Random operators and resonances

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Talk 1. Spectra of conservative systems modeled by random Schrödinger operators. Continuation resonances in open systems.

Illia M. Karabash

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ikarabas@uni-bonn.de

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1 Spectrum, resolvent set, and eigenvalues.

Let X be a (complex) Hilbert space with the inner product $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_X$. Let $A : \operatorname{dom} A \subseteq X \to X$ be a (linear) operator in X with a domain (of definition) $\operatorname{dom} A$, which is assumed to be a linear subspace of X.

By
$$\ker A := \{f \in \operatorname{dom} A : \ Af = 0\},$$
we denote the kernel of A. By $\mathcal{L}(X)$, we denote the Banach space of all bounded operators $A : X \to X$ (with $\operatorname{dom} A = X$) assuming that $\mathcal{L}(X)$ is equipped with the standard operator norm.

Definition 1.1 (resolvent set)**.**
The resolvent set $\rho(A)$ of A is the set of points $k \in \mathbb{C}$ such that (s.t.) the following three conditions hold true:

(a) $A - k$ is invertible, i.e., $\ker(A - kI) = \{0\}$;

(b) $(A - k) \operatorname{dom} A = X$, i.e., the operator $A - k$ maps $\operatorname{dom}(A - k) := \operatorname{dom} A$ onto the whole X;

(c) $(A - k)^{-1} \in \mathcal{L}(X)$.

The $\mathcal{L}(X)$-valued function $k \mapsto (A - k)^{-1}$ is analytic on $\rho(A)$ and is called the resolvent of A.

Definition 1.2 (spectrum)**.**
The set $\sigma(A) := \mathbb{C} \backslash \rho(A)$ is called the spectrum of A.

Proposition 1.1.
The set $\rho(A)$ is open. Consequently, $\sigma(A)$ is closed.

Definition 1.3.

(a) A number $k \in \mathbb{C}$ is called an eigenvalue of A if $\ker(A - k) \neq \{0\}$. The elements of $\ker(A-k)\backslash\{0\}$ are called eigenvectors of A associated with the eigenvalue k.

(b) Following [K],

$$\text{we denote the set of all eigenvalues of } A \text{ by } \varepsilon(A).$$

(c) The geometric multiplicity of an eigenvalue $k \in \sigma(A)$ is the dimensionality $\dim\ker(A-k)$ of the (linear) subspace $\ker(A-k)$ of the associated eigenvectors. The algebraic multiplicity of an eigenvalue k is $\sup_{n\in\mathbb{N}} \dim\ker(A-k)^n$.

(d) An eigenvalue k_0 of A is called an isolated eigenvalue if k_0 is an isolated point of $\sigma(A)$.

Obviously,

$$\varepsilon(A) \subseteq \sigma(A).$$

It is an easy exercise to construct an operator T in $\ell^2(\mathbb{N})$ such that

$$\varepsilon(T) \neq \overline{\varepsilon(T)} = \sigma(T),$$

where $\overline{S}$ is a closure of a set S.

Remark 1.1.
Often, $\varepsilon(A)$ is called the point spectrum and other notations, like $\sigma_{\mathrm{p}}(A)$ or $\sigma_{\mathrm{pp}}(A)$ are used for it. The reason why [K] introduces a bit nonstandard notation $\varepsilon(A)$ for the set of eigenvalues is that the notation $\sigma_{\mathrm{pp}}(A)$ is especially ambiguous. For example, $\sigma_{\mathrm{pp}}(A) = \varepsilon(A)$ in [RS1, KM82], but $\sigma_{\mathrm{pp}}(A)$ equals to the closure $\overline{\varepsilon(A)}$ and is called a pure point spectrum in [K] (and in many other sources).

2 Selfadjoint operators and their spectra

Let X_1 and X_2 be Hilbert spaces.

Definition 2.1 (adjoint operator)**.**
Assume that $T : \operatorname{dom} T \subseteq X_1 \to X_2$ is densely defined, i.e., $\overline{\operatorname{dom} T} = X_1$. Then the adjoint operator $T^* : \operatorname{dom} T^* \subseteq X_2 \to X_1$ is defined in the following way:

(a) $\operatorname{dom} T^*$ consists of all $v \in X_2$ with the property that there exists $f_v \in X_1$ s.t.

$$\langle v|Tu\rangle_{X_2} = \langle f_v|u\rangle_{X_1} \qquad \forall u \in \operatorname{dom} A; \tag{2.1}$$

(b) $T^* v = f_v$.

The assumption $\overline{\operatorname{dom} T} = X_1$ in Definition 2.1 ensures that f_v in (2.1) is unique.

Definition 2.2 (selfadjoint and nonnegative operators)**.**(a) An operator A is called selfadjoint if $A = A^*$ (in particular, this equality includes the equality of domains $\operatorname{dom} A = \operatorname{dom} A^*$).

(b) An operator $A : \operatorname{dom} A \subseteq X \to X$ is called nonnegative if $\langle u, Au\rangle \geqslant 0$ for all $u \in \operatorname{dom} A$. In this case, we write $A \geqslant 0$.

(c) An operator A is called semibounded (from below) if $A - cI \geqslant 0$ for a certain constant $c \in \mathbb{R}$. One writes $A \geqslant c$ in this case.

Under the assumption $A \in \mathcal{L}(X)$, $\quad A = A^* \quad \Leftrightarrow$

$$\langle v, Au \rangle_X = \langle Av, u \rangle_X \qquad \forall u, v \in \operatorname{dom} A = X.$$

For densely defined unbounded operators, this equivalence is not true.

Proposition 2.1.
Let $A = A^*$. Then:

(a) $\varepsilon(A) \subseteq \sigma(A) \subseteq \mathbb{R}$

(b) If $A \geqslant c$ for a constant $c \in \mathbb{R}$, then $\sigma(A) \subseteq [c, +\infty)$.

(c) Geometric and algebraic multiplicities of an eigenvalue $k \in \varepsilon(A)$ are equal (for selfadjoint operators, one can speak simply about multiplicities).

(d) Let $k_1, k_2 \in \varepsilon(A)$ and $k_1 \neq k_2$. Let $u_j \in \ker(A - k_j)$, $j = 1, 2$. Then $u_1 \perp u_2$ (i.e., $\langle u_1, u_2 \rangle_X = 0$).

(e) Assume additionally that X is separable. Then $\varepsilon(A)$ is at most countable.

Definition 2.3 (closed operator, e.g., [Kato])**.**
An operator $T : \operatorname{dom} A \subseteq X_1 \to X_2$ is called closed if its graph

$$\operatorname{Gr} T = \{ \{u, Tu\} \ : \ u \in \operatorname{dom} T\} \text{ is a closed subspace of the orthogonal sum } X_1 \oplus X_2.$$

Theorem 2.1 (von Neumann, e.g., [Kato])**.**
Let $T : \operatorname{dom} T \subseteq X_1 \to X_2$ be densely defined and closed. Then T^*T is a nonnegative selfadjoint operator in X_1.

3 Classifications of spectra with Schrödinger operators as examples.

Let $d \in \mathbb{N}$ and let $G \subseteq \mathbb{R}^d$ be a domain, i.e., G is a nonempty connected open set in $\mathbb{R}^d$. We denote by $L^2(G) = L^2(G, \mathbb{C})$ the Hilbert space of $\mathbb{C}$-valued L^2-functions.

Example 3.1.(a) The operator

$$\mathbf{grad} : H^1(G) \subset L^2(G) \to L^2(G, \mathbb{C}^d) \quad \text{defined by } u \mapsto \nabla u$$

is a closed densely defined unbounded operator, and so is its restriction

$$\mathbf{grad}_0 : H^1_0(G) \subset L^2(G) \to L^2(G, \mathbb{C}^d).$$

(b) By Theorem 2.1, the two nonnegative Laplace operators

$$(-1)\Delta^{\mathrm{D}} = \mathbf{grad}_0^* \, \mathbf{grad}_0 \quad \text{and} \quad (-1)\Delta^{\mathrm{N}} = \mathbf{grad}^* \, \mathbf{grad} \text{ are selfadjoint in } L^2(G).$$

These operators are associated with Dirichlet and Neumann boundary conditions on ∂G, respectively.

(c) Let $V \in L^\infty(G,\mathbb{R})$ be a real valued L^∞-potential. Let us consider in $L^2(G)$ the Schrödinger operator

$$(H^{\mathrm{D}}u)(x) = -\Delta^{\mathrm{D}}u(x) + V(x)u(x) \quad \text{with } \operatorname{dom} H = \operatorname{dom}(-\Delta^{\mathrm{D}}).$$

Then $H^{\mathrm{D}} = (H^{\mathrm{D}})^*$. Analogously, one defines $H^{\mathrm{N}} = (H^{\mathrm{N}})^*$. These operators are semibounded,

$$H^{\mathrm{D(N)}} \geqslant \operatorname{ess\,inf}_{x\in G} V(x).$$

These statements follow from the fact that the multiplication operator $u \mapsto Vu$ is bounded and selfadjoint in $L^2(G)$ for $V \in L^\infty(G,\mathbb{R})$.

(d) If $G = \mathbb{R}^d$, then $(-1)\Delta^{\mathrm{D}} = (-1)\Delta^{\mathrm{N}}$ is the standard nonnegative Laplace operator $-\Delta = -\Delta^*$ in $L^2(\mathbb{R}^d)$. Besides,

$$\sigma(-\Delta) = \overline{\mathbb{R}}_+ = [0,+\infty).$$

What is the set of eigenvalues $\varepsilon(-\Delta)$ of this operator?

(e) Let $G = \mathbb{R}^d$ and $V \in L^\infty(\mathbb{R}^d,\mathbb{R})$. Then $H^{\mathrm{D}} = H^{\mathrm{N}}$ and we denote this selfadjoint Schrödinger operator by

$$H = -\Delta + V, \qquad \operatorname{dom} H = \operatorname{dom}\Delta = H^2(\mathbb{R}^d).$$

Definition 3.1 (discrete and essential spectra, [Kato, RS4])**.**

(a) The discrete spectrum $\sigma_{\mathrm{disc}}(A)$ of A is the set of all isolated eigenvalues of finite algebraic multiplicity [Kato, RS4].

(b) The essential spectrum $\sigma_{\mathrm{ess}}(A)$ is defined by $\sigma_{\mathrm{ess}}(A) := \sigma(A)\backslash\sigma_{\mathrm{disc}}(A)$ [RS4].

Theorem 3.1 (e.g., [Leis])**.**
Let G be a bounded domain with Lipschitz boundary ∂G. Let $V \in L^\infty_{\mathbb{R}}(G,\mathbb{R})$. Then the semibounded selfadjoint operators H^{D} and H^{N} have purely discrete spectra, i.e.,

$$\sigma(H^{\mathrm{D}}) = \sigma_{\mathrm{disc}}(H^{\mathrm{D}}) \quad \textit{and} \quad \sigma(H^{\mathrm{N}}) = \sigma_{\mathrm{disc}}(H^{\mathrm{N}}).$$

Consequently, their spectra can be written as nondecresing sequencies of eigenvalues

$$\{k_n^{\mathrm{D(N)}}\}_{n\in\mathbb{N}} \subset \mathbb{R} \quad \textit{with} \quad \lim k_n^{\mathrm{D(N)}} = +\infty.$$

Such Schrödinger operators in $G_L = [-L,L]^d$ can be used to study the case of the whole space $\mathbb{R}^d$ by means of density of states. A discrete version of this approach will be the subject of talks no.5-6.

These sequences $\{k_n^{\mathrm{D(N)}}\}_{n\in\mathbb{N}}$ can be technically approached via the Courant–Fischer–Weyl min-max principle: if $A = A^* \geqslant c$ and $\sigma(A) = \sigma_{\mathrm{disc}}(A)$, then $\sigma(A) = \{k_n\}_{n\in\mathbb{N}}$ with

$$k_1 = \inf\{\langle u, Au\rangle \ : \ u \in \operatorname{dom} A,\ \|u\|_X = 1\},$$

$$k_{n+1} = \sup_{v_1,\dots,v_n\in X} \inf\{\langle u, Au\rangle \ : \ u \in \operatorname{dom} A,\ \|u\|_X = 1, u \perp v_k, k = 1,\dots,n\}, \qquad n \in \mathbb{N},$$

see, e.g., [RS4, K].

Theorem 3.2 (e.g., [RS4]).
Assume that $V \in L^\infty(\mathbb{R}^d, \mathbb{R})$ *is periodic in the following sense: there exists a basis* $\{a_j\}_{j=1}^d$ *of* $\mathbb{R}^d$ *s.t.* $V(x) = V(x + a_j)$ *a.e. in* $\mathbb{R}^d$. *Then the spectrum of the operator* $H = -\Delta + V$ *in* $L^2(\mathbb{R}^d)$ *has the following properties:*

(a) $\sigma(H) = \sigma_{\mathrm{ess}}(H)$*;*

(b) $\sigma(H) = \bigcup_{j\in\mathbb{N}}[l_j, m_j]$ *with* $\lim l_j = \lim m_j = +\infty$*;*

(c) $\varepsilon(H) = \varnothing$.

Theorem 3.3 (e.g., [RS4]).
Let $V_0 \in L^\infty_{\mathrm{comp}}(\mathbb{R}^d, \mathbb{R})$*, where the subscript "comp" means that* V_0 *has compact support in* $\mathbb{R}^d$. *For a constant* $c \in \mathbb{R}$*, let us consider the selfadjoint Schrödinger operator*

$$H = -\Delta + V = -\Delta + V_0 + c.$$

Then:

(a) $\sigma_{\mathrm{disc}}(H) = \sigma(H) \cap (-\infty, c) = \{k_j\}_{j=1}^n$ *with* $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

(b) $\sigma_{\mathrm{ess}}(H) = [c, +\infty)$ *and* $\varepsilon(H) \cap \sigma_{\mathrm{ess}}(H) = \varnothing$.

One sees that $\sigma_{\mathrm{ess}}(H)$ describes something that happens with the potential $V(x)$ as $|x| \to \infty$. However, the definition of the essential spectrum $\sigma_{\mathrm{ess}}(A)$ does not provide much details about the spectral effects corresponding to the set $\sigma_{\mathrm{ess}}(A)$.

Definition 3.2.
A selfadjoint operator A in X is said to have a purely point spectrum if there exists an orthonormal basis in X consisting of eigenvectors of A.

Example 3.2.
Let $\{r_n\}_{n\in\mathbb{N}} = \mathbb{Q}$ be a certain enumeration of all rational numbers. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis in a separable Hilbert space X. Let us define a selfadjoint operator A by

$$Af = \sum_{n\in\mathbb{N}} r_n \langle e_n, f\rangle e_n$$

for all $f \in X$ s.t. $\sum_{n\in\mathbb{N}} |r_n\langle e_n, f\rangle|^2 < \infty$. Then $A = A^*$ and

$$\mathbb{R} = \overline{\varepsilon(A)} = \sigma(A) = \sigma_{\mathrm{ess}}(A).$$

This operator has a purely point spectrum, but $\sigma_{\mathrm{disc}}(A) = \varnothing$.

This pure point construction seems to be pathological, but occurs to be typical for 1-d Schrödinger operators with reasonable random potentials.

Theorem 3.4 (spectral Anderson localization for 1-d random alloy model [DSS02]).
Let $V_0 \in L^\infty_{\mathrm{comp}}(\mathbb{R}, \mathbb{R})$ *be the single-site potential such that* $\operatorname{supp} V_0 \subseteq [-1/2, 1/2]$ *and* $V_0 \neq 0$ *in the* L^∞*-sense. Let* $\xi_n : \Omega \to \mathbb{R}$, $n \in \mathbb{Z}$, *be independent identically distributed (i.i.d.) random variables such that the support of their common distribution measure is bounded and contains at least two points. Let* $H = -\frac{d^2}{dx^2} + V_\omega$ *be the Schrödinger operator with the random potential*

$$V_\omega(x) = \sum_{n\in\mathbb{Z}} \xi_n(\omega) V_0(x - n), \qquad x \in \mathbb{R}, \qquad \omega \text{ in the probability space } \Omega.$$

Then, almost surely (a.s.), H has a purely point spectrum and all eigenfunctions of A decay exponentially at $\pm\infty$.

On the mathematical level, this effect was discovered and rigorously proved first by Goldsheid, Molchanov, & Pastur [GMP77]. On the Physics level, this effect is related to Anderson localization [A58, AW].

The set $\sigma(H)$ for such models with ergodic potentials is actually deterministic in the sense that there exists a deterministic closed set $S \subseteq \mathbb{R}$ s.t. $\sigma(H) = S$ a.s. (but this does not imply that $\varepsilon(H)$ is deterministic). This property and ergodic operators in general will be the subject of Talks no.2-4.

For every $A = A^*$, there exist an orthogonal decompositions of X and of A

$$X = X_{\text{pp}} \oplus X_{\text{ac}} \oplus X_{\text{sc}}, \qquad A = A|_{X_{\text{pp}}} \ \oplus\ A|_{X_{\text{ac}}} \ \oplus A|_{X_{\text{sc}}},$$

such that $A|_{X_{\text{pp}}}$ has purely point spectrum $\sigma_{\text{pp}}(A) = \sigma(A|_{X_{\text{pp}}})$, $A|_{X_{\text{ac}}}$ is an operator with purely absolutely continuous spectum $\sigma_{\text{ac}}(A) = \sigma(A|_{X_{\text{ac}}})$, and $A|_{X_{\text{sc}}}$ is an operator with purely singular continuous spectum $\sigma_{\text{sc}}(A) = \sigma(A|_{X_{\text{ac}}})$. The rigorous definitions of these types of spectra will be a part of Talk no. 4.

Remark 3.1.
The sets $\sigma_{\text{pp}}(A)$, $\sigma_{\text{ac}}(A)$, and $\sigma_{\text{sc}}(A)$ are not necessarily disjoint. Using multiplication operators similar to Example 3.2, it is easy to construct a (rather pathological) example s.t. $\sigma_{\text{pp}}(A) = \sigma_{\text{ac}}(A)$. It is a bit more difficult to construct an operator A s.t. $\sigma_{\text{pp}}(A) = \sigma_{\text{ac}}(A) = \sigma_{\text{sc}}(A)$.

Theorem 3.2 about periodic potentials can be strengthened [RS4] to

$$\sigma(H) = \sigma_{\text{ac}}(H).$$

In Theorem 3.3 with a compact perturbation of a constant potential equal to c,

$$\sigma_{\text{ac}}(H) = \sigma_{\text{ess}}(H) = [c, +\infty).$$

Theorem 3.4 with a random 1-d potential implies that

$$\varnothing = \sigma_{\text{ac}}(H) = \sigma_{\text{sc}}(H) \quad \text{ with probability 1.}$$

It is expected presently that for $d \geqslant 2$ spectral Anderson localization generally does not hold for the whole $\sigma(H)$. This is a long-standing open problem.

The most of the seminar is devoted for the tools and notions related to Anderson localization, but these tools are considered mainly for the simplified 1-d discrete Anderson model

$$(H_\omega^{\text{disc}} u)(n) = (\Delta_{\text{disc}} u)(n) + V_\omega(n) u(n) = u(n+1) + u(n-1) + V_\omega(n) u(n), \quad n \in \mathbb{Z},$$

in $X = \ell^2(\mathbb{Z})$. We follow the lines of the lecture notes of Kirsch [K] and Chapter 9 of extended lecture notes [CFKS].

However, our ultimate goal is to reach in one or two last talks the theory of random continuation resonances for spatial cut-off versions of H_ω^{disc} following the paper of Klopp

[K16]. This theory is essentially built on the same tools as the theory of the 1-d Andeson localization.

Let us introduce briefly continuation resonances for the continuous Schrödinger operator

$$H = -\Delta + V, \qquad V \in L^{\infty}_{\text{comp}}(\mathbb{R}^d, \mathbb{R}),$$

in odd-dimensional spaces $\mathbb{R}^d$, $d = 1, 3, 5, \ldots$. We replace the spectral parameter k with $k = \lambda^2$ and consider for $\lambda \in \mathbb{C}_+ := \{\operatorname{Im} z > 0\}$ one more version of the resolvent-function

$$\mathcal{R}_{\mathcal{H}}(\lambda) = (\mathcal{H} - \lambda^2)^{-1}, \qquad \mathcal{R}_{\mathcal{H}} : (\mathbb{C}_+ \backslash \{\sqrt{k_j}\}_{j=1}^n) \to \mathcal{L}(L^2(\mathbb{R}^d)).$$

By Theorem 3.3, $\mathcal{R}_{\mathcal{H}}(\lambda)$ does not exists as an $\mathcal{L}(L^2(\mathbb{R}^d))$-valued function for $\lambda \in \mathbb{R}$. However, it is possible to continue $\mathcal{R}_{\mathcal{H}}(\lambda)$ from $\mathbb{C}_+$ through $\mathbb{R}$ to $\mathbb{C}_- = \{\operatorname{Im} z < 0\}$ as a meromorphic $\mathcal{L}(L^2_{\text{comp}}(\mathbb{R}^d), L^2_{\text{loc}}(\mathbb{R}^d))$-valued function $\mathcal{R}^{\text{cont}}_{\mathcal{H}}(\lambda)$. The (continuation) resonances are the poles of this generalized meromorphic continuation.

The Physics meaning of resonances is connected with the description of the long-time behaviors of solutions and with the rate of decay of energy contained inside of $\operatorname{supp} V$.

When the operator $H_\omega = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\omega$ becomes random, the set of resonances becomes random and can be sometimes described by a locally finite stochastic point processes SPP on $\mathbb{C}$. The paper of Klopp [K16] describes asymptotic properties of these stochastic point processes for 1-d discrete Schrödinger operators.

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