V3F1 Stochastic Processes – Problem Sheet 1

Distributed April 5th, 2019. At most in groups of 2. Solutions have to be handed in before 4pm on Thursday April 11th into the marked post boxes opposite to the maths library. Please clearly specify your names and your tutorial group on top of your homework.

Exercise 1. [3+2+2 Pts] Let $(\Omega, \mathcal{F}, \mu)$ a σ-finite measure space and define the outer measure μ^*: $\mathcal{P}(\Omega) \to [0, \infty]$ as in the lecture while the *inner measure* $\mu_*: \mathcal{P}(\Omega) \to [0, \infty]$ as

$$\mu_*(A) = \sup \{\mu(F): F \in \mathcal{F}: A \supseteq F\}.$$

a) Let $\mathcal{M}(\mu) = \{A \in \mathcal{P}(\Omega): \text{for all } B \in \mathcal{F} \text{ with } \mu(B) < \infty, \mu_*(A \cap B) = \mu^*(A \cap B)\}$. Prove that $\mathcal{M}(\mu)$ is a σ-algebra which contains all the sets with outer measure zero and $\mathcal{F}$.

b) Prove that μ is a measure on $\mathcal{M}(\mu)$. The measure space $(\Omega, \mathcal{M}(\mu), \mu)$ is called the completion of the original measure space.

c) Let μ be a probability measure and assume that there exists $G \in \mathcal{P}(\Omega)$ such that $\mu^*(G) = 1$. Let $\mathcal{G} = \sigma(G \cap F: F \in \mathcal{F})$. Prove that $(G, \mathcal{G}, \mu^*)$ is a probability space.

Exercise 2. [2+3 Pts]

a) Let $\mathbb{P}$ and $\mathbb{Q}$ be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ that agree on all intervals of the form $(x, y]$, where $-\infty \leqslant x \leqslant y < \infty$. Prove that the two probability measures are equal.

b) Let $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)_{i=1,2}$ two probability spaces. Let $\Omega = \Omega_1 \times \Omega_2$ and $\mathcal{F}$ be the σ-algebra on Ω generated by events of the form $A \times B$ with $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$. Prove that there is only one measure μ on $(\Omega, \mathcal{F})$ such that $\mathbb{P}(A \times B) = \mathbb{P}_1(A)\mathbb{P}_2(B)$.

Exercise 3. [3 Pts] Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space and $f: (\Omega, \mathcal{F}) \to (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ be a measurable function. Denote by λ the Lebesgue measure on $\mathbb{R}$. Use Fubini's theorem to prove that for any $1 \leqslant p < \infty$

$$\int f^p \mathrm{d}\mu = \int_0^\infty p t^{p-1} \mu(f > t) \lambda(\mathrm{d}t).$$