V4F1 Stochastic Analysis – Problem Sheet 0

Version 1. Tutorial classes: Wed April 13th 8–10 Chunqiu Song | Wed April 13th 12–14 Min Liu. This sheet will be discussed during the tutorial. Nothing to handle in.

Discuss the proof of these statements.

Lemma 1. *Let $\kappa: \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be a continuous non-decreasing function such that $\kappa(0) = 0$ and*

$$\int_{0+} \frac{\mathrm{d}\xi}{\kappa(\xi)} = +\infty,$$

Moreover let $\phi: [0, a] \to \mathbb{R}_+$ be a continuous function such that

$$\phi(x) \leqslant \int_0^x \kappa(\phi(y)) \mathrm{d}y, \qquad x \in [0, a].$$

Then $\phi(x) = 0$ for all $x \in [0, a]$.

Theorem 2. (Yamada–Watanabe) *Pathwise uniqueness holds for the one dimensional SDE*

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t, \qquad X_0 = x \in \mathbb{R},$$

provided there exists a positive increasing function $\rho: \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ and a positive, increasing and concave function $\kappa: \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that

$$|b(x) - b(y)| \leqslant \kappa(|x - y|), \qquad |\sigma(x) - \sigma(y)| \leqslant \rho(|x - y|),$$

and

$$\int_{0+} \frac{\mathrm{d}\xi}{\kappa(\xi)} = +\infty = \int_{0+} \frac{\mathrm{d}\xi}{\rho^2(\xi)}.$$

(Note that this implies that path-wise uniqueness in one dimensions holds if b is Lipshitz and σ Hölder continuous of index $1/2$).

Lemma 3. *If $(B_t)_{t \geqslant 0}$ is a m-dimensional Brownian motion adapted to a filtration $(\mathscr{F}_t)_{t \geqslant 0}$ then $B_t - B_s$ is independent of $\mathscr{F}_s$.*

(Note that $(\mathscr{F}_t)_{t \geqslant 0}$ is not necessarily the filtration generated by $(B_t)_{t \geqslant 0}$, but only contains it)

Lemma 4. *Let $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t \geqslant 0}, X, B)$ a weak solution of an SDE in $\mathbb{R}^n$ driven by an m-dimensional Brownian motion B. Let $\mathbb{Q}_\omega$ be the regular conditional distribution of (X, B) given $\mathscr{F}_0$ where we consider (X, B) as a random variable in $\mathscr{C}^{n+m} = C(\mathbb{R}_+, \mathbb{R}^n \times \mathbb{R}^m)$. Call (Y, Z) the canonical process on $\mathscr{C}^{n+m}$ with the understanding that Y is $\mathbb{R}^n$-valued and Z is $\mathbb{R}^m$-valued. Let $(\mathscr{H}_t)_{t \geqslant 0}$ be the canonical filtration on $\mathscr{C}^{n+m}$ and $\mathscr{H} = \sigma(\mathscr{H}_t : t \geqslant 0)$ then for $\mathbb{P}$-amost all $\omega \in \Omega$ the data $(\mathscr{C}^{n+m}, \mathscr{H}, \mathbb{Q}_\omega, (\mathscr{H}_t)_{t \geqslant 0}, Y, Z)$ is a weak solution to the same SDE.*