehat{\Omega})$-function $x \mapsto \psi(x) x_j$.
Let $\widehat{g} \in H^{-1}(\widehat{\Omega})$ be defined by $\widehat{T}_0(\psi x_j)$, and let $g \in H^{-1}(\Omega)$ be the restriction of the functional $\widehat{g}$, i.e.,

$$g = \widehat{g}|_{H_0^1(\Omega)}.$$

Let

$$\widehat{w}_k = \widehat{L}_{A_{n_k}}^{-1} \widehat{g}.$$

So $\widehat{w}_k \in H_0^1(\widehat{\Omega})$. The sequence of "oscillating test functions" corresponding to a coordinate $j \in \{1, \ldots, d\}$ is defined by

$$w_k = \widehat{w}_k|_\Omega \qquad \text{(see Exercise 5.1).}$$

Step 3. Let us show now that the sequence of functions w_k satisfies properties (a)-(c) of Lemma 5.1.

Since $\widehat{L}_{A_{n_k}}^{-1} \xrightarrow{w} \widehat{T}_0^{-1}$, we have $\widehat{w}_k = \widehat{L}_{A_{n_k}}^{-1} \widehat{g} \xrightarrow{w} \widehat{T}_0^{-1} \widehat{g} = \psi x_j$ in $H_0^1(\widehat{\Omega})$. Hence, Corollary 5.3 implies $w_k \rightharpoonup x_j$ in $H^1(\Omega)$. This proves (a).

Considering the equality $\widehat{L}_{A_{n_k}} \widehat{w}_k = \widehat{g}$ in the space $H^{-1}(\widehat{\Omega})$ and coupling it with the test functions $v \in H_0^1(\Omega)$, one gets (b).

Since $\widehat{Q}_{n_k} \xrightarrow{w} \widehat{Q}_0$, we have

$$\widehat{Q}_{A_{n_k}} \widehat{g} = A_{n_k} \operatorname{grad} \widehat{L}_{A_{n_k}}^{-1} \widehat{g} = A_{n_k} \operatorname{grad} \widehat{w}_k \xrightarrow{w} \widehat{Q}_0 \widehat{g}$$

with the weak convergence in the L^2-sense. We test this weak convergence on vector-fields $v \in L^2(\Omega, \mathbb{R}^d)$ extended to $\widehat{v} \in L^2(\widehat{\Omega}, \mathbb{R}^d)$ by 0 in $\widehat{\Omega} \backslash \Omega$. With the use of Exercise 5.1, this shows that

$$\langle A_{n_k} \nabla \widehat{w}_k \mid \widehat{v} \rangle_{L^2} = \langle A_{n_k} (\nabla \widehat{w}_k)|_\Omega \mid v \rangle_{L^2} = \langle A_{n_k} \nabla (\widehat{w}_k|_\Omega) \mid v \rangle_{L^2} = \langle A_{n_k} \nabla w_k \mid v \rangle_{L^2},$$

and that $\langle A_{n_k} \nabla w_k \mid v \rangle_{L^2}$ converges to $\langle (\widehat{Q}_0 \widehat{g})|_\Omega \mid v \rangle_{L^2}$ as $k \to \infty$ for all $v \in L^2(\Omega, \mathbb{R}^d)$. Thus, $A_{n_k} \nabla w_k \rightharpoonup a^j$ in $L^2(\Omega, \mathbb{R}^d)$ with

$$a^j = (\widehat{Q}_0 \widehat{g})|_\Omega.$$

This completes the proof of Lemma 5.1.

6 Compensated compactness and G-convergence for 2nd order elliptic equations.

6.1 Compensated compactness.

Let V be a Hilbert space. Recall that the notation $v_n \to v$ in V means the strong convergence of the sequence $\{v_n\}$ to v in the space V.

Exercise 6.1.

(a) Assume that $u_n \to u$ in V and $v_n \rightharpoonup v$ in V. Then $\langle u_n|v_n\rangle \to \langle u, v\rangle$.

(b) Assume that $u_n \to u$ in $L^2(\Omega)$ and $v_n \rightharpoonup v$ in $L^2(\Omega)$. Then $u_n v_n \rightharpoonup uv$ in $L^1(\Omega)$.

Here $u_n v_n$ denotes the $L^1(\Omega)$-function $w_n : x \mapsto u_n(x)v_n(x)$ defined for almost all $x \in \Omega$ as a pointwise product.

Example 6.1 (convergence counterexample, product of weakly convergent sequences)**.**
In $L^2(\Omega)$, we consider the sequence $u_n(x) = \sin(nx_1)$, $n \in \mathbb{N}$. Then, we know that $u_n \rightharpoonup \mathbb{0}$ in $L^2(\Omega)$. Let us take the 2nd copy of this weakly convergent sequence, $v_n = u_n$, $n \in \mathbb{N}$. Then

$$u_n v_n = u_n^2 = \sin^2(nx_1) \rightharpoonup \frac{1}{2}\mathbb{1} \text{ in } L^2(\Omega) \text{ and in } L^1(\Omega) \text{ (Exercise).}$$

Here $\mathbb{1}$ is the constant function equal to 1 for all x. Summarizing, we see that the weak limits $u = v = \text{w-}\lim u_n = \text{w-}\lim v_n$ are $\mathbb{0}$, but $u_n v_n \rightharpoonup \frac{1}{2}\mathbb{1} \neq uv$.

Exercise 6.2.

(a) Construct two weakly convergent in $L^2(\Omega)$ sequences $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ such that $\{\langle u_n|v_n\rangle_{L^2}\}_{n\in\mathbb{N}}$ is not convergent.

(b) Construct two sequencies $\{u_n\}_{n\in\mathbb{N}}$, $\{v_n\}_{n\in\mathbb{N}} \subset L^2(\Omega) \cap L^\infty(\Omega)$ such that $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ are weakly convergent in $L^2(\Omega)$, but $\{u_n v_n\}_{n\in\mathbb{N}}$ is not weakly convergent in $L^2(\Omega)$ (in $L^1(\Omega)$).

While it is generally impossible to pass to the limit for various products of weakly convergent sequences, it is still quite desirable in many cases (some of them we see below). The question is if this is possible to do under certain additional assumptions.

Let us define for $v = (v_1, \ldots, v_d) \in L^2(\Omega, \mathbb{R}^d)$ an operator

$$\operatorname{curl} : L^2(\Omega, \mathbb{R}^d) \to H^{-1}(\Omega, \mathbb{R}^{d\times d})$$

as

$$\operatorname{curl} v = \big(\partial_{x_j} v_i - \partial_{x_i} v_j\big)_{i,j=1}^d ,$$

where $H^{-1}(\Omega, \mathbb{R}^{d\times d})$ is perceived as the space of $d \times d$-matrices having the entries from the space $H^{-1}(\Omega)$.

The following compensated compactness result, which essentially stems from works of Murat & Tartar, is given in the very general formulation of the monograph [JKO12].

Lemma 6.1 (div-curl lemma, essentially Murat & Tartar, 1978)**.**
Let $\{u_n\}_{n\in\mathbb{N}}$, $\{v_n\}_{n\in\mathbb{N}} \subset L^2(\Omega, \mathbb{R}^d)$ be such that

$$u_n \rightharpoonup u \text{ in } L^2(\Omega, \mathbb{R}^d),$$
$$v_n \rightharpoonup v \text{ in } L^2(\Omega, \mathbb{R}^d),$$
$$\{\operatorname{div} u_n\}_{n\in\mathbb{N}} \text{ is a relatively compact subset of } H^{-1}(\Omega),$$
$$\{\operatorname{curl} v_n\}_{n\in\mathbb{N}} \text{ is a relatively compact subset of } H^{-1}(\Omega, \mathbb{R}^{d\times d}).$$

Then, for every $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega \varphi(x)\langle u_n(x)|v_n(x)\rangle_{\mathbb{R}^d}\mathrm{d}x \to \int_\Omega \varphi(x)\langle u(x)|v(x)\rangle_{\mathbb{R}^d}\mathrm{d}x.$$

(That is, $\{\langle u_n(x)|v_n(x)\rangle_{\mathbb{R}^d}\}_{n\in\mathbb{N}}$ converges in the sense of distributions.)

We take this lemma without a proof (a proof can be found in [JKO12, Section 4.2]).

Exercise 6.3.

(a) For arbitrary $u \in H^1(\Omega)$, the following equality holds

$$\operatorname{curl} \nabla u = \mathbb{0}, \text{ where } \mathbb{0} = 0_{\mathbb{R}^{d\times d}}.$$

(b) The equality curl grad $= 0$ holds in the sense of linear operators in the space of distributions.

6.2 The proof of Theorem 2.3 (on the H-compactness of $L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$).

Let $\{A_n\}_{n\in\mathbb{N}} \subset L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$. We know from Corollary 4.1 and Lemma 5.1 that we may pass to a subsequence, which we again reindex by $n \in \mathbb{N}$, such that

$$\text{operators } L_{A_n} = -\operatorname{div} A_n \operatorname{grad} \quad \text{are G-convergent to a certain } T_0 \in \mathcal{E}_{\alpha,\beta}(H_0^1(\Omega))$$

and

$$Q_n = A_n \operatorname{grad} L_n^{-1} \text{ are weakly convergent to a certain } Q_0 \in LC(H^{-1}(\Omega), L^2(\Omega, \mathbb{R}^d))$$

as $n \to \infty$. Moreover, this subsequence can be choosen such that additionally for each $j = 1, \dots, d$, there exists a sequence of "oscillatory test functions" $\{w_n^j\}_{n\in\mathbb{N}} \subset H^1(\Omega)$ with the properties:

- $w_n^j \rightharpoonup x_j$ in $H^1(\Omega)$,
- $-\nabla \cdot (A_n \nabla w_n^j) = g_j \; \forall n \in \mathbb{N}$ for a certain $g_j \in H^{-1}(\Omega)$,
- $A_n \nabla w_n^j \rightharpoonup a^j$ in $L^2(\Omega, \mathbb{R}^d)$ for a certain vector-field $a^j \in L^2(\Omega, \mathbb{R}^d)$.

We introduce a $d \times d$-matrix-function $A_* = (a^1, \dots, a^d) \in L^2(\Omega, \mathbb{R}^{d\times d})$. Let us take an arbitrary $f \in H^{-1}(\Omega)$. Then

$$u_n = L_n^{-1} f \rightharpoonup T_0^{-1} f = u_0 \quad \text{in } H_0^1(\Omega),$$
$$A_n \nabla u_n \rightharpoonup Q_0 f \quad \text{in } L^2(\Omega, \mathbb{R}^d).$$

Our aim is to prove that $A_* \in L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$ and $T_0 = -\operatorname{div} A_* \operatorname{grad}$, which can be summarized as $A_n \xrightarrow{\mathrm{H}} A_*$.

The proof of Theorem 2.3 is split in several steps (we follow mainly [A02] with minor differences).

Step 1. We want to show that $A_* \operatorname{grad} T_0^{-1} f \in L^2(\Omega, \mathbb{R}^d)$ *and that*

$$f = -\operatorname{div} A_* \operatorname{grad} T_0^{-1} f \qquad \textit{(this is a precursor of } T_0 = -\operatorname{div} A_* \operatorname{grad}\textit{)}.$$

For every vector $\xi \in \mathbb{R}^d$, we construct $w_n^\xi = \sum_{j=1}^d \xi_j w_n^j$, which satisfies

$$w_n^\xi \rightharpoonup \xi \cdot x \ \text{ in } H^1(\Omega).$$

Since $A_n(x) \in \mathcal{M}_{\alpha,\beta}$ for almost all $x \in \Omega$, we have for almost all $x \in \Omega$

$$\langle A_n(\nabla u_n - \nabla w_n^\xi) | (\nabla u_n - \nabla w_n^\xi) \rangle_{\mathbb{R}^d} \geqslant 0$$

We apply to this inequality the div-curl lemma (Lemma 6.1). Let us show that the assumptions of the div-curl lemma hold true. By Exercise 6.3, $\{\operatorname{curl} \nabla(u_n - w_n^\xi)\}_{n\in\mathbb{N}} = \{\mathbb{0}\}$ is a one point compact set in $H^{-1}(\Omega, \mathbb{R}^{d\times d})$. Besides,

$$u_n \rightharpoonup u_0 \text{ in } H_0^1(\Omega), \text{ and so } \nabla u_n \rightharpoonup \nabla u_0 \text{ in } L^2(\Omega, \mathbb{R}^d),$$
$$w_n^\xi \rightharpoonup \xi \cdot x \text{ in } H_0^1(\Omega), \text{ and so } \nabla w_n^\xi \rightharpoonup \mathbb{1}\xi \text{ in } L^2(\Omega, \mathbb{R}^d);$$

note that we have used here that $\operatorname{grad} \in LC(H^1(\Omega), L^2(\Omega, \mathbb{R}^d))$.
Hence, $\nabla(u_n - w_n^\xi) \rightharpoonup \nabla u_0 - \xi$ in $L^2(\Omega, \mathbb{R}^d)$. On the other hand, denoting $g = \sum_{j=1}^d \xi_j g_j$, we see that $\{\operatorname{div} A_n \nabla(u_n - w_n^\xi)\}_{n\in\mathbb{N}} = \{-f + g\}$ is a one point compact set in $H^{-1}(\Omega)$. Additionally, we have the weak $L^2(\Omega, \mathbb{R}^d)$-convergence

$$A_n \nabla(u_n - w_n^\xi) \rightharpoonup Q_0 f - A_* \xi.$$

So all assumptions of the div-curl lemma are satisfied.

The the div-curl lemma implies that, for every nonnegative $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega \varphi \, \langle Q_0 f - A_* \xi | \nabla u_0 - \xi \rangle_{\mathbb{R}^d} \mathrm{d}x \geqslant 0.$$

Hence, for every $\xi \in \mathbb{R}^d$, there exists a subset $\Omega^\xi \subset \Omega$ of the full measure $\operatorname{meas}(\Omega^\xi) = \operatorname{meas}(\Omega)$ such that

$$\langle (Q_0 f)(x) - A_*(x)\xi | \nabla u_0(x) - \xi \rangle_{\mathbb{R}^d} \geqslant 0, \qquad x \in \Omega^\xi. \tag{6.1}$$

Here and in what follows we fix concrete functions-representatives of $Q_0 f$, A_*, and ∇u_0 in the corresponding L^p-spaces of the equivalence classes.

Let $\Xi = \{\xi^k\}_{k\in\mathbb{N}}$ be a dense countable subset of $\mathbb{R}^d$. Then $\widetilde{\Omega} = \bigcap_{k\in\mathbb{N}} \Omega^{\xi^k}$ also has the properties that (6.1) holds for $x \in \widetilde{\Omega}$ and $\operatorname{meas}(\widetilde{\Omega}) = \operatorname{meas}(\Omega)$. Since Ξ is dense in $\mathbb{R}^d$, for every $x \in \widetilde{\Omega}$, we can approximate for every $\xi \in \mathbb{R}^d$ the number

$$\langle (Q_0 f)(x) - A_*(x)\xi | \nabla u_0(x) - \xi \rangle_{\mathbb{R}^d}$$

by nonnegative numbers

$$\langle (Q_0 f)(x) - A_*(x)\xi^k | \nabla u_0(x) - \xi^k \rangle_{\mathbb{R}^d}.$$

Thus, (6.1) holds for all $x \in \widetilde{\Omega}$ and all $\xi \in \mathbb{R}^d$.

Let us fix now an arbitrary $x^0 \in \widetilde{\Omega}$ and take $\xi = \nabla u(x^0) - ty$ with $t > 0$ and $y \in \mathbb{R}^d$. We obtain from (6.1)

$$\langle (Q_0 f)(x^0) - A_*(x^0) \nabla u_0(x^0) + tA_*(x^0)y | ty \rangle_{\mathbb{R}^d} \geqslant 0$$

for all $t > 0$ and all $y \in \mathbb{R}^d$. Considering the limit as $t \to 0 + 0$, we get

$$\langle (Q_0 f)(x^0) - A_*(x^0) \nabla u_0(x^0) | y \rangle_{\mathbb{R}^d} \geqslant 0 \qquad \forall y \in \mathbb{R}^d.$$

Considering this inequality with $y = \pm \widetilde{y}$ for all $\widetilde{y} \in \mathbb{R}^d$, we see that

$$Q_0 f(x) = A_*(x) \nabla u_0(x) \qquad \forall x \in \widetilde{\Omega}. \tag{6.2}$$

In particular, (6.2) implies

$$A_* \nabla u_0 \in L^2(\Omega, \mathbb{R}^d).$$

Since $A_n \nabla u_n \rightharpoonup Q_0 f$ in $L^2(\Omega, \mathbb{R}^d)$,

$$f = -\nabla \cdot (A_n \nabla u_n) \;\rightharpoonup\; -\nabla (Q_0 f) = -\nabla \cdot (A_* \nabla u_0) \;\text{ in } H^{-1}(\Omega).$$

That is,

$$-\nabla \cdot (A_* \nabla u_0) = f = T_0 u_0.$$

Summarizing, we proved that $A_* \nabla (T_0^{-1} f) \in L^2(\Omega, \mathbb{R}^d)$ and $-\operatorname{div} A_* \operatorname{grad} T_0^{-1} f = f$ for all $f \in H^{-1}(\Omega)$.

Step 2. Let us show that $A_* \in L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$.

For all nonnegative $\varphi \in C_0^\infty(\Omega)$, we have

$$\int_\omega \varphi (A_n \nabla w_n^\xi) \cdot (\nabla w_n^\xi) \mathrm{d}x \geqslant \alpha \int_\Omega \varphi | \nabla w_n^\xi|^2 \mathrm{d}x.$$

Applying to this inequality the arguments with the div-curl lemma similar to those of Step 1, we obtain

$$\langle A_*(x) \xi | \xi \rangle_{\mathbb{R}^d} \geqslant \alpha |\xi|^2 \;\text{ for all } \xi \in \mathbb{R}^d \text{ and almost all } x \in \Omega. \tag{6.3}$$

In particular, (6.3) implies that $(A_*(x))^{-1}$ exists for almost all $x \in \Omega$.
Analogously, for all nonnegative $\varphi \in C_0^\infty(\Omega)$,

$$\int_\omega \varphi (A_n \nabla w_n^\xi) \cdot (A_n^{-1} A_n \nabla w_n^\xi) \mathrm{d}x \geqslant \beta \int_\Omega \varphi |A_n \nabla w_n^\xi|^2 \mathrm{d}x,$$

and with the help of the div-curl lemma we get

$$\langle (A_*(x))^{-1} y | y \rangle_{\mathbb{R}^d} \geqslant \beta |y|^2 \;\text{ for all } y \in \mathbb{R}^d \text{ and for almost all } x \in \Omega.$$

Thus, $A_* \in L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$.

Step 3. The equality $f = -\operatorname{div} A_* \operatorname{grad} T_0^{-1} f$ for all $f \in H^{-1}(\Omega)$ implies that the homeomorphism $(-\operatorname{div}) A_* \operatorname{grad} : H_0^1(\Omega) \to H^{-1}(\Omega)$ is actually the operator T_0. Together with the weak convergences $L_n^{-1} \xrightarrow{w} T_0^{-1}$ and $A_n \operatorname{grad} L_n^{-1} \xrightarrow{w} Q_0 = A_* \operatorname{grad} T_0^{-1}$, this implies the H-convergence $A_n \xrightarrow{\mathrm{H}} A_*$.

In Step 2 of the proof Theorem 2.3, we passed to limits in the following estimates with nonnegative $\varphi \in C_0^\infty(\Omega)$,

$$\int_\omega \varphi(A_n \nabla w_n^\xi) \cdot (\nabla w_n^\xi)\mathrm{d}x \geqslant \alpha \int_\Omega \varphi|\nabla w_n^\xi|^2\mathrm{d}x,$$
$$\int_\omega \varphi(A_n \nabla w_n^\xi) \cdot (A_n^{-1} A_n \nabla w_n^\xi)\mathrm{d}x \geqslant \beta \int_\Omega \varphi|A_n \nabla w_n^\xi|^2\mathrm{d}x.$$

For the limit in the right hand side, we used implicitly the following lemma.

Lemma 6.2.
Let $u_n \rightharpoonup u$ in $L^2(\Omega, \mathbb{R}^d)$. Let $\varphi \in L^\infty(\Omega)$ be almost everywhere nonnegative. Then

$$\liminf_{n\to\infty} \int_\Omega \varphi|u_n|^2\mathrm{d}x \geqslant \int_\Omega \varphi|u|^2\mathrm{d}x.$$

Proof. The proof follows easily from the lower semicontinuity of L^2-norm applied to the weakly L^2-convergent sequence $\{\varphi^{1/2}u_n\}_{n\in\mathbb{N}}$. □

6.3 G-convergence for 2nd order elliptic equations.

Recall that

$$\mathcal{M}_{\alpha,\beta}^{\mathrm{sym}} = \{M \in \mathcal{M}_{\alpha,\beta} \ : \ M = M^\top\},$$

where $M^\top$ is the transpose of the matrix M, and that, for $A \in L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$, we have

$$L_A = -\operatorname{div} A \operatorname{grad} \in LC(H_0^1(\Omega), H^{-1}(\Omega)) \quad \text{and} \quad L_A \in \mathcal{E}_{\alpha,\beta}(H_0^1(\Omega)).$$

Definition 6.1.
A sequence $\{A_n\} \subset L^\infty(\Omega, \mathcal{M}_{\alpha,\beta}^{\mathrm{sym}})$ is said to be G-convergent to a G-limit

$$A_* \in L^\infty(\Omega, \mathcal{M}_{\alpha,\beta}^{\mathrm{sym}}) \quad \text{(with the notation } A_n \xrightarrow{\mathrm{G}} A_*)$$

if $L_{A_n} \xrightarrow{\mathrm{G}} L_{A_*}$ in the abstract sense of Definition 4.2 (i.e., in the sense that $L_{A_n}^{-1} \xrightarrow{w} L_{A_*}^{-1}$).

Theorem 6.1.
Let $\{A_n\} \subset L^\infty(\Omega, \mathcal{M}_{\alpha,\beta}^{\mathrm{sym}})$. Then

$$A_n \xrightarrow{\mathrm{G}} A_* \quad \Longleftrightarrow \quad A_n \xrightarrow{\mathrm{H}} A_*.$$

Proposition 6.1.
If $A_n \xrightarrow{\mathrm{H}} A_*$, then $A_n^\top \xrightarrow{\mathrm{H}} A_*^\top$.

We do not prove this theorem and this proposition in the course (for the proofs, see [A02, Section 1.3.2]).

Corollary 6.1.
Let $\{A_n\} \subset L^\infty(\Omega, \mathcal{M}_{\alpha,\beta}^{\mathrm{sym}})$. Then there exist $A_* \in L^\infty(\Omega, \mathcal{M}_{\alpha,\beta}^{\mathrm{sym}})$ and a subsequence $\{A_{n_k}\}$ such that $A_{n_k} \xrightarrow{\mathrm{G}} A_*$.

Proof. The proof follows easily from Theorems 2.3 and 6.1 combined with Proposition 6.1. ☐

Remark 6.1.
Let us summarize the results about the G-convergence of $\{A_n\} \subset L^\infty(\Omega, \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta})$.

(a) $L^\infty(\Omega, \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta})$ *is a compact subset of* $(L^\infty(\Omega, \mathcal{M}_{\alpha,\beta}), \rho_H)$.

(b) On $L^\infty(\Omega, \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta})$*, the H-convergence can be defined in a simpler, but equivalent way, as the G-convergence.*

Historically, the homogenization-convergence was first introduced on $L^\infty(\Omega, \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta})$ as G-convergence by Spagnolo in 1968. Later the H-convergence was introduced by Murat & Tartar in 1977-78 as an appropriately modified generalization for the generally non-symmetric case of $L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$ (see [MT78, A02, JKO12]).

7 Convergence of eigenvalues.

7.1 Preparational results for the eigenvalue convergence.

We always assume $\rho \in L^\infty(\Omega, [\rho_-, \rho_+])$ with $0 < \rho_- \leqslant \rho_+ < +\infty$, and $A \in L^\infty(\Omega, \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta})$. Recall that the eigenvalues of the operator $L_{A,\rho} = -\frac{1}{\rho} \operatorname{div} A \operatorname{grad}$ associated with the Dirichlet boundary condition $u|_{\partial\Omega} = 0$ can be numbered as $\{\lambda_k\}_{k\in\mathbb{N}} = \{\lambda_k(A,\rho)\}_{k\in\mathbb{N}}$ non-decreasingly taking multiplicities into account. Due to the Dirichlet boundary condition, the corresponding eigenfunctions u_k belong to the space $H^1_0(\Omega)$.

Theorem 7.1 (min-max principle)**.**
Let $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{K} = \mathbb{C}$. *Let* $\rho \in L^\infty(\Omega, [\rho_-, \rho_+])$ *and* $A \in L^\infty(\Omega, \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta})$. *Then*

$$\lambda_k = \min_{\substack{E \subset H^1_0(\Omega) \\ \dim E = k}} \max_{u \in E \setminus \{\mathbb{0}\}} \frac{\int_\Omega \langle A \nabla u | \nabla u \rangle_{\mathbb{K}^d} dx}{\int_\Omega |u|^2 \rho \mathrm{d}x},$$

where the minimum is taken over all subspaces with the finite dimension $\dim E$ *equal to* k.

Remark 7.1.

(a) This is one of the forms of Courant-Fischer-Weyl min-max principle, which is also related to the Cauchy interlacing theorem (see [K13, Section I.6.10]).

(b) In particular, Theorem 7.1 implies

$$\lambda_1 \geqslant \alpha \min_{u \neq \mathbb{0}} \frac{|u|^2_{H^1_0}}{\|u\|^2_{L^2_\rho}} > 0.$$

Remark 7.2.
Let $1 \leqslant p < q \leqslant +\infty$. *Let* p' *and* q' *are conjugate exponents defined as usual, e.g.,* $1/p + 1/p' = 1$.

(a) Recall that Ω is a domain in $\mathbb{R}^d$, and so, Ω is bounded. Hence, the Hölder inequality implies the continuous embedding

$$L^q(\Omega) \hookrightarrow L^p(\Omega).$$

(b) $u_n \overset{*}{\rightharpoonup} u$ *in* $L^q(\Omega) \quad \Rightarrow \quad u_n \rightharpoonup u$ *in* $L^p(\Omega)$.

(c) If $u_n \rightharpoonup u$ *in* $L^p(\Omega)$ *and* $\{u_n\}_{n\in\mathbb{N}}$ *is bounded in* $L^q(\Omega)$*, then* $u_n \overset{*}{\rightharpoonup} u$ *in* $L^q(\Omega)$*. This follows from the density of* $L^{p'}(\Omega)$ *in* $L^{q'}(\Omega)$ *and Proposition 3.1.*

In Exercise 11.1, we have seen that $u_n \to u$ in $L^2(\Omega)$ and $v_n \rightharpoonup v$ in $L^2(\Omega)$ implies $u_n v_n \rightharpoonup uv$ in $L^1(\Omega)$. This statement can be easily extended.

Exercise 7.1.
Let $1 \leqslant p < \infty$.

(a) If $u_n \to u$ in $L^p(\Omega)$ and $v_n \overset{*}{\rightharpoonup} v$ in $L^{p'}(\Omega)$, then $u_n v_n \rightharpoonup uv$ in $L^1(\Omega)$.

(b) If $u_n \to u$ in $L^p(\Omega)$ and $v_n \overset{*}{\rightharpoonup} v$ in $L^\infty(\Omega)$, then $u_n v_n \rightharpoonup uv$ in $L^p(\Omega)$.
Hint: use Remark 7.2 (c).

Remark 7.3.

(a) The compact embedding

$$H_0^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$$

implies the compact embedding

$$L^2(\Omega) \hookrightarrow\hookrightarrow H^{-1}(\Omega).$$

(b) The following implications are consequences of (a):

$$\begin{aligned} u_n \rightharpoonup u \quad &\text{in } H_0^1(\Omega) \quad \Rightarrow \quad u_n \to u \quad \text{in } L^2(\Omega); \\ v_n \rightharpoonup v \quad &\text{in } L^2(\Omega) \quad \Rightarrow \quad v_n \to v \quad \text{in } H^{-1}(\Omega). \end{aligned}$$

Proposition 7.1.
Let $\{A_n\}_{n\in\mathbb{N}} \subset L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$*. Let* $f_n \to f$ *in* $H^{-1}(\Omega)$ *and* $A_n \xrightarrow{\mathrm{H}} A_*$*. Then solutions* $u_n \in H_0^1(\Omega)$ *to* $L_{A_n} u = f_n$ *converge weakly in* $H_0^1(\Omega)$ *to the solution* $u_* \in H_0^1(\Omega)$ *to the problem* $L_{A_*} u = f$*.*

Proof. For any $v \in H^{-1}(\Omega)$, we can write

$$\begin{aligned} |\langle u_n - u_*, v\rangle| = |\langle L_{A_n}^{-1} f_n - L_{A_*}^{-1} f, v\rangle| &\leqslant |\langle L_{A_n}^{-1}(f_n - f), v\rangle| + |\langle (L_{A_n}^{-1} - L_{A_*}^{-1}) f, v\rangle| \\ &\leqslant \alpha^{-1} |f_n - f|_{H^{-1}} |v|_{H^{-1}} + |\langle (L_{A_n}^{-1} - L_{A_*}^{-1}) f, v\rangle|. \end{aligned}$$

The assumptions $f_n \to f$ in $H^{-1}(\Omega)$ and $A_n \xrightarrow{\mathrm{H}} A_*$ imply that the right hand side converges to 0, and so imply $u_n \rightharpoonup u_*$. □

7.2 Convergence of eigenvalues.

Let $\{A_n\}_{n\in\mathbb{N}} \subset L^\infty(\Omega, \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta})$ and $\{\rho_n\}_{n\in\mathbb{N}} \subset L^\infty(\Omega, [\rho_-, \rho_+])$.

Theorem 7.2.
Assume that $A_n \xrightarrow{\mathrm{H}} A_*$ *and* $\rho_b \xrightarrow{*} \rho_*$. *Then there exists an (increasing) subsequence of indices* $\{n_j\}_{j\in\mathbb{N}}$ *of* $\mathbb{N}$, *which we (with an abuse of notation) replace by indices* $\{j\}$, *such that, for every* $k \in \mathbb{N}$,

(a) $\lambda_k^j = \lambda_k(A_j, \rho_j)$ *converge to a certain number* $\widetilde{\lambda}_k \in \mathbb{R}_+$ *as* $j \to \infty$,

(b) there exists $L^2_{\rho_j}$*-normalized solutions* $u_k^j \in H_0^1(\Omega)$ *to* $L_{A_j} u = \lambda_k^j \rho_j u$ *such that*

$$u_k^j \rightharpoonup \widetilde{u}_k \text{ in } H_0^1(\Omega) \quad \text{and} \quad u_k^j \to \widetilde{u}_k \text{ in } L^2(\Omega) \quad \text{as } j \to \infty$$

for a certain $L^2_{\rho_*}$*-normalized solution* $\widetilde{u}_k \in H_0^1(\Omega)$ *to*

$$L_{A_*} u = \widetilde{\lambda}_k \rho_* u,$$

(c) and, additionally, the following orthonormal equalities hold for all $k, \ell \in \mathbb{N}$

$$\langle u_k^j | u_\ell^j \rangle_{L^2_{\rho_j}} = \delta_{k\ell}, \qquad \langle \widetilde{u}_k | \widetilde{u}_\ell \rangle_{L^2_{\rho_*}} = \delta_{k\ell},$$

where $\delta_{k\ell}$ *is the Kronecker-delta.*

Proof. The min-max principle (Theorem 7.1) implies that

$$\lambda_k^- \leqslant \lambda_k^n \leqslant \lambda_k^+, \tag{7.1}$$

where $\lambda_k^\mp$ are eigenvalues of multiples of the Laplacian $\frac{-\alpha}{\rho_+}\Delta$ and $\frac{-1}{\beta\rho_-}\Delta$.

Let $\{u_k^n\}_{k\in\mathbb{N}}$ be an $L^2_{\rho_n}$-orthonormal basis of eigenfunctions of $L_n = -\frac{1}{\rho_n} \operatorname{div} A_n \operatorname{grad}$ associated with $\{\lambda_k^n\}_{k\in\mathbb{N}}$. Then

$$\lambda_k^+ \geqslant \lambda_k^n = \langle A_n \nabla u_k^n | \nabla u_k^n \rangle_{L^2} \geqslant \alpha |u_k^n|^2_{H_0^1}.$$

So (7.1) implies that $\{u_k^n\}_{n\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$ and contains a weakly convergent in $H_0^1(\Omega)$ subsequence.
Passing to convergent subsequences, one can see that for every $K \in \mathbb{N} \cup \{\infty\}$ there exists a subsequence $\{n_j^K\}_{j\in\mathbb{N}} \subset \mathbb{N}$ such that for all $k \leqslant K$, as $j \to \infty$,

$$\lambda_k^{n_j^K} \to \widetilde{\lambda}_k, \tag{7.2}$$

$$u_k^{n_j^K} \rightharpoonup \widetilde{u}_k \text{ in } H_0^1(\Omega), \tag{7.3}$$

$$u_k^{n_j^K} \to \widetilde{u}_k \text{ in } L^2(\Omega), \tag{7.4}$$

for certain $\widetilde{\lambda}_k \in \mathbb{R}_+$ and $\widetilde{u}_k \in H_0^1(\Omega)$. The existence of such subsequencies $\{n_j^K\}_{j\in\mathbb{N}}$ is first proved iteratively for $K \in \mathbb{N}$, and then using a diagonal subsequence we obtain $\{n_j^\infty\}_{j\in\mathbb{N}}$ for $K = \infty$.

Let us put $n_j = n_j^{\infty}$ for all $j \in \mathbb{N}$. For this subsequence statements (7.2)-(7.4) are valid for all $k \in \mathbb{N}$.

It follows now from (7.4) and Exercise 7.1 (b) that $\langle \widetilde{u}_k | \widetilde{u}_\ell \rangle_{L^2_{\rho*}} = \delta_{k\ell}$ for all $k, \ell \in \mathbb{N}$, and, in particular, $\|\widetilde{u}_k\|_{L^2_{\rho*}} = 1$ for all k. Exercise 7.1 (b) implies also

$$\lambda_k^{n_j} \rho_{n_j} u_k^{n_j} \rightharpoonup \widetilde{\lambda}_k \rho_* \widetilde{u}_k \quad \text{in } L^2(\Omega),$$

and so

$$\lambda_k^{n_j} \rho_{n_j} u_k^{n_j} \to \widetilde{\lambda}_k \rho_* \widetilde{u}_k \quad \text{in } H^{-1}(\Omega),$$

as $j \to \infty$. Combining this with proposition 7.1 now implies that $u_k^{n_j} \rightharpoonup u_k$ in $H_0^1(\Omega)$, where u_k is the H_0^1-solution to

$$-\nabla \cdot (A_* \nabla u_k) = \widetilde{\lambda}_k \rho_* \widetilde{u}_k.$$

However, (7.3) implies $u_k^{n_j} \rightharpoonup \widetilde{u}_k$ in $H_0^1(\Omega)$. Thus, $u_k = \widetilde{u}_k$ is the $L^2_{\rho*}$-normilized eigenfunction of L_{A_*,ρ_*} associated with $\widetilde{\lambda}_k$. □

Let us summarize. We have converging sequences $\rho_n \overset{*}{\rightharpoonup} \rho_*$ and $A_n \overset{H}{\longrightarrow} A_*$. Furthermore, the eigenvalues of $L_n = \frac{1}{\rho_n} L_{A_n} = -\frac{1}{\rho_n} \operatorname{div} A_n \operatorname{grad}$ are numbered non-decreasingly taking into account multiplicities as $\{\lambda_k^n\}_{k\in\mathbb{N}} = \{\lambda_k(A_n, \rho_n)\}_{k\in\mathbb{N}}$, and $\{u_k^n\}_{k\in\mathbb{N}}$ is the corresponding $L^2_{\rho_n}$-orthonormal basis of eigenfunctions. We proved uniform (in n) bounds

$$0 < \lambda_k^- \leqslant \lambda_k^n \leqslant \lambda_k^+$$

and the existence of an (increasing) subsequence $\{n_j\}_{j\in\mathbb{N}} \subset \mathbb{N}$ with the properties that, as $j \to \infty$, there exist certain limits of subsequences

$$\lambda_k^{n_j} \to \widetilde{\lambda}_k, \quad u_k^{n_j} \rightharpoonup \widetilde{u}_k \text{ in } H_0^1(\Omega), \text{ and } \; u_k^{n_j} \to \widetilde{u}_k \text{ in } L^2(\Omega).$$

Exercise 12.1 (b) implies that

$$\rho_{n_j} u_k^{n_j} \rightharpoonup \rho_* \widetilde{u}_k \text{ in } L^2(\Omega). \tag{7.5}$$

Hence, passing to the limits for inner products of eigenfunctions of L_n, we get

$$\langle \rho_* \widetilde{u}_k | \widetilde{u}_\ell \rangle_{L^2} = \delta_{k,\ell}, \quad \text{where } \delta_{k\ell} \text{ is the Kronecker-delta.}$$

Formula (7.5) also implies the convergence of the right-hand sides $f_j := \lambda_k^{n_j} \rho_{n_j} u_k^{n_j}$ of the equations $L_{A_n} u_k^{n_j} = \lambda_k^{n_j} \rho_{n_j} u_k^{n_j}$,

$$\lambda_k^{n_j} \rho_{n_j} u_k^{n_j} \to \widetilde{\lambda}_k \rho_* \widetilde{u}_k \text{ in } H^{-1}(\Omega).$$

From this, using proposition 12.2, we obtain that $\{\widetilde{u}_k\}$ is an $L^2_{\rho*}$-orthonormal system of eigenfunctions of

$$L_* = \frac{1}{\rho_*} L_{A_*}.$$

This summarized Theorem 7.2 and its proof.

We have not proved yet that $\{\widetilde{u}_k\}$ is a complete system of eigenfunctions. That is, we still need to prove that, passing to a limit, we have not missed any eigenvalue (or multiplicity) and any eigenfunction of the orthonormal basis for L_*.

In what follows we simplify the notation for subsequences indexed by $\{n_j\}$ indexing them with n, i.e., we write A_n, λ_k^n, etc. instead of A_{n_j}, $\lambda_k^{n_j}$, etc.

Theorem 7.3. *(a)* $\widetilde{\lambda}_k = \lambda_k(A_*, \rho_*)$ *for all* $k \in \mathbb{N}$.

(b) $\{\widetilde{u}_k\}_{k\in\mathbb{N}}$ *is an* $L^2_{\rho_*}$*-orthonormal system of eigenfunctions of* L_*.

Proof. Let $\{\lambda_k\}_{k\in\mathbb{N}} = \{\lambda_k(A_*, \rho_*)\}_{k\in\mathbb{N}}$ be the eigenvalues of L_* numbered non-decreasingly taking into account multiplicities.

Step 1. Taking into account Theorem 7.2, statements (a) and (b) of Theorem 7.3 are equivalent. Besides, in order to prove statement (a) it is enough to prove that

$$\widetilde{\lambda}_k \leqslant \lambda_k \quad \forall k \in \mathbb{N}. \tag{7.6}$$

Step 2. Let us prove (7.6) by *reductio ad absurdum.*

Assume that (7.6) is not valid. Then there exists a smallest $k_0 \in \mathbb{N}$ such that

$$\lambda_{k_0} < \widetilde{\lambda}_{k_0}$$

(i.e., this is the first time when we jumped over a certain actual eigenvalue of L_*).
Note that, for eigenvalues with smaller indices $k < k_0$, we have $\widetilde{\lambda}_k = \lambda_k$, and that $\{\widetilde{u}_k\}_{k=1}^{k_0-1}$ is the corresponding $L^2_{\rho_*}$-orthonormal system of eigenfunctions of L_*. Moreover, there exists an $L^2_{\rho_*}$-normalized eigenfunction u_{k_0} of L_* that corresponds to λ_{k_0} and is orthogonal to $\operatorname{span}\{\widetilde{u}_k\}_{k=1}^{k_0-1}$.
Put $w_n = L_{A_n}^{-1}(\lambda_{k_0}\rho_* u_{k_0})$ for all n. The H-convergence $A_n \xrightarrow{\mathrm{H}} A_*$ yields

$$w_n \rightharpoonup L_{A_8}^{-1}(\lambda_{k_0}\rho_* u_{k_0}) = u_{k_0} \text{ in } H_0^1(\Omega). \tag{7.7}$$

This implies

$$w_n \to u_{k_0} \text{ in } L^2(\Omega), \tag{7.8}$$

$$\rho_n w_n \rightharpoonup \rho_* u_{k_0} \text{ in } L^2(\Omega). \tag{7.9}$$

Then, for $k < k_0$, we have

$$\langle A_n \nabla w_n | \nabla u_k^n \rangle_{L^2} = \langle L_{A_n} w_n | u_k^n \rangle_{L^2} = \langle \lambda_{k_0}\rho_* u_{k_0} | u_k^n \rangle_{L^2} \to \lambda_{k_0} \langle \rho_* u_{k_0} | \widetilde{u}_k \rangle_{L^2} = 0. \tag{7.10}$$

The convergencies (7.8) and $u_k^n \to \widetilde{u}_k$ in L^2 (and so in $L^2_{\rho_*}$) imply that the determinant of the $L^2_{\rho_*}$-Gram-matrix of the system $S_n = \{u_1^n, \ldots, u_{k_0-1}^n, w_n\}$ converges to 1 as $n \to \infty$. So, for large enough n the system of function S_n is linearly independent. Hence, the min-max principle implies

$$\lambda_{k_0}^n \leqslant \max_{\substack{v \in \operatorname{span} S_n \\ v \neq \mathbb{0}}} \frac{\langle A_n \nabla v | \nabla v \rangle_{L^2}}{\langle \rho_n v | v \rangle_{L^2}}. \tag{7.11}$$

Let $v_n = \sum_{1\leqslant k<k_0} c_k^n u_k^n + c_{k_0}^n w_n$ be a maximizer corresponding to the maximum in (7.11), which can be chosen such that $\sum_{1\leqslant k\leqslant k_0}(c_k^n)^2 = 1$. Passing to (sub)subsequences if necessary we can ensure that $c_k^n \to c_k$ for all $1 \leqslant k \leqslant k_0$ (here we keep indexing the subsequences by n).
The limit of $\langle A_n \nabla v | \nabla v\rangle_{L^2}$ can be calculated using (7.10) and the formulae

$$\langle A_n \nabla u_k^n | \nabla u_k^n\rangle_{L^2} = \langle \lambda_k^n \rho_n u_k^n | u_k^n\rangle_{L^2} = \lambda_k^n \delta_{k\ell},$$
$$\langle A_n \nabla w_n | \nabla w_n\rangle_{L^2} = \lambda_{k_0}\langle \rho_* u_{k_0} | w_n\rangle_{L^2} \to \lambda_{k_0}.$$

Namely,

$$\lim_{n\to\infty} \langle A_n \nabla u_k^n | \nabla u_k^n\rangle_{L^2} = \sum_{1\leqslant k<k_0} \widetilde{\lambda}_k c_k^2 + \lambda_{k_0} c_{k_0}^2.$$

Similarly,

$$\lim_{n\to\infty} \langle \rho_n v_n | v_n\rangle_{L^2} = \sum_{1\leqslant k\leqslant k_0} c_k^2 = 1.$$

Thus, (7.11) implies

$$\widetilde{\lambda}_{k_0} \leqslant \lim \frac{\langle A_n \nabla v_n | \nabla v_n\rangle_{L^2}}{\langle \rho_n v_n | v_n\rangle_{L^2}} \leqslant \lambda_{k_0},$$

which contradicts the initial assumption of Step 2 that $\lambda_{k_0} < \widetilde{\lambda}_{k_0}$. This proves (7.6), and due to Step 1 concludes the proof of the theorem. □

Returning to the original notation with indexing of subsequences by n_j, we see from Theorems 7.2 and (7.3) that there exists a subsequence $\{n_j\}$ such that

$$\lambda_k^n \to \lambda_k(A_*, \rho_*) \qquad \forall k.$$

However, this argument can be applied to any subsequence $\{\lambda_k^{n_j}\}$. This means that every subsequence $\{\lambda_k^{n_j}\}$ contains a subsubsequence converging to $\lambda_k(A_*, \rho_*)$. In other words, $\lambda_k(A_*, \rho_*)$ is the only partial limit of $\{\lambda_k^n\}$.

Thus, we proved the following theorem.

Theorem 7.4.
If $\rho_n \xrightarrow{*} \rho_*$ and $A_n \xrightarrow{\mathrm{H}} A_*$, then $\lambda_k(A_n, \rho_n) \to \lambda_k(A_*, \rho_*)$.

Theorems 7.2 and 7.4 together imply the complete theorem on convergence of eigenvalues (Theorem 2.2).

Remark 7.4.
The theorems of this type on convergence of eigenvalues are originated from [BM76, K79] for the case $\rho_n \equiv 1$. The proof above is essentially a combination of arguments of [K79] and [A02, Section 1.3.3]. A proof of somewhat stronger result via a general abstract theory of eigenvalue convergence can be found in [JKO12].

8 Lamination, periodic homogenization, and G_θ-closure.

8.1 H-limits of layered structures.

Assume that matrix-valued functions $A_n(x) = A_n(x_1, x_2, \ldots, x_d)$, where x is in a domain $\Omega \subset \mathbb{R}^d$, have the form

$$A_n(x) = a_n(x_1) I_{\mathbb{R}^d}, \qquad x \in \Omega,$$

with certain scalar functions $a_n \in L^\infty(\mathbb{R}, [\alpha, \beta^{-1}])$, $n \in \mathbb{N}$.

The following theorem follows from the results of Murat & Tartar [MT78, Section 4].

Theorem 8.1.
The H-convergence $a_n(x_1) I_{\mathbb{R}^d} \xrightarrow{\mathrm{H}} A_*$ to a certain $A_* = (A_*^{j,k})_{j,k=1}^d \in L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$ is equivalent to the combination of the following conditions

(a) $A_*^{j,j}(x) = a_*^{j,j}(x_1)$ *for certain* $a_*^{j,j} \in L^\infty(\mathbb{R}, [\alpha, \beta^{-1})$, $1 \leqslant j \leqslant d$*; while, for non-diagonal entries,* $A_*^{j,k} \equiv 0$ *for all* $j \neq k$*;*

(b) $\dfrac{1}{a_n} \xrightarrow{*} \dfrac{1}{a_*^{1,1}}$ *in* $L^\infty(\Omega_1)$*, where* Ω_1 *is the orthogonal projection of* Ω *on the* x_1*-axis;*

(c) $a_n \xrightarrow{*} a_*^{j,j}$ *in* $L^\infty(\Omega_1)$ *for* $2 \leqslant j \leqslant d$.

This theorem is without proof in this course (for the proof see [A02, Section 1.3.5]).

Remark 8.1.

(a) We see from Theorem 8.1 that scalar matrix-functions A_n (which physically correspond to isotropic materials) can have a non-scalar H-limit A_* (i.e., the homogenized material can be anisotropic, which means that it has different effective properties in different directions). In the 3-D case, this effect appears in Theorem 8.1 whenever

$$(\text{w*- }\lim a_n^{-1})^{-1} = a_*^{1,1} \neq a_*^{2,2} = a_*^{3,3} = \text{w*- }\lim a_n.$$

It is an easy exercise to construct a sequence $\{a_n\}_{n\in\mathbb{N}}$ that leads to this effect.

(b) In the 3-D context of the conductance, Theorem 8.1 has an analogy with the elementary rules of school physics for the total resistance for several resistors connected in series or in parallel.

If we interpret $a_n(\cdot)$ as a function describing the conductivity of the layers, and interpret $r_n = \frac{1}{a_n}$ as the function describing the spatially varying resistivity, then in the direction x_1, the layers are placed in series. Therefore, the mean resistivity in the direction x_1 is calculated via the arithmetic mean and corresponds to weak--convergence of resistivities $r_n = \frac{1}{a_n}$. As it was discussed after Remark 3.3, weak and weak-* L^p-convergencies can be interpreted as convergencies 'in arithmetic average' if $1 < p \leqslant \infty$. That is why, the rule $\dfrac{1}{\text{w*- }\lim 1/a_n} = a_*^{1,1}$ for the limiting conductivity $a_*^{1,1}$ in the direction x_1 can be interpreted as a convergence of a_n in the sense of 'harmonic average'. The function $\underline{a} = (\text{w*- }\lim a_n^{-1})^{-1}$ is called in [A02] the harmonic mean (corresponding to the sequence $\{a_n\}$).*

From the point of view of directions x_2 and x_3, the layers of $A_n(x) = a_n(x_1)I_{\mathbb{R}^d}$ are placed in parallel. This corresponds to the computation of the average resistivity via the harmonic mean, and so to the computation of the average conductivity via the arithmetic mean. The conductivities in the directions x_2 and x_3 are given via the convergence 'in arithmetic average', i.e., as the weak- L^∞-limit*

$$a_*^{2,2} = a_*^{3,3} = \overline{a} := \text{w*-}\lim a_n.$$

The function $\overline{a} := \text{w*-}\lim a_n$ is interpreted here as the arithmetic mean corresponding to the sequence $\{a_n\}$.

8.2 Laminates and locality of H-convergence.

Exercise 8.1.
Let $f \in L^\infty(\mathbb{R})$ be a T-periodic function, i.e., $f(t+T) = f(t)$ for almost all $t \in \mathbb{R}$, where $T > 0$. Then, for every measurable subset S of $\mathbb{R}$, the family of functions $f_\epsilon(t) = f(t/\epsilon)$, $t \in \mathbb{R}$, $\epsilon > 0$, have the $L^\infty(S)$-weak-* limit as $\epsilon \to 0$ equal to the constant function with the value $\frac{1}{T}\int_0^T f(s)\mathrm{d}s$, i.e.,

$$f_\epsilon \xrightarrow{*} \left(\frac{1}{T}\int_0^T f(s)\mathrm{d}s\right)\mathbb{1} \quad \text{in} \quad L^\infty(S).$$

Here, in comparison to the notation that we used before, $\epsilon \to 0$ corresponds to $\frac{1}{n} \to 0$ for $n \in \mathbb{N}$ going to $+\infty$.

Remark 8.2.

(a) *Let $a \in L^\infty(\mathbb{R}, [\alpha, \beta^{-1}])$ be a T-periodic function. Theorem 8.1 implies that, in every domain $\Omega \subset \mathbb{R}^d$, the family $A_\epsilon(x) = a(x_1/\epsilon)I_{\mathbb{R}^d}$ H-converges as $\epsilon \to 0$ to the following constant matrix valued function*

$$\mathbb{1}\begin{pmatrix} \underline{a_\epsilon} & 0 & \dots & 0 \\ 0 & \overline{a_\epsilon} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \overline{a_\epsilon} \end{pmatrix},$$

where $\underline{a_\epsilon}$ denote the 'harmonic mean' of the homogenization theory associated with the family $\{a_\epsilon(t)\}_{\epsilon>0} = \{a_\epsilon(t/\epsilon)\}_{\epsilon>0}$

$$\underline{a_\epsilon} := \frac{1}{\text{w*-}\lim_{\epsilon\to 0}\frac{1}{a_\epsilon}}$$

and $\overline{a_\epsilon}$ denotes the 'arithmetic mean'

$$\overline{a_\epsilon} := \text{w*-}\lim_{\epsilon\to 0} a_\epsilon.$$

The limits can be understood in the sense of $L^\infty(\mathbb{R})$. Due to Exercise 8.1, we have now an explicit formulae

$$\underline{a_\epsilon} = \left(\frac{1}{T}\int_0^T \frac{1}{a(t)}\mathrm{d}t\right)^{-1}, \qquad \overline{a_\epsilon} = \frac{1}{T}\int_0^T a(t)\mathrm{d}t.$$

(a) *Periodic piecewise-constant structures are called laminates. Statement (a) allows one to find H-limits of laminates of isotropic media. The following theorem works also for laminates of anisotropic media.*

Theorem 8.2 ([MT78], see also [A02])**.**
Assume that $\{A_n\}_{n\in\mathbb{N}} \subset L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$ *be such that, for all* n, $A_n(\cdot) = (A_n(\cdot))_{i,j=1}^d$ *depend only on the variable* x_1. *Then*

$$A_n(\cdot) \xrightarrow{\mathrm{H}} A_*(\cdot) = (A_*^{i,j})_{i,j=1}^d$$

*if and only if the combination of the following weak-** $L^\infty(\Omega)$*-convergencies takes place:*

(a) $\dfrac{1}{A_n^{1,1}} \overset{*}{\rightharpoonup} \dfrac{1}{A_*^{1,1}}$;

(b) *for* $2 \leqslant j \leqslant d$,

$$\frac{A_n^{1,j}}{A_n^{1,1}} \overset{*}{\rightharpoonup} \frac{A_*^{1,j}}{A_*^{1,1}};$$

(c) *for* $2 \leqslant i \leqslant d$,

$$\frac{A_n^{i,1}}{A_n^{1,1}} \overset{*}{\rightharpoonup} \frac{A_*^{i,1}}{A_*^{1,1}};$$

(d) *for all* $2 \leqslant i \leqslant d$ *and* $2 \leqslant i \leqslant d$,

$$A_n^{i,j} - \frac{A_n^{1,j} A_n^{i,1}}{A_n^{1,1}} \overset{*}{\rightharpoonup} A_*^{i,j} - \frac{A_*^{1,j} A_*^{i,1}}{A_*^{1,1}}.$$

This theorem remains without proof in this course (for the proof see [A02]).

Remark 8.3.

(a) *Note that* $A_*(\cdot)$ *depends only on* x_1.

(b) *In the case of an isotropic medium* $A_n(x) = a_n(x_1) I_{\mathbb{R}^d}$, *Theorem 8.2 implies immediately Theorem 8.1.*

Remark 8.4.
Lamination limits can be iterated, typically, in different directions. This is possible due to the locality of H-convergence, which is rigorously formalized in the next theorem.

Theorem 8.3 (locality of H-convergence, [MT78, T85], see also [A02])**.**
Assume that $\{A_n\}_{n\in\mathbb{N}} \subset L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$ *and* $A_* \in L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$. *Then the following statements are equivalent:*

(a) $A_n \xrightarrow{\mathrm{H}} A_*$ *in* Ω.

(b) $A_n \xrightarrow{\mathrm{H}} A_*$ *in every domain* ω *such that* $\overline{\omega} \subset \Omega$.

(c) For every two sequences $\{E_n\}_{n\in\mathbb{N}}, \{D_n\}_{n\in\mathbb{N}} \subset L^2(\Omega, \mathbb{R}^3)$ the combination of the following properties:

$$D_n = A_n E_n \text{ almost everywhere in } \Omega \text{ for all } n \in \mathbb{N}, \tag{8.1}$$

$$D_n \rightharpoonup D_*, \quad E_n \rightharpoonup E_* \text{ in } L^2(\Omega, \mathbb{R}^d), \tag{8.2}$$

$$\{\nabla \times E_n\}_{n\in\mathbb{N}} \text{ is a relatively compact subset of } H^{-1}(\Omega, \mathbb{R}^d), \tag{8.3}$$

$$\{\nabla \cdot D_n\}_{n\in\mathbb{N}} \text{ is a relatively compact subsets of } H^{-1}(\Omega), \tag{8.4}$$

implies

$$D_* = A_* E_* \qquad \text{almost everywhere in } \Omega.$$

We do not prove this theorem in the course. Statement (b) is the original definition of H-convergence in [MT78]. Statement (a) is a simplified formulation of [A02]. The equivalence of (b) and (c) takes its origin in [T85].

8.3 Periodic homogenization

An explicit PDE formula for H-limit is available for the general periodic homogenization.

Following [A02], let us take the unit cube $Y = (0,1)^d \subset \mathbb{R}^d$ as a periodic cell. One can identify Y with the unit d-dimensional torus. By $L^p_\#(Y)$, the space of L^p-functions on the unit torus is denoted. Then also the following identification is possible

$$L^p_\#(Y) = \{f \in L^p_{\text{loc}}(\mathbb{R}^d) : \quad f \text{ is } Y\text{-periodic }\}$$

and, as the norm in $L^p_\#(Y)$, the norm of $L^p(Y)$ is taken. Using a similar identification, we define

$$H^1_\#(Y) = \{f \in H^1_{\text{loc}}(\mathbb{R}^d) : \ f \text{ is } Y\text{-periodic }\}$$

with the norm $\|\cdot\|_{H^1(Y)}$.

The quotient space (factor-space) $H^1_\#(Y)/\mathbb{R}$ is defined as the space of classes of $H^1_\#(Y)$-functions equal up to an additive constant.

Let $A \in L^\infty_\#(Y, \mathcal{M}_{\alpha,\beta})$. In a domain $\Omega \subset \mathbb{R}^d$, we consider the family of matrix functions

$$A_\epsilon(x) = A(x/\epsilon), \qquad x \in \Omega, \tag{8.5}$$

indexed by $\epsilon > 0$ (only sufficiently small values of ϵ are important here).

Theorem 8.4 (H-limit for periodic homogenization)**.**
The family $\{A_\epsilon\}_{\epsilon>0}$ defined by (8.5) H-converges as $\epsilon \to 0$ to a constant matrix-valued function $\mathbb{1}A_*$. The matrix $A_* = (A_*^{i,j})_{i,j=1}^d \in \mathcal{M}_{\alpha,\beta}$ can be calculated by the formula

$$A_*^{i,j} = \int_Y \langle\, A(y)(e_i + \nabla w_i(y)) \mid e_j + \nabla w_j(y)\, \rangle_{\mathbb{R}^d} \mathrm{d}y,$$

where $\{e_i\}_{i=1}^d$ is the standard orthonormal basis in $\mathbb{R}^d$ and w_i, $i = 1, \ldots, d$, are the unique solutions to the periodic problem

$$-\operatorname{div}\left[A(y)(e_i + \nabla w_i(y))\right] = 0, \qquad y \in Y,$$
$$w_i \in H^1_\#(Y)/\mathbb{R}.$$

We do not prove this theorem in the course.

8.4 G- and G_θ-closure problems for two-phase composites.

Let $0 < \alpha = \alpha_1 < \alpha_2 = \beta^{-1}$. Let $M_1, M_2 \in \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta} = \mathcal{M}^{\mathrm{sym}}_{\alpha_1,\alpha_2^{-1}}$ be two (constant) symmetric matrices. We assume that M_1 and M_2 correspond to two homogeneous media I and II.

Example 8.1.
Our main example will be a pair of different isotropic media $M_1 = \alpha_1 I_{\mathbb{R}^d}$ and $M_2 = \alpha_2 I_{\mathbb{R}^d}$.

Let us consider several problems concerning the description of H-limits for mixtures of materials M_1 and M_2.

Problem 8.1 (G-closure problem).
Consider the family $L^\infty(\Omega, \{0,1\})$ of all indicator-functions $\chi = \chi_\omega$ for all possible measurable subsets $\omega \subset \Omega$. For each such χ, we define the composite structure

$$A^\chi(x) = \chi(x)M_1 + (1-\chi(x))M_2 \in L^\infty(\Omega, \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta}).$$

The family of all such structures is denoted by

$$\mathbb{F}^{M_1,M_2,\Omega} = \{A^\chi : \ \chi \in L^\infty(\Omega, \{0,1\})\}.$$

The problem is to find the H-closure of $\mathbb{F}^{M_1,M_2,\Omega}$, which we denote by

$$\mathbb{F} := \overline{\mathbb{F}^{M_1,M_2,\Omega}}^H.$$

Problem 8.1′ (periodic G-closure problem).
For $\chi \in L^\infty_\#(Y, \{0,1\})$, let us consider a Y-periodic structure on $\mathbb{R}^d$

$$A^\chi_\#(y) = \chi(y)M_1 + (1-\chi(y))M_2 \in L^\infty_\#(Y, \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta}).$$

Then, for an arbitrary domain $\Omega \subset \mathbb{R}^d$,

$$A^\chi_\#\left(\frac{x}{\epsilon}\right) \xrightarrow{\mathrm{H}} \mathbb{1}M_\chi \tag{8.6}$$

with a constant matrix $M_\chi \in \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta}$, which can be determined by the fomulae of Theorem 8.4. The problem is to characterize the set of all possible H-limits M_χ for such periodic homogenization, i.e., to find

$$\mathbb{P} = \{M \in \mathcal{M}^{\mathrm{sym}}_{\alpha,\beta} : \ M = M_\chi \ \text{ for a certain } \chi \in L^\infty_\#(Y, \{0,1\})\}.$$

It occurs that Problem 8.1 can be reduced in a certain sense to Problem 8.1′. However, it is better to understand this connection via two other useful problems with fixed ratios of materials, i.e., via G_θ-closure problems (see [A02] and [C00]).

Proposition 8.1.
The weak- L^∞-closure of $L^\infty(\Omega, \{0,1\})$ is the closed L^∞-ball $L^\infty(\Omega, [0,1])$.*

In the course, we leave this proposition without a proof. Proposition 8.1 follows, e.g., from the Krein-Milman theorem: *a compact convex subset of a Hausdorff locally convex linear topological space is equal to the closed convex hull of its extreme points.*

Problem 8.2 (G_θ-closure problem).
Let $\theta \in L^\infty(\Omega, [0,1])$. The problem is to find the family $\mathbb{F}_\theta$ of all possible H-limits A_* of families $\{A^{\chi_\epsilon}\}_{\epsilon>0}$ satisfying $\chi_\epsilon \overset{*}{\rightharpoonup} \theta$ in $L^\infty(\Omega)$ as $\epsilon \to 0$, where all $\chi_\epsilon \in L^\infty(\Omega, \{0,1\})$.

The meaning of the function θ is that $\theta(x)$ is the "local proportion" of materials M_1 and M_2 at $x \in \Omega$.

The G-closure problem is obviously reduced to the G_θ-closure problem.

Proposition 8.2.

$$\mathbb{F} = \bigcup_{\theta \in L^\infty(\Omega,[0,1])} \mathbb{F}_\theta.$$

The proof is a simple exercise.

Problem 8.2′ (periodic G_θ-closure problem).
Let $\theta \in [0,1]$ be a fixed number. Consider $\chi \in L^\infty_\#(Y, \{0,1\})$ additionally satisfying the assumption

$$\int_Y \chi(y)\mathrm{d}y = \theta. \tag{8.7}$$

The problem is to find the set $\mathbb{P}_\theta$ of all H-limits $\mathbb{1}M_\chi$ in the sense of (8.6) produced by $\chi \in L^\infty_\#(Y, \{0,1\})$ such that the ratio-assumption (8.7) holds, i.e., to find

$$\mathbb{P}_\theta = \{M_\chi : \ \chi \in L^\infty_\#(Y, 0, 1) \text{ satisfies (8.7)}\}.$$

The G_θ-closure problem can be essentially reduced to the periodic G_θ-closure problem in the following way.

Theorem 8.5 ([T85]).
For a constant $\theta \in [0,1]$, we denote (following [A02])

$$G_\theta := \overline{\mathbb{P}_\theta},$$

where the closure is taken in the usual sense of the space $\mathbb{R}^{d\times d}$ of matrices. Then, for an arbitrary function $\theta \in L^\infty(\Omega, [0,1])$, the following formula holds

$$\mathbb{F}_\theta = \{A_* \in L^\infty(\Omega, \mathcal{M}^{\text{sym}}_{\alpha,\beta}) : \ A_*(x) \in G_{\theta(x)} \ \ \textit{almost everywhere in } \Omega\}.$$

The discussion of this theorem can be found in [A02] together with references to a more general result going back to Dal Maso & Kohn and to its proof by Raitums.

Recall that $0 < \alpha_1 < \alpha_2$. In the case where $M_1 = \alpha_1 I_{\mathbb{R}^d}$ and $M_2 = \alpha_2 I_{\mathbb{R}^d}$, an explicit description of the family $G_\theta = \overline{\mathbb{P}_\theta}$ was found independently by Murat & Tartar [M83, MT85, T85] and Lurie & Cherkaev [LC82, LC84, LC86] (see also [A02, Theorem 2.2.13] and the remarks afterwards).

Theorem 8.6 ([MT85, T85, LC86]).
Let $\theta \in [0,1]$ be a constant. Let $M_1 = \alpha_1 I_{\mathbb{R}^d}$ and $M_2 = \alpha_2 I_{\mathbb{R}^d}$. Then the set G_θ defined in Theorem 8.5 is the convex set of all symmetric $d \times d$-matrices M such that their eigenvalues $\lambda_1, \ldots, \lambda_d$ (numbered taking their multiplicities into account) satisfy the following inequalities:

$$\lambda_\theta^- \leqslant \lambda_j \leqslant \lambda_\theta^+, \qquad j = 1, \ldots, d; \tag{8.8}$$

$$\sum_{j=1}^{d} \frac{1}{\lambda_j - \alpha_1} \leqslant \frac{1}{\lambda_\theta^- - \alpha_1} + \frac{d-1}{\lambda_\theta^+ - \alpha_1}; \tag{8.9}$$

$$\sum_{j=1}^{d} \frac{1}{\alpha_2 - \lambda_j} \leqslant \frac{1}{\alpha_2 - \lambda_\theta^-} + \frac{d-1}{\alpha_2 - \lambda_\theta^+}, \tag{8.10}$$

where λ_θ^- and λ_θ^+ are weighted harmonic and arithmetic means of α_1 and α_2:

$$\lambda_\theta^- = \left(\theta \frac{1}{\alpha_1} + (1-\theta)\frac{1}{\alpha_2}\right)^{-1}, \qquad \lambda_\theta^+ = \theta\alpha_1 + (1-\theta)\alpha_2.$$

This theorem is without proof in this course. A proof and additional remarks about the description of the set G_θ can be found in [A02, Section 2.2].

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