V4F1 Stochastic Analysis – Problem Sheet 9

Tutorial classes: Wed June 29th 8–10 Chunqiu Song | Wed June 29th 12–14 Min Liu. The sheet has to be handled in the lecture of Thursday June 23rd. At most in groups of two.

Exercise 1. [Pts 4] Prove that if B is a Brownian motion, then we have the relation $L_t^{|B|,0} = 2L_t^{B,0}$ where $L_t^{X,a}$ denotes the local time in $a \in \mathbb{R}$ of the semimartingale X.

Exercise 2. [Pts 2+4] Let $y: \mathbb{R}_+ \to \mathbb{R}$ be a continuous function and let

$$a(t) = \sup_{s \in [0,t]} (y(s))_- = \sup_{s \in [0,t]} (-y(s) \vee 0), \qquad z(t) = y(t) + a(t).$$

a) Prove that a, z are continuous functions and a is non-decreasing.

b) Prove that a is of bounded variation and that $\int_0^\infty \mathbb{1}_{z_s > 0} \mathrm{d}a_s = 0$. (Hint: use the fact that $\mathrm{d}a_s$ is a Borel measure).

Exercise 3. [Pts 2+2+2] Prove (the upper bound of) Burkholder–Davis–Gundy inequality. Let M be a continuous local martingale (with $M_0 = 0$). For any $p \geqslant 2$ we have

$$\mathbb{E}[\sup_{t \leqslant T} |M_t|^p] \leqslant C_p \mathbb{E}[([M]_T^{p/2})]$$

where C_p is a universal constant depending only on p.

a) Assume that the martingale M is bounded. Use Itô formula on $t \mapsto (\varepsilon + |M_t|^2)^{p/2}$ to prove that

$$\mathbb{E}[\sup_{t \leqslant T} |M_t|^p] \leqslant \int_0^T \mathbb{E}[|M_t|^{p-2} \mathrm{d}[M]_t].$$

(why we need $\varepsilon > 0$?)

b) Use Hölder's and Doob's inequality to conclude.

c) Remove the assumption of boundedness.

Exercise 4. [Pts 2+2+2] Let us continue with the setting of Exercise 3 and prove now a complementary lower bound when $p \geqslant 4$, that is

$$\mathbb{E}[([M]_T^{p/2})] \leqslant C_p \mathbb{E}[\sup_{t \leqslant T} |M_t|^p].$$

where again C_p is a universal constant depending only on p (not the same as that of the upper bound).

a) Use the relation

$$[M]_T = M_T^2 - 2\int_0^T M_s \mathrm{d}M_s$$

to estimate $\mathbb{E}[([M]_T^{p/2})]$ and then use the BDG upper bound for the stochastic integral.

b) Prove that if we let $N_T = \int_0^T M_s \mathrm{d}M_s$ then for any $\varepsilon > 0$ there exists $\lambda_\varepsilon > 0$ such that

$$[N]_T^{1/2} \leqslant \lambda_\varepsilon \sup_{t \leqslant T} |M_t| + \varepsilon [M]_T$$

c) Conclude by choosing ε small enough.