**Institute for Applied Mathematics – SS19**
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# V3F1 Stochastic Processes – Problem Sheet 9

Distributed May 31st, 2019. At most in groups of 2. Solutions have to be handed in before noon on Thursday June 6th into the marked post boxes opposite to the maths library. Please clearly specify your names and your tutorial group on top of your homework.
**(revised the 4/6/2019 to correct some typos)**

**Exercise 1.** (The classic secretary problem) [Pts 1+1+1+1+1+2+2+1] Our aim to to choose the best among $N$ objects. We inspect each of them in turn and we have the possibility to take it or to pass to the next. In this second eventuality we loose the right to go back to the discarded object. If we arrive to the last one without choosing any of the previous then we take it. We assume that the are able to strictly order the objects according to some preference criterion. However this absolute ordering is not know at the start and after having seen $n$ objects we are only able to establish their *relative* order. We want to determine a strategy which will allow us to maximize the probability of choosing the best among the $N$ objects.

To model probabilistically this situation we choose as probability space the set $\Omega$ of all permutations of $N$ objects. Therefore $\omega \in \Omega$ can be identified with a vector of $N$ different integers in $\{1, \ldots, N\}$ and $\omega(i)$ is the absolute rank of the $i$-th object. On $\Omega$ we consider the uniform distribution $\mathbb{P}$. At each step we observe the relative rank $X_n(\omega)$ of the $n$-th object among the first $n$ objects. Therefore $X_n(\omega) \in \{1, \ldots, n\}$ and $X_N(\omega) = \omega(N)$. Let $\Xi = \{(x_1, \ldots, x_N) \in \{1, \ldots, N\}^N : 1 \leqslant x_k \leqslant k, \quad k = 1, \ldots, N\}$ and $\mathscr{F}_n = \sigma(X_1, \ldots, X_n)$. The goal is to find a stopping time $T : \Omega \to \{1, \ldots, N\}$ for the filtration $(\mathscr{F}_n)_{1 \leqslant n \leqslant N}$ which maximizes the probability $\mathbb{P}(\omega(T) = 1)$.

a) Show that $X = (X_1, \ldots, X_n) : \Omega \to \Xi$ is a bijection and that

$$\mathbb{P}(X_1 = x_1, \ldots, X_N = x_N) = \frac{1}{N!}, \qquad x \in \Xi.$$

b) Show that for all $0 \leqslant n \leqslant N$ we have $\mathbb{P}(X_n = j) = 1/n$ for $j = 1, \ldots, n$ and that the r.v. $X_1, \ldots, X_N$ are independent.

c) Show that for all stopping times $T$ as above we have $\mathbb{P}(\omega(T) = 1) = \mathbb{E}[Y_T]$ where $Y_k = \mathbb{P}(\omega(k) = 1 | \mathscr{F}_k)$.

d) Show that $Y_k = \mathbb{1}_{X_k = 1}(k/N)$.

e) Show that an optimal stopping time $S$ is given by

$$S = \inf\{1 \leqslant k \leqslant N : \mathbb{E}(Z_{k+1}) \leqslant k/N, X_k = 1\},$$

where $(Z_k)_{k \in \{1, \ldots, N\}}$ is the Snell envelope of $(Y_k)_{k \in \{1, \ldots, N\}}$.

f) Show that $\mathbb{E}[Z_k]$ is decreasing in $k$ and therefore that there exists $r$ such that

$$S = T_r := \inf\{t \in \{r, \ldots, N\} : X_t = 1\} \cup \{N\}.$$

g) Show that for all $r \in \{1, \ldots, N\}$:

$$G_N(r) = \mathbb{E}[Y_{T_r}] = \mathbb{P}(\omega(T_r) = 1) = \frac{r-1}{N} \sum_{k=r}^{N} \frac{1}{k-1}.$$

h) Show that $\lim_{N \to \infty} G_N(xN) = -x \log x$ for $x \in (0, 1)$ and conclude that asymptotically it is optimal to stop to the first relatively best object after having inspected $\approx N/e$ objects.

**Exercise 2.** (The game Googol) [Pts 1+2+1+1+1+1+2+1] The Pareto law $\mathscr{P}a(\alpha, x)$ is the probability distribution on $\mathbb{R}$ with density

$$f(z) = \alpha x^{\alpha} z^{-\alpha-1} 1_{z \geqslant x}$$

with respect to the Lebesgue measure. Let $\Theta$ be a r.v. with law $\mathscr{P}a(\alpha, 1)$ for some $\alpha > 0$. Moreover let $(X_1, \ldots, X_n)$ a vector such that, conditionally on $\Theta$, are independent and with uniform law on $[0, \Theta]$. Let $M_0 = X_0 = 1$ and for all $j \leqslant n$ let $M_j = \max(X_1, \ldots, X_j) = \max(M_{j-1}, X_j)$. Let $\mathscr{F}_n = \sigma(X_1, \ldots, X_n)$. We play a stopping game where we observe in sequence the values of the $X$'s and we win if let last observed value is bigger than all the previous values. We want to compute an optimal strategy for this game.

a) Are the r.v. $X_1, \ldots, X_n$ independent? Why?

b) Show that the conditional law of $\Theta$ given $X_1, \ldots, X_k$ is $\mathscr{P}a(k + \alpha, M_k)$ for all $k \in \{1, \ldots, n\}$.

c) Show that $Y_k := \mathbb{P}(X_k = M_n | \mathscr{F}_k) = \frac{k + \alpha}{n + \alpha} \mathbb{1}_{X_k = M_k}$.

d) Show that the event $\{X_k = M_k\}$ is independent of $\mathscr{F}_{k-1}$.

e) Show that we can write the probability of winning by using the stopping time $T: \Omega \to \{1, \ldots, n\}$ as $\mathbb{E}[Y_T]$.

f) Show that $\mathbb{E}[Z_k | \mathscr{F}_{k-1}]$ is a constant r.v. which is a decreasing function of $k \in \{1, \ldots, n\}$.

g) Show that the optimal strategy is to wait until some time $r$ and then to stop at the first $j \geqslant r$ such that $X_j = M_j$ where here $r \in \{1, \ldots, n-1\}$. Let us call $T_r$ such a rule.

h) Show that

$$\mathbb{P}(X_{T_r} = M_n) = \frac{r - 1 + \alpha}{n + \alpha} \sum_{j=r}^{n} \frac{1}{j - 1 + \alpha}$$

and therefore that an optimal stopping strategy is given by the number $r$ which maximizes this expression.