V4F1 Stochastic Analysis – Problem Sheet 1

Tutorial classes: Wed April 20th 8–10 Chunqiu Song | Wed April 20th 12–14 Min Liu. The sheet has to be handled in the lecture of Thursday April 14th. At most in groups of two.

Exercise 1. [Pts 4+2] **(Martingale problem)** Let $b: \mathbb{R}^n \to \mathbb{R}^n$, $\sigma: \mathbb{R}^n \to \mathbb{R}^{n\times n}$ locally bounded coefficients. Let $a(x) = \sigma(x)\sigma(x)^T \in \mathbb{R}^{n\times n}$ and for all $f \in C^2(\mathbb{R}^n)$ let

$$\mathscr{L} f(x) = b(x) \cdot \nabla f(x) + \frac{1}{2}\mathrm{Tr}[a(x)\nabla^2 f(x)], \qquad x \in \mathbb{R}^n$$

where $\nabla^2 f(x)$ is the $\mathbb{R}^{n\times n}$ matrix of second derivatives of f.

a) Prove that the following conditions are equivalent

   i. For any $f \in C^2(\mathbb{R}^d)$, the process $M_t^f = f(X_t) - f(X_0) - \int_0^t \mathscr{L} f(X_s)\mathrm{d}s$ is a local martingale.

   ii. For any $v \in \mathbb{R}^d$, the process $M_t^v = v \cdot X_t - v \cdot X_0 - \int_0^t v \cdot b(X_s)\mathrm{d}s$ is a local martingale with quadratic variation
   $$[M^v]_t = \int_0^t v \cdot a(X_s) v \mathrm{d}s.$$

   iii. For any $v \in \mathbb{R}^d$ the process
   $$Z_t^v = \exp\left(M_t^v - \frac{1}{2}\int_0^t v \cdot a(X_s) v \mathrm{d}s\right)$$
   is a local martingale.

   [Hint: use the fact that linear combinations of exponentials are dense in C^2 w.r.t. uniform convergence on compacts for the functions and its first two derivatives (assumed wihtout proof)]

b) Show that any of the conditions *i*,*ii*,*iii* implies that
   $$(f(X_t)/f(X_0))\exp\left(-\int_0^t \frac{\mathscr{L} f}{f}(X_s)\mathrm{d}s\right)$$
   is a local martingale for every stricly positive C^2 function f.

Exercise 2. [Pts 2+2+2] Let $(B_t)_{t\geqslant 0}$ be a one dimensional Brownian motion. Find the SDEs satisfied by the following processes: (for all $t \geqslant 0$)

a) $X_t = B_t/(1+t)$,

b) $X_t = \sin(B_t)$

c) $(X_t, Y_t) = (a\cos(B_t), b\sin(B_t))$ where $a, b \in \mathbb{R}$ with $ab \neq 0$

Exercise 3. [Pts 2+2+2+2] **(Variation of constants)** Consider the nonlinear SDE

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + c(t)X_t \mathrm{d}B_t, \qquad X_0 = x,$$

where $f: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $c: \mathbb{R}_+ \to \mathbb{R}$ are continuous deterministic functions.

a) Find an explicit solution Z_t in the case $f = 0$ and $Z_0 = 1$.

b) Use the Ansatz $X_t = C_t Z_t$ to show that X solves the SDE provided C solves an ODE with random coefficients.

c) Apply this method to solve the SDE
   $$\mathrm{d}X_t = X_t^{-1}\mathrm{d}t + \alpha X_t \mathrm{d}B_t, \qquad X_0 = x$$
   where α is a constant.

d) Apply the method to study the solution of the SDE
   $$\mathrm{d}X_t = X_t^{\gamma}\mathrm{d}t + \alpha X_t \mathrm{d}B_t, \qquad X_0 = x > 0$$
   where α and γ are constants. For which values of γ do we get explosion ,i.e. the solution tends to $+\infty$ for finite time?