V4F1 Stochastic Analysis – Problem Sheet 4

Tutorial classes: Wed May 11th 8–10 Chunqiu Song | Wed May 11th 12–14 Min Liu. The sheet has to be handled in the lecture of Thursday May 5th. At most in groups of two.

Exercise 1. [Pts 2] **(Brownian motion on the unit sphere)** Let $Y_t = B_t/|B_t|$ where B is a Brownian motion in $\mathbb{R}^n$ and $n > 2$. Prove that the time–changed process

$$Z_a = Y_{T_a}, \qquad T = A^{-1}, \qquad A_t = \int_0^t |B_s|^{-2} \mathrm{d}s,$$

is a diffusion taking values in the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ with generator

$$\mathscr{L} f(x) = \frac{1}{2}\left(\Delta f(x) - \sum_{i,j} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right) - \frac{n-1}{2} \sum_i x_i \frac{\partial f}{\partial x_i}(x), \qquad x \in S^{n-1}.$$

where Δ is the Laplacian in $\mathbb{R}^n$ and where diffusion here means continuous time process solving the martingale problem for this generator.

Exercise 2. [Pts 2+2+2+1+1] **(Polar points of Brownian motion for $d \geqslant 2$)** Let (X, Y) be a Brownian motion on $\mathbb{R}^2$ starting at $(0, 0)$. Let

$$(M_t, N_t) := e^{X_t}(\cos(Y_t), \sin(Y_t)).$$

We will assume without proof that

$$\int_0^\infty e^{2X_s} \mathrm{d}s = +\infty, \qquad a.s.$$

a) Prove that (M, N) is a Brownian motion on $\mathbb{R}^2$ changed of time (starting from where?) ;
b) Compute the Euclidean norm $|(M_t, N_t)|$ of the vector (M_t, N_t) and deduce that a Brownian motion B in $\mathbb{R}^2$ never visit the point $(-1, 0)$, that is

$$\mathbb{P}(\exists t > 0 : B(t) = (-1, 0)) = 0.$$

c) Conclude that B never visit any given point $x \neq (0, 0)$.
d) Use the Markov property to deduce from (c) that $\mathbb{P}(\exists t > 0 : B(t) = (0, 0)) = 0$. [Hint: consider $\mathbb{P}(\exists t \geqslant 1/n : B(t) = (0, 0))$ as $n \to 0$.]
e) Prove that a Brownian motion in $\mathbb{R}^d$ with $d > 2$ does not visit any given point $x \in \mathbb{R}^d$.

Exercise 3. [Pts 2+2+2+1+1] **(Transience of Brownian motion in $d \geqslant 3$)** Let X be a Brownian motion in $\mathbb{R}^3$ starting from $a \in \mathbb{R}^3 \neq 0$. We say that a process Y is transient if $|Y_t| \to \infty$ as $t \to \infty$ almost surely.

a) Prove that the process $M_t = 1/|X_t|$ is a positive local martingale.
b) Prove that $M_\infty = \lim_{t\to\infty} M_t$ exists almost surely.
c) Compute $\mathbb{E}[M_t]$ and deduce that $M_\infty = 0$. This implies that X is transient.
d) Show that whatever the starting point is, X is always transient.
e) Prove that a Brownian motion in $\mathbb{R}^d$ with $d \geqslant 3$ is transient.

Exercise 4. [Pts 2] **(Conformal invariance of Brownian motion)** Let $f: \mathbb{C} \to \mathbb{C}$ be an holomorphic function and $Z = X + iY$ be a planar Brownian motion (with the identification of $\mathbb{C}$ with $\mathbb{R}^2$). Prove that the process $M_t = f(Z_t)$ is a continuous local martingale with values in $\mathbb{C}$. Deduce that it is a complex Brownian motion changed of time. This property is called conformal invariance of Brownian motion.