Note 2 – version 2 – 2022.5.20 – Please send remarks and corrections to gubinelli@iam.uni-bonn.de

SDE techniques

We develop some techniques to study solutions of SDEs, in particular changing the drift via absolutely continuous changes of measures, how solutions of SDEs change upon conditioning and the concept of local time with its relation to stochastic calculus.

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1 Girsanov's theorem

Equivalence of measures in a filtered probability space, Girsanov transformation, applications of Girsanov formula: Doob's transform, change of measure, weak solution to SDE via Girsanov.

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geqslant 0})$ be a filtered probability space with a right-continuous filtration and let $\mathbb{P}, \mathbb{Q}$ two probability measures on this space. Assume that $\mathbb{Q} \ll \mathbb{P}$ and define the positive martingale

$$Z_t := \mathbb{E}[H | \mathscr{F}_t], \qquad t \geqslant 0,$$

where $H = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ is the Randon-Nikodym derivative of $\mathbb{Q}$ to wrt. to $\mathbb{P}$, i.e. the unique random variable $H \in L^1(\mathbb{P})$ such that $\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}(H \mathbb{1}_A)$.

Note that $(Z_t)_{t \geqslant 0}$ is uniformly integrable and $Z_\infty = \lim_{t\to\infty} Z_t = \mathbb{E}[H|\mathscr{F}_\infty]$ in $L^1(\mathbb{P})$ and a.s.

Bayes formula: if $X \in L^1(\mathbb{Q}) \cap L^1(\mathbb{P})$ and $X \in \mathscr{F}_t$ then for any $t \geqslant s$

$$\mathbb{E}_{\mathbb{Q}}[X|\mathscr{F}_s] = \frac{\mathbb{E}_{\mathbb{P}}[Z_t X|\mathscr{F}_s]}{Z_s}, \qquad \mathbb{Q}-a.s. \tag{1}$$

this is well defined when $Z_s > 0$ and note that $\mathbb{Q}(Z_s = 0) = \mathbb{E}_{\mathbb{P}}[Z_s \mathbb{1}_{Z_s=0}] = 0$.

Define $T = \inf\{s \geqslant 0 : Z_s = 0\}$ and recall that on $T < \infty$ we have that $Z_s = 0$ for all $s \geqslant T$, then $\mathbb{Q}(T < \infty) = \mathbb{E}_{\mathbb{P}}[Z_T \mathbb{1}_{Z_T=0}] = 0$ and if $\mathbb{P} \ll \mathbb{Q}$ we have also that $\mathbb{P}(T < \infty) = 0$ so $Z_t > 0$ for all $t > 0$ $\mathbb{P}$-a.s.

Remark 1. There is no reason in general that the martingale $(Z_t)_{t\geqslant 0}$ is continuous. Think for example to the filtration $(\mathscr{F}_t)_{t\geqslant 0}$ generated by a Poisson process $(N_t)_{t\geqslant 0}$.

We are going to **assume** all along that $(Z_t)_{t\geqslant 0}$ is continuous and that $\mathbb{P} \sim \mathbb{Q}$.

Lemma 2. *$(X_t)_{t\geqslant 0}$ is a $\mathbb{Q}$-martingale iff $(Z_t X_t)_{t\geqslant 0}$ is a $\mathbb{P}$-martingale. The same is true also for local martingales.*

Proof. We will prove only one of the directions. Assume tha ZX is a $\mathbb{P}$-martingale, then by Bayes formula (1) we have $\mathbb{E}_{\mathbb{Q}}[X_t|\mathscr{F}_s] = Z_s^{-1}\mathbb{E}_{\mathbb{P}}[Z_t X_t|\mathscr{F}_s] = X_s$ so $(X_t)_{t\geqslant 0}$ is a martingale (check that indeed X_t is in $L^1(\mathbb{Q})$ for any $t \geqslant 0$). Assume now that the stopped process $(ZX)^T$ is a $\mathbb{P}$-martingale for some stopping time T, moreover observe that for any $A \in \mathscr{F}_s$ we have for $s < t$,

$$\mathbb{E}_{\mathbb{Q}}[X_s^T \mathbb{1}_A] = \mathbb{E}_{\mathbb{P}}[Z_s^T X_s^T \mathbb{1}_A] = \mathbb{E}_{\mathbb{P}}[Z_{t\wedge T} X_t^T \mathbb{1}_A] = \mathbb{E}_{\mathbb{Q}}[X_t^T \mathbb{1}_A]$$

(the first and last equality can be obtained by considering the partition $1 = \mathbb{1}_{T<s} + \mathbb{1}_{T\geqslant s}$ and the same for t) so

$$\mathbb{E}_{\mathbb{Q}}[X_t^T|\mathscr{F}_s] = X_s^T.$$

By localization if (XZ) is a $\mathbb{P}$-local martingale this shows that X is a $\mathbb{Q}$-local martingale. In this part of the proof we just used that $\mathbb{Q} \ll \mathbb{P}$ but in order to prove the converse one has to use that $\mathbb{P} \ll \mathbb{Q}$ to have $Z_t > 0$ always. □

Assume that $(X_t)_{t\geqslant 0}$ is a continuous spositive local martingale which is almost surely $X_t > 0$ then we can define the continuous local martingale

$$L_t = \log(X_0) + \int_0^t X_s^{-1} \mathrm{d}X_s$$

and note that $(X_t)_{t\geqslant 0}$ is the solution to the SDE $\mathrm{d}X_t = X_t \mathrm{d}L_t$ and a solution Y to this equation is given by

$$Y_t = \exp\left(L_t - \frac{1}{2}[L]_t\right) = \mathscr{E}(L)_t$$

and by using Ito formula one can check that the process $(Y_t^{-1} X_t)_{t\geqslant 0}$ is constant, therefore $X_t = X_0 Y_t$. The process $\mathscr{E}(L)$ is called the stochastic exponential of the continuous local martingale L. Via the stochastic exponential we can associate a continuous local martingale L to any continuous stricly positive local martingale X.

So we can write $Z_t = \mathscr{E}(L)_t$ (which defines L given Z).

Take $(M_t)_{t\geqslant 0}$ to be a $\mathbb{P}$-local martingale and let $\tilde{M} := M - [L, M]$ then by Ito formula

$$\mathrm{d}(Z\tilde{M})_t = Z_t \mathrm{d}\tilde{M}_t - \tilde{M}_t \mathrm{d}Z_t + \mathrm{d}[Z, \tilde{M}]_t = Z_t \mathrm{d}M_t - \tilde{M}_t \mathrm{d}Z_t + \underbrace{\mathrm{d}[Z, M]_t - Z_t \mathrm{d}[L, M]_t}_{=0} = Z_t \mathrm{d}M_t - \tilde{M}_t \mathrm{d}Z_t$$

where we used that $[Z, M]_t = [Z, \tilde{M}]_t$ and that $\mathrm{d}Z_t = Z_t \mathrm{d}L_t$. Therefore $Z\tilde{M}$ is a $\mathbb{P}$-local martingale and by the previous lemma we have that $\tilde{M}$ is a $\mathbb{Q}$-local martingale. Therefore we proved that

Theorem 3. (Girsanov) *Assume $\mathbb{Q} \sim \mathbb{P}$ and define $Z = \mathscr{E}(L)$ as above. Then if M is a $\mathbb{P}$-local martingale, the process $\tilde{M} := M - [L, M]$ is a $\mathbb{Q}$-local martingale. In particular since $[M] = [\tilde{M}]$ we have that if M is a Brownian motion then $\tilde{M}$ is also a Brownian motion.*

Remark 4. Note that $\tilde{L} = L - [L]$ so we have

$$Z_t^{-1} = \exp(-L_t + [L]_t / 2) = \exp(-\tilde{L}_t - [\tilde{L}]_t / 2) = \mathscr{E}(-\tilde{L}_t)$$

and is easy to check that

$$\mathbb{E}\left[\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\middle|\mathscr{F}_t\right] = Z_t^{-1}, \qquad t \geqslant 0.$$

So the relation between $\mathbb{Q}$ and $\mathbb{P}$ is simmetric, indeed $\tilde{M} = M - [L, M] = M - [\tilde{L}, \tilde{M}]$ and $M = \tilde{M} - [(-\tilde{L}), \tilde{M}]$.

Remark 5. By Girsanov's theorem we see that equivalent measures agree on classifying a process as a semimartingale. Indeed if $X = X_0 + M + V$ is a $\mathbb{P}$ semimartingale then X is also a $\mathbb{Q}$-semimartingale with decomposition $X = X_0 + \tilde{M} + \tilde{V}$ where $\tilde{M} = M - [L, M]$ and $\tilde{V} = V + [L, M]$.

In many applications we have a measure $\mathbb{P}$ and a positive continuous martingale $(Z_t)_{t\geqslant 0}$ with which we can define a new measure $\mathbb{Q}$ such that

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{F}_t} = Z_t, \qquad t \geqslant 0.$$

This is enough to define the measure $\mathbb{Q}$ on $\mathscr{F}_\infty = \vee_{t\geqslant 0}\mathscr{F}_t$. If this is not the full $\mathscr{F}$ then we can simply let $\mathrm{d}\mathbb{Q} = Z_\infty \mathrm{d}\mathbb{P}$ where $Z_\infty = \lim_{t\to\infty} Z_t$ provided the martingale is uniformly integrable.

Note that Z is uniformly integrable iff $\mathbb{Q} \ll \mathbb{P}$.

However in many aplications we only have that $(Z_t)_{t\geqslant 0}$ is a martingale but not uniformly integrable. In that case we can apply Girsanov's theorem on any bounded interval $[0, T]$ so we can also deduce that it extends to this situation.

Example 6. (Brownian motion with drift) Let $\gamma \in \mathbb{R}^n$ and B to be a n-dimensional Brownian motion, define the process $L_t = \gamma \cdot B_t$

$$Z_t = \exp\left(L_t - \frac{1}{2}[L]_t\right) = \exp\left(\gamma \cdot B_t - \frac{1}{2}|\gamma|^2 t\right), \qquad t \geqslant 0,$$

is a strictly positive continuous local martingale and it defines a new measure $\mathbb{Q}$ on $\mathscr{F}_\infty$ under which the process

$$\tilde{B}_t^\alpha = B_t^\alpha - [L, B^\alpha]_t = B_t^\alpha - \gamma^\alpha t, \qquad \alpha = 1, \ldots, n, t \geqslant 0,$$

is a $\mathbb{Q}$-Brownian motion. So under $\mathbb{Q}$ the process B is Brownian motion with a drift γ. The measure $\mathbb{Q}$ is not absolutely continuous wrt. $\mathbb{P}$. Indeed consider the event

$$A = \left\{ \lim_{t\to\infty} \frac{\tilde{B}_t + \gamma t}{t} = \gamma \right\} \in \mathscr{F}_\infty$$

for which we have (by the law of iterated log) $\mathbb{Q}(A) = 1$ while $\mathbb{P}(A) = 0$ unless $\gamma = 0$. And similarly one shows that $\mathbb{P}, \mathbb{Q}$ are singular. This is linked to the fact that $(Z_t)_{t\geqslant 0}$ is not uniformly integrable.

2 Doob's transform

Let $(X_t, B_t)_{t\geqslant 0}$ be the solution of an SDE with Markovian drift $b: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ and diffusion coefficient $\sigma: \mathbb{R}_+ \times \mathbb{R}^n \to \mathscr{L}(\mathbb{R}^m, \mathbb{R}^n)$ where B is the Brownian motion driving the SDE.

Let $h \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_{>0})$ be a stricly positive function such that

$$(\partial_t + \mathscr{L}) h(t,x) = 0,$$

for all $t \in [0, t_*]$ and $x \in \mathbb{R}^n$ where $\mathscr{L}$ is the generator of the SDE, i.e. $\mathscr{L} = b \cdot \nabla + \frac{1}{2}\mathrm{Tr}[\sigma \sigma^T \nabla^2]\big)$.

By Ito formula the process $Z_t := h(t, X_t)$ is a positive local martingale. Let us assume that $(Z_t)_{t \in [0, t_*]}$ is a (true) martingale and that $Z_0 = h(0, X_0) = 1$ (this can be always arranged by normalizing h). Then we can use the process $(Z_t)_t$ to define a new measure

$$\mathrm{d}\mathbb{Q} := Z_{t_*} \mathrm{d}\mathbb{P}.$$

(If needed we can extend $Z_t = Z_{t_*}$ if $t > t_*$). Note that by construction the process Z is continuous and $Z_0 = 1$.

By using Girsanov's theorem we know that the process

$$\tilde{B} = B - [B, L]$$

is a $\mathbb{Q}$-Brownian motion where L is the only local martingale such that $Z = \mathscr{E}(L)$. Since $\mathrm{d}Z_t = Z_t \mathrm{d}L_t$ we have that

$$\mathrm{d}Z_t = \sigma(t, X_t)^T \nabla h(t, X_t) \cdot \mathrm{d}B_t, \qquad \mathrm{d}L_t = Z_t^{-1} \mathrm{d}Z_t = \frac{\sigma^T(t, X_t) \nabla h(t, X_t)}{h(t, X_t)} \cdot \mathrm{d}B_t = \sigma^T(t, X_t) \nabla \log h(t, X_t) \cdot \mathrm{d}B_t$$

for $t \leqslant t_*$ and $\mathrm{d}Z_t = 0$ if $t > t_*$. Therefore

$$\mathrm{d}\tilde{B}_t = \mathrm{d}B_t - \sigma^T(t, X_t) \nabla \log h(t, X_t) \mathrm{d}t, \qquad t \in [0, t_*],$$

and $\mathrm{d}\tilde{B}_t = \mathrm{d}B_t$ if $t > t_*$. As consequence the process X solves now a new SDE (under $\mathbb{Q}$)

$$\mathrm{d}X_t = \underbrace{[b(t, X_t) + \sigma(t, X_t) \sigma^T(t, X_t) \nabla \log h(t, X_t)] \mathrm{d}t}_{\tilde{b}(t, X_t)\mathrm{d}t} + \sigma(t, X_t) \mathrm{d}\tilde{B}_t, \qquad t \in [0, t_*]$$

with the same diffusion coeffient σ but a new drift

$$\tilde{b}(t,x) = b(t,x) + \mathbb{1}_{t \in [0, t_*]} (\sigma \sigma^T \nabla \log h)(t,x), \qquad t \geqslant 0, x \in \mathbb{R}^n.$$

This construction is called *Doob's h-transform.*

Exercise 1. Try to perform the same construction for a martingale problem, i.e. not relying on the process B but only on X. I.e. starting from a measure $\mathbb{P}$ on the canonical path space $C(\mathbb{R}_+; \mathbb{R}^n)$ solving the martingale problem for $\mathscr{L}$ construct a new measure $\mathbb{Q}$ which solves a new martingale problem with a modified drift as above.

Example 7. Take $h(t,x) = \exp\big(\gamma \cdot x - \frac{1}{2}|\gamma|^2\big)$ where $\gamma \in \mathbb{R}^n$ and $t \geqslant 0$. Then the Doob's h-transformed process of a Brownian motion with this funciton gives a Brownian motion with drift.

If $(Z_t)_t$ is only a martingale in an open interval $I = [0, t_*)$ with possibly $t_* = +\infty$. Then we can still define $\mathbb{Q}$ on $\mathscr{F}_t$ to be given by $\mathrm{d}\mathbb{Q}|_{\mathscr{F}_t} := Z_t \mathrm{d}\mathbb{P}|_{\mathscr{F}_t}$ and check that this gives a well-defined probability measure on $\mathscr{F}_\infty = \vee_{t \geqslant 0} \mathscr{F}_t$. In this case is natural to restict all the measures to $\mathscr{F}_\infty$ i.e. to require $\mathscr{F}_\infty = \mathscr{F}$.

Remark 8. We do not need to require that h is positive everywhere (actually this will not be the case in the applications). What we need is that the process $Z_t = h(t, X_t)$ is a local martingale, i.e. $(\partial_t + \mathscr{L}) h(t, X_t) = 0$ a.s. and for almost every $t \geqslant 0$ and that $Z_t > 0$ almost surely. If h is not stricly positive we can always define the stopping time $T = \inf \{t \geqslant 0 \colon Z_t = 0\}$, then the stopped process $(Z_t^T)_{t \geqslant 0}$ is a positive local martingale and some condition is needed to ensure that it is a martingale. Remember that we require that $Z_0 = 1$ and by construction $(Z_t)_{t \geqslant 0}$ is continuous. In this setting one can perform the Doob's transform up to the random stopping time T. Note that under the measure $\mathbb{Q}$ we always have $T = +\infty$ almost surely.

3 Diffusion bridges

We use now Doob's transform to describe the regular conditional law of a Markovian diffusion $(X_t)_{t \geqslant 0}$ conditioned on the event that $X_T = y$ with $T > 0$ and deterministic, and $y \in \mathbb{R}^n$. I will assume also that $X_0 = x_0$. We need to assume that the process $(X_t)_{t \geqslant 0}$ is a Markov process with transition density given by

$$\mathbb{P}(X_t \in \mathrm{d}x' | X_s = x) = p(s, x; t, x') \mathrm{d}x', \qquad s < t \in [0, T], x, x' \in \mathbb{R}^n,$$

for some measurable and positive function p. Note that we cannot take $s = t$ here. Recall that $\mathbb{P}(X_t \in \mathrm{d}y | X_s = x)$ means the regular conditional probability kernel for the conditional law of X_t given X_s.

Define now the function

$$h^y(s, x) := \frac{p(s, x; T, y)}{p(0, x_0; T, y)}, \qquad s \in [0, T), x \in \mathbb{R}^n.$$

Let $Z_t^y := h^y(t, X_t)$, this is non-negative process, and it is also a martingale, indeed by the Markov property of X

$$\mathbb{E}[Z_t^y | \mathscr{F}_s] = \mathbb{E}[h^y(t, X_t) | \mathscr{F}_s] = \mathbb{E}[h^y(t, X_t) | X_s] = \int_{\mathbb{R}^n} h^y(t, x') p(s, X_s; t, x') \mathrm{d}x'$$

$$= \frac{1}{p(0, x_0; T, y)} \int_{\mathbb{R}^n} p(s, X_s; t, x') p(t, x'; T, y) \mathrm{d}x' = \frac{p(s, X_s; T, y)}{p(0, x_0; T, y)} = Z_s^y$$

by Chapman–Kolmogorov equations (the consistency condition for the transition density of a Markov process).

We want to define a probability kernel $(\mathbb{Q}^y)_{y\in\mathbb{R}^n}$ on $(\Omega,\mathscr{F})$ such that they are the regular conditional probabiliy of $\mathbb{P}$ given X_T, that is they have to satisfy

$$\mathbb{P}(A)=\mathbb{E}[\mathbb{P}(A|X_T)]=\mathbb{E}[\mathbb{Q}^{X_T}(A)]=\int_{\mathbb{R}^n}\mathbb{Q}^y(A)\mathbb{P}(X_T\in \mathrm{d}y)=\int_{\mathbb{R}^n}\mathbb{Q}^y(A)p(0,x_0;T,y)\mathrm{d}y$$

for all $A\in\mathscr{F}$. Take $A\in\mathscr{F}_s$ for some $s<T$, by Markov property we have for any bounded measurable g,

$$\mathbb{E}[\mathbb{1}_A g(X_T)]=\mathbb{E}[\mathbb{1}_A\,\mathbb{E}[g(X_T)|\mathscr{F}_s]]=\mathbb{E}[\mathbb{1}_A\,\mathbb{E}[g(X_T)|X_s]]=\mathbb{E}\Big[\mathbb{1}_A\int_{\mathbb{R}^n}g(y)p(s,X_s;T,y)\mathrm{d}y\Big]$$

$$=\int_{\mathbb{R}^n}g(y)\,\mathbb{E}[\mathbb{1}_A p(s,X_s;T,y)]\mathrm{d}y$$

since

$$\mathbb{E}[g(X_T)|X_s]=\int_{\mathbb{R}^n}g(y)p(s,X_s;T,y)\mathrm{d}y.$$

This means that we have $\mathbb{P}(A|X_T)=q(X_T)$ and we can take

$$q(y):=\mathbb{E}\Big[\mathbb{1}_A\frac{p(s,X_s;T,y)}{p(0,x_0;T,y)}\Big],$$

since we have proven that

$$\mathbb{E}[q(X_T)g(X_T)]=\mathbb{E}[\mathbb{1}_A g(X_T)]=\int_{\mathbb{R}^n}g(y)q(y)\,p(0,x_0;T,y)\mathrm{d}y.$$

As a consequence we can take

$$\mathbb{Q}^y(A):=\mathbb{E}\Big[\mathbb{1}_A\frac{p(s,X_s;T,y)}{p(0,x_0;T,y)}\Big],\qquad A\in\mathscr{F}_s$$

and have that $y\mapsto\mathbb{Q}^y$ indentify a well-defined probability kernel on $\mathscr{F}_{T-}$ since for any $A\in\mathscr{F}_{T-}$ the function $y\mapsto\mathbb{Q}^y(A)$ is measurable in y and for any y, $\mathbb{Q}^y$ is a probability in A.

Remark 9. Is it possible with some care to extend $\mathbb{Q}^y$ to the full $\mathscr{F}$, but we refrain to do so here.

We have now the formula

$$\mathbb{P}(A|X_T)=\mathbb{Q}^{X_T}(A),\qquad A\in\mathscr{F}_{T-}.$$

I want now to describe better the measure $\mathbb{Q}^y$ (at least up to time T), we observe that $\mathbb{Q}^y$ is obtained as the Doob's h-transform of $\mathbb{P}$ in the interval $[0,T)$ with $h=h^y$ function

$$h^y(s,x):=\frac{p(s,x;T,y)}{p(0,x_0;T,y)},\qquad s\in[0,T),x\in\mathbb{R}^n.$$

As a consequence we can show that the process X under $\mathbb{Q}^y$ satisfies an SDE provided I can apply Ito formula to h^y, that is I have to require that $(s,x) \mapsto p(s,x;T,y)$ is $C^{1,2}([0,T) \times \mathbb{R}^n)$. Given that Doob's transform give that X under $\mathbb{Q}^y$ solves the new SDE (or an equivalent martingale problem)

$$\mathrm{d}X_t = b(t,X_t)\mathrm{d}t + \sigma\sigma^T \nabla \log h^y(t,X_t)\mathrm{d}t + \sigma(t,X_t)\mathrm{d}B_t, \qquad t \in [0,T).$$

Is easy to see from specific examples that the function $\sigma\sigma^T \nabla \log h^y(t,x)$ is singular when $t \nearrow T$.

Exercise 2. Compute the SDE satisfied by a n-dimensional Brownian motion when we condition it to reach the point y at time $T > 0$.

Observe that under $\mathbb{Q}^y$ we have that

$$\mathbb{Q}^y\left(\lim_{t \uparrow T} X_t = z\right) = \mathbb{1}_{z=y}.$$

for any $y, z \in \mathbb{R}^n$. Observe also that

$$\mathbb{P}\left(\lim_{t \uparrow T} X_t = y\right) = \mathbb{P}(X_T = y) = 0$$

since X_T has density $p(0,x_0;T,\cdot)$. So the measures $\mathbb{Q}^y$ are all singular wrt. $\mathbb{P}$.

4 Doob's transform/Conditioning

Last lecture: we described the law of a diffusion $(X_t)_{t \geqslant 0}$ conditioned to reach a point y at a time T.

More precisely, take $\mathbb{P}$ to be the law of a diffusion $(X_t)_{t \geqslant 0}$,. The goal was to identify the probability kernel $y \in \mathbb{R}^n \mapsto \mathbb{P}^y \in \Pi(\mathscr{C}^n)$ such that we can disintegrate $\mathbb{P}$ as

$$\mathbb{P}(A) = \int_{\mathbb{R}^n} \mathbb{P}^y(A)\mathbb{P}(X_T \in \mathrm{d}y) = \mathbb{E}[\mathbb{P}^{X_T}(A)], \qquad A \in \mathscr{F}, \tag{2}$$

where $\mathbb{P}(X_T \in \mathrm{d}y)$ represent the law of X_T under $\mathbb{P}$. We define the law of X conditioned to reach a point y at a time T as the law of X under $\mathbb{P}^y$. Being the event $\{X_T = y\}$ of zero probability for $\mathbb{P}$ in general, this is a reasonable way to define this event. We see indeed that $\mathbb{P}^{X_T}(A) = \mathbb{E}[\mathbb{1}_A | X_T]$.

We had to assume that the process $(X_t)_{t \geqslant 0}$ is Markov wrt. the given filtration $(\mathscr{F}_t)_{t \geqslant 0}$ and that it has a transition probability given by the density $p(s,x;s',x')$ so that

$$\mathbb{P}(X_{s'} \in \mathrm{d}x' | X_s = x) = p(s,x;s',x')\mathrm{d}x', \qquad s < s', x, x' \in \mathbb{R}^n.$$

We can the introduce the *martingale* $Z_t^y = h^y(t,X_t)$ $t \in [0,T)$, given by

$$h^y(t,x) = \frac{p(t,x;T,y)}{p(0,x_0;T,y)}, \qquad t \in [0,T),$$

where we assume that $X_0 = x_0 \in \mathbb{R}^n$ and that $p(t,x;T,y) > 0$ for all x and $t \in [0,T)$. (this can be obtained by first conditioning on X_0 and then performing the construction). Usually $p(t,x;T,y)$ is not well defined when $t \to T$. E.g. in the case of Brownian motion one has

$$p(t,x;T,y) = (2\pi(T-t))^{-d/2}\exp\left(-\frac{|x-y|^2}{2(T-t)}\right).$$

Then one use this to construct the Doob's transformed measure $\mathbb{P}^y$ on $\mathscr{F}_{T-}$ by letting

$$\mathrm{d}\mathbb{P}^y|_{\mathscr{F}_t} = Z_t^y \mathrm{d}\mathbb{P}|_{\mathscr{F}_t}, \quad t \in [0,T).$$

And one can check that this definition satisfy (2) for $A \in \mathscr{F}_{T-}$. Now if $A_2 \in \sigma(X_t : t \geqslant T)$ we have

$$\mathbb{P}(A_2) = \mathbb{E}[\mathbb{E}[\mathbb{1}_{A_2}|\mathscr{F}_T]] = \mathbb{E}[\mathbb{E}[\mathbb{1}_{A_2}|X_T]] = \mathbb{E}[\varphi^{A_2}(X_T)]$$

with $\varphi^A(x) = \mathbb{E}[\mathbb{1}_A | X_T = x]$. So now consider also an event $A_1 \in \sigma(X_t : t < T)$. In this case we have

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{E}[\mathbb{1}_{A_1}\mathbb{E}[\mathbb{1}_{A_2}|\mathscr{F}_T]] = \mathbb{E}[\mathbb{1}_{A_1}\varphi^{A_2}(X_T)] = \int_{\mathbb{R}^n} \mathbb{P}^y(A_1)\varphi^{A_2}(y)\mathbb{P}(X_T \in \mathrm{d}y).$$

Let assume that we have proven that

$$\mathbb{P}^y\left(\lim_{t \uparrow T} X_t = x\right) = \mathbb{1}_{x=y}, \qquad x \in \mathbb{R}^n,$$

then we can write

$$\mathbb{P}(A_1 \cap A_2) = \int_{\mathbb{R}^n} \underbrace{\mathbb{E}^y[\mathbb{1}_{A_1}\varphi^{A_2}(X_T)]}_{\mathbb{P}^y(A_1 \cap A_2)} \mathbb{P}(X_T \in \mathrm{d}y).$$

So this shows us that we can define

$$\mathbb{P}^y(A_1 \cap A_2) = \mathbb{E}^y[\mathbb{1}_{A_1}\varphi^{A_2}(X_T)] = \mathbb{E}^y[\mathbb{1}_{A_1}]\varphi^{A_2}(y).$$

This defines $\mathbb{P}^y$ in $\sigma(X_t : t < T) \vee \sigma(X_t : t \geqslant T) = \sigma(X_t : t \geqslant 0)$. So $\mathbb{P}^y$ can be used to define the conditional law of X.

One can then show that if the process X satisfies the SDE

$$\mathrm{d}X_t = b(t,X_t)\mathrm{d}t + \sigma(t,X_t)\mathrm{d}B_t, \qquad t \geqslant 0$$

then under $\mathbb{P}^y$ the process X satisfies the SDE (provided $h^y(t,x)$ is $C^{1,2}$ for any $t < T$)

$$\mathrm{d}X_t = [b(t,X_t) + (\sigma\sigma^T \nabla \log h^y)(t,X_t)]\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t, \qquad t < T$$

and

$$\mathrm{d}X_t = b(t,X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t, \qquad t \geqslant T.$$

Recall that under $\mathbb{P}^y$ we have $X_{T-} = X_T = y$.

Remark 10. This approach can be exteded to condition a diffusion to reach a sequence of states $y_1, \ldots, y_n$ at given times $T_1 < \cdots < T_n$.

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5 Condition a diffusion to not leave a domain

Consider the following situation: we want to condition a one dimensional Brownian motion $(B_t)_{t \geqslant 0}$ to stay positive for all times $t \geqslant 0$. This event has probability zero (since eventually BM will visit zero and by strong Markov property will have $1/2$ probability to go negative + Borel-Cantelli). So the idea is to use less singular conditioning to arrive to describe this event.

Assume $B_0 = x_0 > 0$. In this case the convenient thing to do is to fix $R > x_0$ and ask consider the stopping time

$$T_R := \inf \{t \geqslant 0: B_t \notin [0, R]\}$$

and the event $E_R := \{B_{T_R} = R\}$. Now we know that $\mathbb{P}(E_R) = x_0 / R \in (0, 1)$. We can then define the conditional probability

$$\mathbb{P}^R(A) := \frac{\mathbb{P}(A \cap E_R)}{\mathbb{P}(E_R)}$$

and now we would like to send $R \to \infty$ and study the limit.

Let $T_x = \inf \{t \geqslant 0: B_t = x\}$. We want to say that

$$\{T_0 = +\infty\} = \cap_{R > 0} \{B_{T_R} = R\}$$

note that $(E_R)_R$ is a descreasing sequence of events.

I want to describe $\mathbb{P}^R$. Let $A \in \mathscr{F}_s$, observe that

$$\mathbb{P}^R(A) = \frac{\mathbb{P}(A \cap E_R)}{\mathbb{P}(E_R)} = \frac{\mathbb{E}[\mathbb{1}_A \mathbb{E}[\mathbb{1}_{E_R} | \mathscr{F}_s]]}{\mathbb{P}(E_R)} = \frac{\mathbb{E}[\mathbb{1}_A \mathbb{E}[\mathbb{1}_{T_R > s} \mathbb{1}_{E_R} | \mathscr{F}_s]] + \mathbb{E}[\mathbb{1}_A \mathbb{E}[\mathbb{1}_{T_R \leqslant s} \mathbb{1}_{E_R} | \mathscr{F}_s]]}{\mathbb{P}(E_R)}$$

$$= \frac{\mathbb{E}[\mathbb{1}_A \mathbb{1}_{T_R > s} \mathbb{P}_{X_s}(E_R)] + \mathbb{E}[\mathbb{1}_A \mathbb{1}_{T_R \leqslant s} \mathbb{1}_{B_{T_R} = R}]}{\mathbb{P}(E_R)}$$

where $\mathbb{P}_x(E_R)$ is the probability of E_R for a BM starting at x at time 0. Note that $\mathbb{P}_0(E_R) = 0$ and $\mathbb{P}_R(E_R) = 1$ therefore setting

$$h(x) = \mathbb{P}_x(E_R) / \mathbb{P}_{x_0}(E_R) = x / x_0$$

we have that

$$\mathbb{P}^R(A) = \mathbb{E}[\mathbb{1}_A \mathbb{1}_{T_R > s} h(B_s)] + \mathbb{E}[\mathbb{1}_A \mathbb{1}_{T_R \leqslant s} h(B_{T_R})] = \mathbb{E}[\mathbb{1}_A h(B_{s \wedge T_R})].$$

Remember that we did this for any $s \geqslant 0$ and $A \in \mathscr{F}_s$. So

$$\mathrm{d}\mathbb{P}^R|_{\mathscr{F}_s} = h(B_{s \wedge T_R}) \mathrm{d}\mathbb{P}|_{\mathscr{F}_s}.$$

If we take $Z_t^R = h(B_{t \wedge T_R})$ then $(Z_t^R)_{t \geqslant 0}$ is a non-negative martingale (indeed $0 \leqslant Z_t^R \leqslant R$).

We have to pay attention to the fact that Z_t^R could touch zero and this happens at the stopping time T_0. After time T_0 the process Z_t^R will stay in zero.

(Next lecture we continue)

Note that

$$Z_t^R = \mathscr{E}(L)_t, \qquad L_t = \int_0^t \mathbb{1}_{s \leqslant T_R} (\log h)'(B_s) \mathrm{d}B_s.$$

By Girsanov's theorem we have that under the measure $\mathbb{P}^R$ the process B satisfy the SDE

$$\mathrm{d}B_t = \frac{\mathbb{1}_{t \leqslant T_R}}{B_t} \mathrm{d}t + \mathrm{d}W_t, \qquad t \geqslant 0,$$

where W is a $\mathbb{P}^R$-Brownian motion.

6 Conditioning Brownian motion to stay positive

$(B_t)_{t \geqslant 0}$ BM. $R > 0$ $T_x := \inf\{t \geqslant 0 : B_t = x\}$ and $S_R := T_R \wedge T_0$. $B_0 = x_0 \in (0, R)$. Note that $\mathbb{P}(T_x < \infty) = 1$ for all $x \in \mathbb{R}$. We introduced the measure

$$\mathbb{Q}^R(A) = \mathbb{E}(\mathbb{1}_A h(B_{s \wedge S_R})), \qquad A \in \mathscr{F}_s$$

with $h(x) = x / x_0$. By Girsanov's theorem we have that under $\mathbb{Q}^R$ $(B_t)_{t \geqslant 0}$ solves the SDE

$$\mathrm{d}B_t = \frac{\mathbb{1}_{t < S_R}}{B_t} \mathrm{d}t + \mathrm{d}W_t, \qquad t \geqslant 0$$

where W is a $\mathbb{Q}^R$ Brownian motion. Important observation

$$\mathbb{Q}^R(T_0 < T_R) = \mathbb{E}(\mathbb{1}_{T_0<T_R} h(B_{S_R})) = \mathbb{E}(\mathbb{1}_{T_0<T_R} h(B_{T_0})) = 0$$

so under $\mathbb{Q}^R$ we will never touch 0 before R and therefore under $\mathbb{Q}^R$ we have $S_R = T_R$ almost surely and that $B_t > 0$ for all $t \in [0, T_R]$.

Now we want to take the limit $R \to \infty$. Observe that if $A \in \mathscr{F}_{T_R}$ then $\mathbb{Q}^R(A) = \mathbb{Q}^{R'}(A)$ for any $R' \geqslant R$ since $T_{R'} > T_R$ and

$$\mathbb{E}_{\mathbb{Q}^{R'}}[\mathbb{1}_A] = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_A h(B_{T_R \wedge S_{R'}})] = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_A \mathbb{E}[h(B_{T_R \wedge S_{R'}})|\mathscr{F}_{T_R}]] = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_A h(B_{T_R \wedge S_R})] = \mathbb{E}_{\mathbb{Q}^R}[\mathbb{1}_A].$$

Thefore for any $R > 0$ and $A \in \mathscr{F}_{T_R}$ we can define a measure $\mathbb{Q}$ by letting $\mathbb{Q}(A) := \lim_{R' \to \infty} \mathbb{Q}^{R'}(A)$. Therefore the measure is well defined on $\vee_{R \geqslant 0} \mathscr{F}_{T_R}$. Observe that $\lim_{R \to \infty} \mathbb{Q}(T_R \leqslant s) = 0$ by continuity of B. For any A $\in \mathscr{F}_s$ we can define $\mathbb{Q}(A) := \lim_{R \to \infty} \mathbb{Q}(A, T_R > s)$ and check that it is the unique extension of $\mathbb{Q}$ which is consistent with it on $\vee_{R \geqslant 0} \mathscr{F}_{T_R}$.

Under $\mathbb{Q}$ the process B satisfies the SDE

$$\mathrm{d}B_t = \frac{1}{B_t}\mathrm{d}t + \mathrm{d}W_t, \qquad t \geqslant 0.$$

Therefore we discovered that under $\mathbb{Q}$ the process B satisfies the SDE defining the Bessel process R of dimension $d = 3$:

$$\mathrm{d}R_t = \frac{d-1}{2}\frac{1}{R_t}\mathrm{d}t + \mathrm{d}W_t, \qquad t \geqslant 0.$$

Recall that the Bessel process $R_t = |X_t|$ is defined as the modulus of a d-dimensiona Brownian motion $(X_t)_{t \geqslant 0}$.

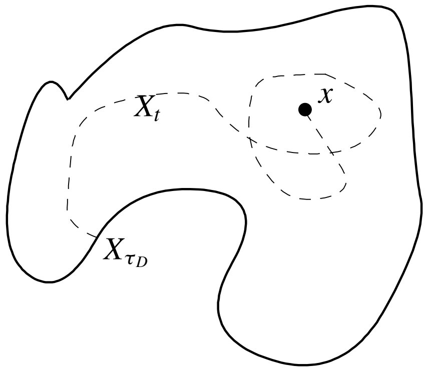
Theorem 11. *The law of a one dimensional Brownian motion conditioned never to hit zero is the same as the law of the modulus of a* 3*d Brownian motion.*

From this identification we can derive the corollary that the BM conditioned not to hit zero is transient and go to infinity with probability 1.

7 Condition a diffusion not to leave a domain

In the previous argument we relien on many properties of the problem (i.e. Brownian motion, one dimension). We would like now to sketch how to solve the conditioning problem for more general processes and more general domains. We will not give all the details or all the necessary assumptions.

Assume $(X_t)_{t \geqslant 0}$ is a Markov diffusion process in $\mathbb{R}^d$ (think to the solution of an SDE with nice coefficients) and let $D \subseteq \mathbb{R}^d$ be a open, connected domain and bounded and let $\tau_D = \inf\{t \geqslant 0 : X_t \notin D\}$. Let $\mathbb{P}_x$ be the law of the Markov process starting from $x \in \mathbb{R}^d$ and let $\mathscr{L}$ the generator of the process.



Let us assume that $\mathbb{P}(\tau_D < \infty) = 1$. If we want to force the process to stay in D forever we are asking something which has probability 0 under $\mathbb{P}$. So we start by approximating the event we want and just ask that the process stay in D up to time $T > 0$ and define $\mathbb{Q}^T$ as the corresponding conditional probability:

$$\mathbb{Q}^T(A) := \frac{\mathbb{E}_{x_0}[\mathbb{1}_A \mathbb{1}_{\tau_D > T}]}{\mathbb{E}[\mathbb{1}_{\tau_D > T}]}.$$

By reasoning as in the previous example we obtain the formula: for any $A \in \mathscr{F}_s$ and $s < T$

$$\mathbb{Q}^T(A) = \frac{\mathbb{E}_{x_0}[\mathbb{1}_A g^{T-s}(X_{s \wedge \tau_D})]}{g^T(x_0)} = \mathbb{E}_{x_0}[\mathbb{1}_A Z_s^T]$$

where

$$g^T(x) := \mathbb{E}_x[\mathbb{1}_{\tau_D > T}].$$

The formula follows by a simple application of the Markov property. It also holds that $Z_s^T := g^{T-s}(X_{s \wedge \tau_D}) / g^T(x_0)$ is a martingale up to time T and $Z_0^T = 1$. Note that we have $\mathbb{Q}^T(\tau_D \leqslant T) = 0$ so the process X will never hit the boundary before T under $\mathbb{Q}^T$. We will assume that $(T, x) \mapsto g^T(x)$ is $C^{1,2}$, under this condition we can perform Doob's transformation and deduce that if under $\mathbb{P}$ the process X satisfies the SDE

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t$$

then under $\mathbb{Q}^T$ it will satisfy the SDE (we denote here $\sigma(t, X_t)^*$ the transpose of $\sigma(t, X_t)$)

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\sigma(t, X_t)^* \frac{\nabla g^{T-t}(X_t)}{g^{T-t}(X_t)}\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t^T, \qquad t \in [0, T]$$

provided $g^t(x) > 0$ for all $t > 0$ and $x \in D$. Note that $g^T(x) = 0$ if $x \in \partial D$. To take $T \to \infty$ in this SDE is now a bit more difficult that in the BM case because the drift depends on T (and therefore the Brownian motion W^T depends on T). So assumptions have to be made and essentially the most important is to require that the existence of the limit

$$\lim_{T \to \infty} \frac{\nabla g^T(x)}{g^T(x)}$$

and moreover that it is given by

$$\lim_{T\to\infty}\frac{\nabla g^T(x)}{g^T(x)}=\frac{\nabla\varphi_0(x)}{\varphi_0(x)}$$

where φ_0 is the eigenfunction of $(-\mathscr{L})$ with Dirichlet boundary conditions on D and with lowest eigenvalue $\lambda_0>0$. The idea is that the function $g^T(x)$ for $T\to\infty$ has the asymptotic expansion

$$g^T(x)=e^{-\lambda_0 T}\varphi_0(x)+o(e^{-\lambda_0 T})$$

uniformly in $x\in D$ and the same for the derivative $\nabla g^T(x)$. Moreover note that the function g if it is $C^{1,2}(\mathbb{R}_+\times D)\cap C(\bar{D})$ then it is a solution to the parabolic PDE

$$\partial_t g^t(x)=\mathscr{L}g^t(x),\qquad x\in D,$$

$$g^t(x)=0,\quad x\in\partial D.$$

In order to be sure that these conditions are met one has to make more precise assumptions on b,σ and on D.

$$-\mathscr{L}\varphi_0=\lambda_0\varphi_0$$

and $\varphi_0(x)=0$ for $x\in\partial D$ and $\varphi_0(x)>0$ for $x\in D$.

In this setting one can prove that the family $(\mathbb{Q}^T)_{T>0}$ weakly convergence to a measure $\mathbb{Q}$ such that the process X is a weak solution to the SDE

$$\mathrm{d}X_t=b(t,X_t)\mathrm{d}t+\sigma(t,X_t)\sigma(t,X_t)^*\frac{\nabla\varphi_0(X_t)}{\varphi_0(X_t)}\mathrm{d}t+\sigma(t,X_t)\mathrm{d}W_t,\qquad t\in[0,T].$$

So in general one can expect that a diffusion conditioned to stay inside a given domain satisfy this SDE as soon as we can solve the Dirichlet problem and the solution is suitably regular.

For details see the work of Pisky on Annals of Probability ('80).

Remark 12. This approach cannot be directly used for conditioning BM to stay positive because in this case the function $g^T(x)$ does not decay exponentially in T as $T\to\infty$. The Brownian motion can stay away from zero by going very far out, and this happens with algebraically decaying probability.

Example 13. Take $(X_t)_{t\geqslant 0}$ to be Brownian motion in $d=1$ and $D=(0,L)$. In this case we can make even precise the above discussion. However the conclusion is that if we condition the BM to stay in D forever it will statisfy the SDE

$$\mathrm{d}X_t=\frac{\nabla\varphi_0(X_t)}{\varphi_0(X_t)}\mathrm{d}t+\mathrm{d}W_t,\qquad t\geqslant 0$$

where φ_0 is the lowest eigenfunction of $-\Delta$ with zero b.c. on $[0,L]$, namely $\varphi_0=\sin(\pi x/L)$. The eigenvalue $\lambda_0=(\pi/L)^2$ describe the exponential decay of the probability $\mathbb{P}_x(\tau_D>T)\approx e^{-\lambda_0 T}\varphi_0(x)$. Therefore we have

$$\mathrm{d}X_t=\frac{\pi}{L}\frac{\cos(\pi X_t/L)}{\sin(\pi X_t/L)}\mathrm{d}t+\mathrm{d}W_t^L,\qquad t\geqslant 0.$$

Now if we take (formally) the limit $L \to \infty$ we have

$$\mathrm{d}X_t = \frac{1}{X_t}\mathrm{d}t + \mathrm{d}W_t, \qquad t \geqslant 0.$$

as we expect from our previous computations for the BM conditioned to stay positive.

Example 14. (Brownian motion in the Weyl chamber) Let $X = (X^1, \ldots, X^n)$ be a family of n independent one dimensional BMs and let $S = \{x \in \mathbb{R}^d : x^1 < x^2 < \cdots < x^n\}$ (Weyl chamber) and take $x_0 \in S$ and $X_0 = x_0$. I want to condition X to stay inside S. Consider the function

$$h(x) = \frac{\prod_{i<j} (x^j - x^i)}{\prod_{i<j} (x_0^j - x_0^i)}, \qquad x \in S$$

which is strictly positive in S and 0 on the boundary of S. One can check that this function is harmonic in S, i.e. $(-\Delta_{\mathbb{R}^n})h(x) = 0$ where $\Delta_{\mathbb{R}^n}$ is the Laplacian in $\mathbb{R}^n$ which is the generator of X. Assuming that this is the relevant eigenfunction for describing the conditioning, we get that X conditioned to stay in S solves the SDE

$$\mathrm{d}X^i = \sum_{j \neq i} \frac{1}{X_t^j - X_t^i}\mathrm{d}t + \mathrm{d}W_t^i, \qquad t \geqslant 0.$$

Some comments about Exercise 2 in Sheet 5. Brownian bridge: BM conditioned to reach a point at a given time. E.g. $y \in \mathbb{R}$ at time $t = 1$. Two ways to construct it. Let $(X_t)_t$ be a Brownian motion in $d = 1$

a) Gaussian approach: define the process

$$X_t^y := X_t + t(X_1 - y),$$

and show that X_1, X^y are independent (via computation of the covariance). So the law μ^y of X^y is the law of the BM conditioned to arrive in y at time 1.

$$\mu^y(\mathrm{d}\omega) := \mathbb{P}(X \in \mathrm{d}\omega | X_t = 1)$$

(we use the fact that BM is a Gaussian process). This method can be used for other Gaussian processes.

b) SDE/Markovian approach (following the method in the lecture) (we use that BM is a Markov process with known transition density, the method works for a large class of Markov processes). The canonical process Y under the conditional law μ^y satisfies the SDE

$$\mathrm{d}Y_t = -\mathbb{1}_{t<1}\frac{Y_t - y}{1 - t}\mathrm{d}t + \mathrm{d}B_t$$

The two descriptions have to agree. Indeed note that Y is a Gaussian process (since the SDE is linear): think how to prove it and then check that covariance and mean agree. But of course the construction itself shows that the law of Y and the law of X^y agree.

Note that $(X_t^y)_{t\in[0,1]}$ is not adapted to the filtratrion $\mathscr{F}^X$ of $(X_t)_{t\geqslant 0}$ but it is adapted to the "enlarged filtration" $\mathscr{H}_t := \mathscr{F}_t^X \vee \sigma(X_1)$.

If we consider the process $(X_t^y)_{t\in[0,1]}$ with respect to its own filtration $(\mathscr{G}_t)_{t\geqslant 0}$ (which is smaller than $(\mathscr{H}_t)_t$) then by considering the associated martingale problem and using point b) we deduce that there should exists a Brownian motion $\tilde{B}$ on the same probability space (since $\sigma(x) > 0$ and we are in one dimension, so our simplified argument given at the beginning of the course works) such that

$$X_t^y = -\mathbb{1}_{t<1}\frac{X_t^y - y}{1-t}\mathrm{d}t + \mathrm{d}\tilde{B}_t,$$

This means that $(X_t^y)_t$ is a semimartingale and if we can determine the drift of X_t^y (by some statistical procedure) we can infer the point y (if we know that it is a BM conditioned to arrive somewhere at time 1). At the same time the BM $(\tilde{B})$ contains the "new information" which cannot be predicted by observing the process (and so it is martingale).

So enlarging the filtration by adding the observation $\sigma(X_1)$ to all the σ-fields transforms the BM into the Brownian Bridge above, however it still remains a semimartingale.

There is a whole subfield of stochastic analysis which try to understand how stochastic processes change when enlarging the filtrations (problem of enlargement of filtrations).

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8 General change of drift

We want now use Girsanov transform to create new processes starting from known ones in more general ways than Doob's transform allows.

For any continuous local martingale $(L_t)_{t\geqslant 0}$ and a probability measure $\mathbb{P}$ on a filtered space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0})$ we can consider the new measure

$$\mathrm{d}\mathbb{Q}^L = \mathscr{E}(L)_\infty \mathrm{d}\mathbb{P}$$

provided $\mathbb{E}[\mathscr{E}(L)_\infty] = 1$. In this case one can show that $(\mathscr{E}(L)_t)_{t\geqslant 0}$ is a martingale and the Girsanov theorem tells us that

$$\tilde{M}_t = M_t - [L, M]_t, \qquad t \geqslant 0$$

is a continuous local $\mathbb{Q}$-martingale for any local $\mathbb{P}$-martingale M.

So a key point here is how we check that

$$\mathbb{E}[\mathscr{E}(L)_\infty] = \mathbb{E}\left[\exp\left(L_\infty - \frac{1}{2}[L]_\infty\right)\right] = 1,$$

in practice indeed is not obvious how to estimate the local martingale L_∞ appearing there.

Example 15. To keep in mind a useful example one could think that $(X_t)_{t\geqslant 0}$ is a $\mathbb{P}$-BM in $\mathbb{R}^n$ and that we are given a vector field $b: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ measurable and let

$$L_t := \int_0^{t\wedge T} b(s, X_s) \mathrm{d}X_s$$

for some T which can be finite or not and maybe a stopping time. In this case we have

$$\mathbb{E}[\mathscr{E}(L)_\infty] = \mathbb{E}\left[\exp\left(\int_0^T b(s, X_s)\mathrm{d}X_s - \frac{1}{2}\int_0^T |b(s, X_s)|^2 \mathrm{d}s\right)\right].$$

In order to estimate this kind of quantities and establish that $(\mathscr{E}(L)_t)_{t\geqslant 0}$ is a martingale it is useful to consider Novikov's condition.

Observe that $(\mathscr{E}(L)_t)_{t\geqslant 0}$ is always a positive local-martingale and is supermartingale (by a Fatou argument) so the point is to show that $\mathbb{E}[\mathscr{E}(L)_\infty] \geqslant 1$ since we know that $\mathbb{E}[\mathscr{E}(L)_\infty] \leqslant 1$.

If you recall the example of the BM with drift in that case one would indeed have $\mathscr{E}(L)_\infty = 0$ a.s.

Novikov condition (in the form given to it by Krylov) is a sufficient criterion to ensure that $\mathbb{E}[\mathscr{E}(L)_\infty] = 1$.

Theorem 16. *(Novikov-Krylov's condition) Let L be a local martingale starting at* 0 *and assume that*

$$\lim_{\varepsilon \downarrow 0} \varepsilon \log \mathbb{E}\left[\exp\left(\frac{1-\varepsilon}{2}[L]_\infty\right)\right] = 0$$

then $\mathbb{E}[\mathscr{E}(L)_\infty] = 1$. *In particular this holds if (this is usually called Novikov's condition)*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}[L]_\infty\right)\right] < \infty.$$

Remark 17. This is not a necessary conditions, it does not care of the sign of L but there are examples where $\mathbb{E}[\mathscr{E}(L)_\infty] = 1$ but $\mathbb{E}[\mathscr{E}(-L)_\infty] < 1$. A finer condition is Kazamaki condition, which reads

$$\mathbb{E}[\exp(L_\infty/2)] < \infty,$$

but this is not easy to check usually since there are not many ways to estimate the exponential of a stochastic integral. To see that Kazamaki is finer than Novikov, observe that

$$\begin{aligned}
\mathbb{E}[\exp(L_\infty/2)] &= \mathbb{E}[\exp(L_\infty/2 - [L]_\infty/4)\exp([L]_\infty/4)] \\
&= \mathbb{E}[\mathcal{E}(L)_\infty^{1/2}\exp([L]_\infty/4)] \\
&\leqslant \mathbb{E}[\mathcal{E}(L)_\infty]^{1/2}\,\mathbb{E}[\exp([L]_\infty/2)]^{1/2} \\
&\leqslant \mathbb{E}[\exp([L]_\infty/2)]^{1/2}
\end{aligned}$$

where we used Hölder and the fact that $\mathbb{E}[\mathcal{E}(L)_\infty] \leqslant 1$. In particular, if $\mathbb{E}[\exp([L]_\infty/2)]$ is finite, then so must be $\mathbb{E}[\exp(L_\infty/2)]$, while the converse is not necessarily true.

Example 18. Continuing Example 15. Novikov's condition reads

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T |b(s,X_s)|^2 \mathrm{d}s\right)\right] < \infty.$$

It is not difficult to show by using the Markov property of the Brownian motion and the assumption that b is of **linear growth**, i.e. that there exists a constant $C<\infty$ such that

$$|b(t,x)|^2 \leqslant C(1+|x|^2), \qquad x\in\mathbb{R}^n, t\geqslant 0,$$

that Novikov's condition is in this case satisfied. Take $T\geqslant 0$ to be a deterministic time. Then we can defined the measure $\mathbb{Q}^T$ as above and under $\mathbb{Q}^T$ the process

$$\tilde{X}_t = X_t - \int_0^{t\wedge T} b(X_s)\mathrm{d}s, \qquad t\geqslant 0$$

is a $\mathbb{Q}^T$ Brownian motion. (pay attention that this equation is for $\mathbb{R}^n$-valued processes). Now the family of measures $(\mathbb{Q}^T, \mathscr{F}_T)_{t\geqslant 0}$ is a consistent family and therefore it admits a unique extension $\mathbb{Q}^\infty$ to $\mathscr{F}_\infty = \vee_T \mathscr{F}_T$. This happens because the process $(\mathscr{E}(L)_t)_{t\geqslant 0}$ with

$$L_t = \int_0^t b(s,X_s)\mathrm{d}X_s$$

is a martingale for $t\geqslant 0$ excluding $t=\infty$ (is not uniformly integrable in general). Under $\mathbb{Q}^\infty$ we have that X satisfy the SDE

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \mathrm{d}B_t \qquad t\geqslant 0.$$

for some $\mathbb{Q}^\infty$–Brownian motion B.

Remark 19. (Another Novikov-type condition) In some situations (including Example 18 above), it is useful to apply the following criterion (see Exercise 1.40 from Revuz-Yor), instead of trying to verify Novikov directly.

Let B be a Brownian motion, H a predictable process and $T>0$ fixed; assume that there exist constants $a,c>0$ such that $\mathbb{E}[\exp(a|H_t|^2)]\leqslant c$ for all $t\in[0,T]$. Then for $L_t := \int_0^t H_s\cdot \mathrm{d}B_s$, it holds

$$\mathbb{E}[\mathcal{E}(L)_T] = 1.$$

Examples of such H include $H_t = b(B_t)$ for b of linear growth as above, but also any Gaussian process (e.g. $H=\tilde{B}$ BM independent of B).

Theorem 20. *If b is of linear growth uniformly in time then there exists a weak solution to the SDE in $\mathbb{R}^n$*

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \mathrm{d}B_t \qquad t\geqslant 0.$$

We can apply the same method to more general equations. Starting from the solution of an SDE of the form

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t \qquad t \geqslant 0$$

and performing the change of drift to a new measure $\mathbb{Q}$ given by the density $(\mathscr{E}(L)_t)_{t \geqslant 0}$ wrt. $\mathbb{P}$ with

$$L_t = \int_0^t c(X_s)\mathrm{d}B_s$$

(assuming $(\mathscr{E}(L)_t)_{t \geqslant 0}$ is a martingale) we obtain that under $\mathbb{Q}$ the process X is the solution of the SDE $(\tilde{B}_t = B_t - \int_0^t c(X_s)\mathrm{d}s)$

$$\mathrm{d}X_t = (b(X_t) + \sigma(X_t)c(X_t))\mathrm{d}t + \sigma(X_t)\mathrm{d}\tilde{B}_t \qquad t \geqslant 0.$$

So we can change the drift only in directions belonging to the image of $\sigma(x)$.

This observation has application in coupling of diffusions.

Exercise 3. Think about how to perform this change of drift in a martingale problem formulation. (the difficulty is that the is no B in view in the martingale problem). Here is meant without going thru the SDE formulation of martingale problem.

Proof. (of Novikov's condition, via Krylov's proof) The goal is to prove that $\mathbb{E}[\mathscr{E}(L)_\infty] = 1$ and by Fatou and a stopping time argument it is enough to check that $\mathbb{E}[\mathscr{E}(L)_\infty] \geqslant 1$.

We start by observing that by Hölder's inequality

$$\mathbb{E}[\mathscr{E}((1-\varepsilon)L)_\infty] = \mathbb{E}\left(\exp\left[(1-\varepsilon)\left(L_\infty - \frac{1}{2}[L]_\infty\right)\right]\exp\left[\frac{\varepsilon(1-\varepsilon)}{2}[L]_\infty\right]\right)$$

$$\leqslant \left\{\mathbb{E}\left(\exp\left[p(1-\varepsilon)\left(L_\infty - \frac{1}{2}[L]_\infty\right)\right]\right)\right\}^{1/p}\left\{\mathbb{E}\left(\exp\left[q\frac{\varepsilon(1-\varepsilon)}{2}[L]_\infty\right]\right)\right\}^{1/q}$$

$$\leqslant \left\{\mathbb{E}\left(\exp\left[L_\infty - \frac{1}{2}[L]_\infty\right]\right)\right\}^{(1-\varepsilon)}\left\{\mathbb{E}\left(\exp\left[\frac{(1-\varepsilon)}{2}[L]_\infty\right]\right)\right\}^{\varepsilon}$$

$$\leqslant \{\mathbb{E}(\mathscr{E}(L)_\infty)\}^{(1-\varepsilon)}\left\{\mathbb{E}\left(\exp\left[\frac{(1-\varepsilon)}{2}[L]_\infty\right]\right)\right\}^{\varepsilon}$$

Now we are in good shape because we just need to control L multiplied with a coefficient less than 1. Therefore is enough to show that $\mathbb{E}[\mathscr{E}((1-\varepsilon)L)_\infty] = 1$ given that we know by assumption that

$$\lim_{\varepsilon\downarrow 0}\left\{\mathbb{E}\left(\exp\left[\frac{(1-\varepsilon)}{2}[L]_\infty\right]\right)\right\}^\varepsilon = 1.$$

Now we use again the Hölder trick and try to prove that $\mathscr{E}((1-\varepsilon)L)_\infty \in L^p$ for some $p > 1$ because in this case I can prove (by localization) that $(\mathscr{E}((1-\varepsilon)L)_t)_{t\geqslant 0}$ is a uniformly integrable martingale and therefore that $\mathbb{E}[\mathscr{E}((1-\varepsilon)L)_\infty] = 1$. So take some $p > 1$ and observe that (to be done rigorously via a localizing sequence L^{T_n})

$$\mathbb{E}[\mathscr{E}((1-\varepsilon)L)_\infty^p] = \mathbb{E}\left[\exp\left(p(1-\varepsilon)L_\infty - \frac{1}{2}p(1-\varepsilon)^2[L]_\infty\right)\right]$$

one can now apply Hölder to split again this expectation into the form

$$\leqslant\left(\mathbb{E}\left[\exp\left(p'p(1-\varepsilon)\left(L_\infty - \frac{1}{2}[L]_\infty\right)\right)\right]\right)^{1/p'}\left(\mathbb{E}\left[\exp\left(q'\frac{c(p,\varepsilon)}{2}[L]_\infty\right)\right]\right)^{1/q'}$$

take p' so that $p'p(1-\varepsilon) = 1$ and observe that (thinking about localization we have $\mathbb{E}[\mathscr{E}(L)_\infty] = 1$)

$$\leqslant(\mathbb{E}[\mathscr{E}(L)_\infty])^{1/p'}\left(\mathbb{E}\left[\exp\left(q'\frac{c(p,\varepsilon)}{2}[L]_\infty\right)\right]\right)^{1/q'}$$

$$\leqslant\left(\mathbb{E}\left[\exp\left(q'\frac{c(p,\varepsilon)}{2}[L]_\infty\right)\right]\right)^{1/q'}$$

and now one check that all the coefficients are such there exists an $\varepsilon' \in (0,1)$ (for suitable choice of p and sufficiently small $\varepsilon > 0$) such that

$$q'\frac{c(p,\varepsilon)}{2} \leqslant \frac{1-\varepsilon'}{2}$$

which implies

$$\mathbb{E}[\mathscr{E}((1-\varepsilon)L)_\infty^p] \leqslant \left(\mathbb{E}\left[\exp\left(\frac{1-\varepsilon'}{2}[L]_\infty\right)\right]\right)^{1/q'}.$$

Using again our assumption we know that $\mathbb{E}\left[\exp\left(\frac{1-\varepsilon'}{2}[L]_\infty\right)\right] < \infty$ for all $\varepsilon' > 0$ so this allow to conclude that

$$\mathbb{E}[\mathscr{E}((1-\varepsilon)L)_\infty^p] < \infty$$

and this finish the proof the theorem. □

Next lecture: uniqueness in law via Girsanov's theorem and maybe the Brownian martingale representation theorem.

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9 Uniqueness in law via Girsanov's theorem

Consider the SDE in $\mathbb{R}^n$ with initial condition $X_0 = x_0 \in \mathbb{R}^n$

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \mathrm{d}B_t, \qquad t \geqslant 0$$

where $b: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a measurable time-dependent vector field. We are going to assume that

$$\int_0^T |b(s, X_s)|^2 \mathrm{d}s < +\infty, \qquad \text{a.s. for all } T \geqslant 0. \tag{3}$$

The goal is to show that under this condition all weak solutions of the SDE have the same law, in other words we want to establish uniqueness in law.

We are going to use Girsanov's transformation to remove the drift by absorbing it into the Brownian motion B.

Assume therefore to be given a weak solution (X, B). Define the increasing sequence of stopping times

$$\tau_n := \inf\left\{ t \geqslant 0 : \int_0^t |b(s, X_s)|^2 \mathrm{d}s \geqslant n \right\}.$$

By assumption we have that $\tau_n \to \infty$ a.s. when $n \to \infty$ by (3). Then we can define a new measure $\mathbb{Q}^n$

$$\frac{\mathrm{d}\mathbb{Q}^n}{\mathrm{d}\mathbb{P}} = \exp\left(-\int_0^{\tau_n} b(s, X_s)\mathrm{d}B_s - \frac{1}{2}\int_0^{\tau_n} |b(s, X_s)|^2 \mathrm{d}s \right)$$

so that the process

$$\tilde{B}_t = B_t - [L, B]_t = B_t + \int_0^{\tau_n \wedge t} b(s, X_s)\mathrm{d}s$$

is a $\mathbb{Q}^n$ Brownian motion. In particular up to the random time τ_n we have $X_t = \tilde{B}_t$. So $(X_t)_{t \in [0, \tau_n]}$ is a Brownian motion (in the sense that the stopped process X^{τ_n} has the law of a Brownian motion stopped at a stopping time).

Let now $A_T \in \mathscr{B}(\mathscr{C}^n \times \mathscr{C}^n)$ such that $\{(X, B) \in A_T\} \in \mathscr{F}_T$ then

$$\mathbb{E}_{\mathbb{P}}[\mathbb{1}_{(X,B)\in A_T} \mathbb{1}_{T \leqslant \tau_n}] = \mathbb{E}_{\mathbb{Q}^n}\left[\mathbb{1}_{(X,B)\in A_T} \mathbb{1}_{T \leqslant \tau_n} \exp\left(\int_0^{\tau_n} b(s, X_s)\mathrm{d}B_s + \frac{1}{2}\int_0^{\tau_n} |b(s, X_s)|^2 \mathrm{d}s \right) \right]$$

$$= \mathbb{E}_{\mathbb{Q}^n}\left[\mathbb{1}_{(X,B)\in A_T} \mathbb{1}_{T \leqslant \tau_n} \exp\left(\int_0^{\tau_n} b(s, X_s)\mathrm{d}X_s - \frac{1}{2}\int_0^{\tau_n} |b(s, X_s)|^2 \mathrm{d}s \right) \right]$$

Moreover note that B is an adapted function of X (by the SDE) so $B=\Phi(X)$ where $\Phi:\mathscr{C}^n\to\mathscr{C}^n$ is some measurable and adapted functional (recall that $\mathscr{C}^n=C(\mathbb{R}_+;\mathbb{R}^n)$). We write also $\tau_n=\tilde{\tau}_n(X)$ to stress that it is a given measurable function $\tilde{\tau}_n:\mathscr{C}^n\to\mathbb{R}_+\cup\{+\infty\}$ of X. Therefore

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[\mathbb{1}_{(X,B)\in A_T}\mathbb{1}_{T\leqslant\tau_n}] &= \mathbb{E}_{\mathbb{Q}^n}\left[\mathbb{1}_{(X,\Phi(X))\in A_T}\mathbb{1}_{T\leqslant\tilde{\tau}_n(X)}\exp\left(\int_0^{\tilde{\tau}_n(X)} b(s,X_s)\mathrm{d}X_s-\frac{1}{2}\int_0^{\tilde{\tau}_n(X)}|b(s,X_s)|^2\mathrm{d}s\right)\right]\\ &=\mathbb{E}_{\mathbb{Q}^n}\left[\mathbb{1}_{(X,\Phi(X))\in A_T}\mathbb{1}_{T\leqslant\tilde{\tau}_n(X)}\mathbb{E}_{\mathbb{Q}^n}\left[\exp\left(\int_0^{\tilde{\tau}_n(X)} b(s,X_s)\mathrm{d}X_s-\frac{1}{2}\int_0^{\tilde{\tau}_n(X)}|b(s,X_s)|^2\mathrm{d}s\right)|\mathscr{F}_T\right]\right]\\ &=\mathbb{E}_{\mathbb{Q}^n}\left[\mathbb{1}_{(X,\Phi(X))\in A_T}\mathbb{1}_{T\leqslant\tilde{\tau}_n(X)}\exp\left(\int_0^{\tilde{\tau}_n(X)\wedge T} b(s,X_s)\mathrm{d}X_s-\frac{1}{2}\int_0^{\tilde{\tau}_n(X)\wedge T}|b(s,X_s)|^2\mathrm{d}s\right)\right]\\ &=\int_{\mathscr{C}^n}\mathbb{1}_{(\omega,\Phi(\omega))\in A_T}\mathbb{1}_{T\leqslant\tilde{\tau}_n(\omega)}\exp\left(\int_0^T b(s,\omega_s)\mathrm{d}\omega_s-\frac{1}{2}\int_0^T|b(s,\omega_s)|^2\mathrm{d}s\right)\mathbb{W}(\mathrm{d}\omega),\end{aligned}$$

where $\mathbb{W}$ is the law of a $\mathbb{R}^n$ valued Brownian motion (i.e. the Wiener measure). So we proved that the probability $\mathbb{P}((X,B)\in A_T,T\leqslant\tau_n)$ can be expressed independently of the given weak solution and therefore if $(X^1,B^1,\mathbb{P}^1)$ and $(X^2,B^2,\mathbb{P}^2)$ are two weak solutions of the SDE then

$$\mathbb{P}^1((X^1,B^1)\in A_T,T\leqslant\tilde{\tau}_n(X^1))=\mathbb{P}^2((X^2,B^2)\in A_T,T\leqslant\tilde{\tau}_n(X^2)) \tag{4}$$

moreover if both these weak solutions satisfy the assumptions on the drift we have that

$$\mathbb{P}^1\left(\lim_n\tilde{\tau}_n(X^1)=\infty\right)=\mathbb{P}^2\left(\lim_n\tilde{\tau}_n(X^2)=\infty\right)=1$$

we can take the limit $n\to\infty$ in (4) and conclude that for any $T\geqslant 0$ and A_T given as above we have

$$\mathbb{P}^1((X^1,B^1)\in A_T)=\mathbb{P}^2((X^2,B^2)\in A_T)$$

which implies uniqueness in law since we can also take $T\to\infty$ to have that

$$\mathbb{P}^1((X^1,B^1)\in A)=\mathbb{P}^2((X^2,B^2)\in A)$$

for any $A\in\mathscr{B}(\mathscr{C}^n\times\mathscr{C}^n)$.

So we proved that

Theorem 21. *The SDE in* $\mathbb{R}^n$

$$\mathrm{d}X_t=b(t,X_t)\mathrm{d}t+\mathrm{d}B_t,\qquad t\geqslant 0$$

where $b:\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}^n$ *is a measurable time-dependent vector field has uniquess in law in the class of weak solutions which satisfy*

$$\int_0^T|b(s,X_s)|^2\mathrm{d}s<+\infty,\qquad \text{a.s. for all } T\geqslant 0. \tag{5}$$

In particular, if b is bounded then we have (unconditional) uniqueness in law for the SDE.

Remark 22. The conclusion of Theorem 21 regarding unconditional uniquess also covers the case of drifts with linear growth (which, combined with Theorem 20, provides a complete result on weak existence and uniqueness in law for the SDE in this case).

We can in fact prove much more: if X is a solution to the SDE, then

$$\begin{aligned}|X_t| &\leqslant |x_0| + \int_0^t |b(s,X_s)|\,\mathrm{d}s + |B_t|\\ &\leqslant |x_0| + Ct + C\int_0^t |X_s|\,\mathrm{d}s + |B_t|\\ &\leqslant C\int_0^t |X_s|\,\mathrm{d}s + \left(|x_0| + CT + \sup_{t\in[0,T]} |B_t|\right) \quad \forall t\in[0,T]\end{aligned}$$

An application of Grönwall's lemma then allows to deduce the *pathwise estimate* (namely valid for a.e. fixed $\omega\in\Omega$)

$$\sup_{t\in[0,T]} |X_t(\omega)| \leqslant e^{CT}\left(|x_0| + CT + \sup_{t\in[0,T]} |B_t(\omega)|\right) \quad \forall T>0.$$

Since for any $p\in[1,\infty)$ it holds $\mathbb{E}[\sup_{t\in[0,T]}|B_t|^p]<\infty$ (e.g. apply Doob's inequality, or the reflection principle), we conclude that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t|^p\right] < \infty \quad \forall T<\infty, p\in[1,\infty)$$

which in particular implies a.s. finiteness of $\int_0^T |b(s,X_s)|^2\,\mathrm{d}s$ (and a lot more).

Exercise 4. Prove that, under the conditions of Theorem 21, the unique weak solution X is a Markov process.

Remark 23. The proof of Theorem 21 works also if $b\colon\mathbb{R}_+\times\mathscr{C}^n\to\mathbb{R}^n$ such that $(b(t,X_t))_{t\geqslant 0}$ is adapted to the filtration generated by X. In this more general context the solution of the SDE it not a Markov process anymore.

Remark that from the proof we have the representation formula

$$\mathbb{E}_{\mathbb{P}}[\mathbb{1}_{(X,B)\in A_T}\mathbb{1}_{T\leqslant\tau_n}] = \int_{\mathscr{C}^n} \mathbb{1}_{(\omega,\Phi(\omega))\in A_T}\mathbb{1}_{T\leqslant\tilde{\tau}_n(\omega)}\exp\left(\int_0^T b(s,\omega_s)\mathrm{d}\omega_s - \frac{1}{2}\int_0^T |b(s,\omega_s)|^2\mathrm{d}s\right)\mathbb{W}(\mathrm{d}\omega)$$

Recalling that the exponential term is integrable (it holds $\mathbb{E}[\mathcal{E}(L)_T]\leqslant 1$, since $\mathcal{E}(L)$ is a supermartingale), we can take by dominated convergence the limit $n\to\infty$ and obtain that

$$\mathbb{P}((X,B)\in A_T) = \int_{\mathscr{C}^n} \mathbb{1}_{(\omega,\Phi(\omega))\in A_T}\exp\left(\int_0^T b(s,\omega_s)\mathrm{d}\omega_s - \frac{1}{2}\int_0^T |b(s,\omega_s)|^2\mathrm{d}s\right)\mathbb{W}(\mathrm{d}\omega).$$

In particular one has the explicit representation formula **(path integral formula)**

$$\mathbb{P}(X\in A_T) = \int_{\mathscr{C}^n} \mathbb{1}_{\omega\in A_T}\exp\left(\int_0^T \langle b(s,\omega_s),\mathrm{d}\omega_s\rangle_{\mathbb{R}^n} - \frac{1}{2}\int_0^T |b(s,\omega_s)|^2_{\mathbb{R}^n}\mathrm{d}s\right)\mathbb{W}(\mathrm{d}\omega) \tag{6}$$

for any $A_T\in\sigma(\omega_t\colon t\in[0,T])\subseteq\mathscr{B}(\mathscr{C}^n)$.

It could be tempting to try to use the formula (6) to simulate a diffusion, indeed by Monte-Carlo methods one could take independent samples $(B^{(k)})_{k\in\mathbb{N}}$ of a Brownian motion and observe that by the law of large numbers one has

$$\mathbb{P}(X\in A_T) = \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} \mathbb{1}_{B^{(k)}\in A_T}\exp\left(\int_0^T \langle b(s,B_s^{(k)}),\mathrm{d}B_s^{(k)}\rangle_{\mathbb{R}^n} - \frac{1}{2}\int_0^T |b(s,B_s^{(k)})|^2_{\mathbb{R}^n}\mathrm{d}s\right)$$

the appeal of this method would be that it is very easy to simulate an (approximate) Brownian motion (i.e. via the Levy construction). Unfortunately is not easy to have a robust approximation of the stochastic integral in the exponent: i.e. if one try to replace it by Riemann sums then the resulting object converge very slowly to its "real value" and moreover it show very wild oscillations due to the fact that the exponential function "amplifies" very large positive fluctuations of its argument (all these problems are "similar" or "of the same nature" of the subtleties related to the integrability of the stochastic exponential $\mathscr{E}(L)$).

A particular situation which is quite nice is when $b(x) = -\nabla V(x)$ with a sufficiently smooth function V. Indeed in this case we have, by Ito formula on the canonical space $\mathscr{C}^n$ with the Wiener measure $\mathbb{W}$:

$$V(\omega_T) = V(\omega_0) + \int_0^T \nabla V(\omega_s)\mathrm{d}\omega_s + \frac{1}{2}\int_0^T \Delta V(\omega_s)\mathrm{d}s$$

provided $V \in C^2(\mathbb{R}^n)$ so that by "integrating by parts" we have

$$\exp\left(\int_0^T \langle b(\omega_s), \mathrm{d}\omega_s\rangle_{\mathbb{R}^n} - \frac{1}{2}\int_0^T |b(\omega_s)|_{\mathbb{R}^n}^2 \mathrm{d}s\right) = \exp\left(-\int_0^T \nabla V(\omega_s)\mathrm{d}\omega_s - \frac{1}{2}\int_0^T |\nabla V(s, \omega_s)|^2 \mathrm{d}s\right)$$

$$= \exp\left(V(\omega_0) - V(\omega_T) - \frac{1}{2}\int_0^T (|\nabla V(\omega_s)|^2 - \Delta V(\omega_s))\mathrm{d}s\right) = \Phi(\omega)$$

and the stochastic integral disappear from the exponent. This make the numerical method more stable since now the functional $\Phi: \mathscr{C}^n \to \mathbb{R}_+$ is easily seen to be continuous in the uniform topology on $\mathscr{C}^n$.

The formula

$$\mathbb{P}(X \in A_T) = \int_{\mathscr{C}^n} \mathbb{1}_{\omega \in A_T} \exp\left(V(\omega_0) - V(\omega_T) - \frac{1}{2}\int_0^T (|\nabla V(\omega_s)|^2 - \Delta V(\omega_s))\mathrm{d}s\right)\mathbb{W}(\mathrm{d}\omega)$$

can be also used to understand other properties of the solutions X of the SDE. Take for example $X_0 = x$ (call $\mathbb{P}_x$ the law of the associated solution to the SDE) and $f: \mathbb{R}^n \to \mathbb{R}$ and observe that

$$\mathbb{E}_x(f(X_T)) = \int_{\mathscr{C}^n} f(\omega_T) \exp\left(V(\omega_0) - V(\omega_T) - \frac{1}{2}\int_0^T (|\nabla V(\omega_s)|^2 - \Delta V(\omega_s))\mathrm{d}s\right)\mathbb{W}_x(\mathrm{d}\omega)$$

where $\mathbb{W}_x$ is the Wiener measure starting from x, i.e. with $\omega_0 = x$ almost surely. So we can express the transition kernel P of the time-homogeneous markov process (X_t) as

$$(P_T f)(x) = \mathbb{E}_x(f(X_T)) = \int_{\mathscr{C}^n} f(\omega_T) \exp\left(V(\omega_0) - V(\omega_T) - \frac{1}{2}\int_0^T (|\nabla V(\omega_s)|^2 - \Delta V(\omega_s))\mathrm{d}s\right)\mathbb{W}_x(\mathrm{d}\omega).$$

$$|(P_T f)(x)| \leqslant \|f e^{-V}\|_\infty e^{V(x)} \exp\left(-\frac{1}{2}\int_0^T \inf_{x \in \mathbb{R}^n} (|\nabla V(x)|^2 - \Delta V(x))\mathrm{d}s\right).$$

So for example, if $\inf_{x\in\mathbb{R}^n}(|\nabla V(x)|^2 - \Delta V(x)) \geqslant 2\alpha > 0$ then we have the exponential decay

$$e^{-V(x)}|(P_T f)(x)| \leqslant e^{-\alpha T}\|f e^{-V}\|_\infty,$$

in other words

$$\|e^{-V}(P_T f)\|_\infty \leqslant e^{-\alpha T}\|f e^{-V}\|_\infty.$$

Exercise 5. Using the path-integral formula show that for any two bounded functions f, g and under appropriate conditions on V:

$$\int (P_T f)(x) g(x) e^{-2V(x)} \mathrm{d}x = \int f(x)(P_T g)(x) e^{-2V(x)} \mathrm{d}x$$

which shows that P_T is symmetric wrt. the measure $e^{-2V(x)}\mathrm{d}x$ and taking $g=1$ show that $e^{-2V(x)}\mathrm{d}x$ properly normalized is an invariant measure for the SDE

$$\mathrm{d}X_t = -\nabla V(X_t)\mathrm{d}t + \mathrm{d}B_t,$$

meaning that if X_0 is taken with probability distribution $\propto e^{-2V(x)}\mathrm{d}x$ then

$$\mathbb{E}[f(X_0)] = \mathbb{E}[f(X_T)],$$

for all $T \geqslant 0$.

Remark on Ex 3 of Sheet 6:

Note the relevant Hilbert space is $L^2(\mathbb{R}^n)$ where

$$\int \overline{f(x)}\nabla_\alpha g(x)\mathrm{d}x = \int \overline{(-\nabla_\alpha) f(x)} g(x)\mathrm{d}x$$

so $\nabla_\alpha^* = -\nabla_\alpha$

$$H(A)f = |\nabla - iA|^2 f + Vf = \sum_{\alpha=1}^n (\nabla_\alpha - iA_\alpha)^*(\nabla_\alpha - iA_\alpha)f + Vf$$

$$= \sum_{\alpha=1}^n (-\nabla_\alpha + iA_\alpha)(\nabla_\alpha - iA_\alpha)f = \sum_{\alpha=1}^n (-\nabla_\alpha\nabla_\alpha f + i\nabla_\alpha(A_\alpha f) + iA_\alpha\nabla_\alpha f + A_\alpha^2 f)$$

$$= -\Delta f + i2A\cdot\nabla f + (i(\nabla\cdot A) + |A|^2)f$$

From the rep. formula by using Jensen's inequality and taking $\psi_0 \geqslant 0$

$$|(e^{-H(A)t}\psi_0)(x)| \leqslant (e^{-H(0)t}\psi_0)(x)$$

$$\psi(t,x) = \sum_{n\geqslant 0} e^{-E_n t}\langle\varphi_n,\psi_0\rangle\varphi_n(x) = e^{-E_0 t}\langle\varphi_0,\psi_0\rangle\varphi_0(x) + e^{-E_0 t}\sum_n e^{-(E_n - E_0)t}\langle\varphi_n,\psi_0\rangle\varphi_n(x)$$

Suggestion: take ψ_0 to be the lowest eigenfunction of either $H(A)$ or $H(0)$.

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10 Ito–Tanaka formula and local times of semimartingales

We want to extend Ito formula to functions which are not C^2.

Let X be a (one-dimensional) semimartingale and $f\colon \mathbb{R} \to \mathbb{R}$ a convex function.

Recall that for f convex there always exists f_-' (the derivative from the left) and it is an increasing function.

Let $\rho \in C^\infty(\mathbb{R})$ which is compactly supported on $\{x<0\}$, for example in $(-1,0)$ and define

$$f_n(x) := n\int \rho(ny) f(x+y)\mathrm{d}y$$

which is a smooth function such that $f_n \to f$ pointwise and for which $f_n'(x) \nearrow f_-'(x)$. By Ito formula

$$f_n(X_t) = f_n(X_0) + \int_0^t f_n'(X_s)\mathrm{d}X_s + \frac{1}{2}A_t^{f_n}$$

with $A_t^{f_n} := \int_0^t f_n''(X_s)\mathrm{d}[X]_s$ a continuous, increasing process. Eventually by using stopping times we can localize the problem so that f_-' is bounded, morover we note that by Doob's inequality we have (where $X = M + V$ is the decomposition of the semimartingale)

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t (f_n'(X_s) - f_-'(X_s))\mathrm{d}M_s\right|^2\right] \lesssim \mathbb{E}\left[\left|\int_0^T (f_n'(X_s) - f_-'(X_s))\mathrm{d}M_s\right|^2\right]$$

$$\lesssim \mathbb{E}\left[\int_0^T (f_n'(X_s) - f_-'(X_s))^2\mathrm{d}[M]_s\right] \to 0$$

by dominated convergence (again maybe put a stopping time to guarantee boundedness). This shows that in probability and uniformly on compact sets (in t)

$$\int_0^t f_n'(X_s)\mathrm{d}M_s \to \int_0^t f_-'(X_s)\mathrm{d}M_s.$$

On the hand, always by dominated convergence (decomposing the finite measure $\mathrm{d}V_s$ into positive and negative parts)

$$\int_0^t f_n'(X_s)\mathrm{d}V_s \to \int_0^t f_-'(X_s)\mathrm{d}V_s.$$

We can conclude that we have the following lemma

Lemma 24. *If X is a continuous semimartingale and f a convex function, then there exists a continuous increasing process $(A_t^f)_t$ such that*

$$f(X_t) = f(X_0) + \int_0^t f_-'(X_s)\mathrm{d}X_s + \frac{1}{2}A_t^f, \qquad t \geqslant 0.$$

We now can take $f(x)$ nice and simple convex functions like $|x-a|$, $(x-a)_\pm$ where $(x)_+ = (x \wedge 0)$ and $(x)_- := (-x)_+$. As a corollary of the previous lemma we then have

Theorem 25. *(Tanaka's formula) For any $a \in \mathbb{R}$ there exists a continuous increasing process $(L_t^a)_{t\geqslant 0}$ such that*

$$|X_t - a| = |X_0 - a| + \int_0^t \operatorname{sgn}(X_s - a)\mathrm{d}X_s + L_t^a$$

$$(X_t - a)_+ = (X_0 - a)_+ + \int_0^t \mathbb{1}_{X_s > a}\mathrm{d}X_s + \frac{1}{2}L_t^a$$

$$(X_t - a)_- = (X_0 - a)_- - \int_0^t \mathbb{1}_{X_s \leqslant a}\mathrm{d}X_s + \frac{1}{2}L_t^a$$

where $\operatorname{sgn}(x) = \mathbb{1}_{x>0} - \mathbb{1}_{x\leqslant 0}$.

Remark 26. This proves in particular that $|X_t - a|$, $(X_t - a)_\pm$ are semimartingales. The process $(L_t^a)_{t\geqslant 0}$ it is called the local time of X at a.

Proof. Each of the formulas derives from the previous lemma by computing the left derivative of the various convex functions. The missing point is to identify the various increasing processes $A^{\operatorname{sgn}(x-a)}$, $A^{(x-a)_+}, A^{(x-a)_-}$. Note that

$$X_t - a = (X_t - a)_+ - (X_t - a)_- = X_0 - a + \int_0^t \underbrace{(\mathbb{1}_{X_s > a} + \mathbb{1}_{X_s \leqslant a})}_{=1}\mathrm{d}X_s + \frac{1}{2}(A_t^{(x-a)_+} - A_t^{(x-a)_-})$$

so we have

$$0 = X_t - X_0 - \int_0^t \mathrm{d}X_s = \frac{1}{2}(A_t^{(x-a)_+} - A_t^{(x-a)_-}) \Rightarrow A_t^{(x-a)_+} = A_t^{(x-a)_-} =: L_t^a.$$

Moreover

$$|X_t - a| = (X_t - a)_+ + (X_t - a)_- + \int_0^t \underbrace{(\mathbb{1}_{X_s > a} - \mathbb{1}_{X_s \leqslant a})}_{= \operatorname{sgn}(X_s - a)} \mathrm{d}X_s + \underbrace{\frac{1}{2}(A_t^{(x-a)_+} + A_t^{(x-a)_-})}_{L_t^a}$$

□

The increasing process $(L_t^a)_{t \geqslant 0}$ is associated with a measure $\mathrm{d}L_t^a$ on $\mathbb{R}_+$ (times) which represents the time the process X "spent" in a up to time t. We are going to make this precise in the following.

By Ito formula wrt. the semimartingale $(|X_t - a|)_{t \geqslant 0}$ (with $X = M + V$)

$$(X_t - a)^2 = (|X_t - a|)^2 = (|X_0 - a|)^2 + 2\int_0^t |X_s - a| \operatorname{sgn}(X_s - a) \mathrm{d}X_s + 2\int_0^t |X_s - a| \mathrm{d}L_s^a + [|X_\cdot - a|]_t$$

$$= (X_0 - a)^2 + 2\int_0^t (X_s - a) \mathrm{d}X_s + 2\int_0^t |X_s - a| \mathrm{d}L_s^a + \int_0^t \underbrace{\operatorname{sgn}(X_s - a)^2}_{=1} \mathrm{d}[M]_s$$

And by comparing with the standard Ito formula

$$(X_t - a)^2 = (X_0 - a)^2 + 2\int_0^t (X_s - a) \mathrm{d}X_s + \underbrace{[X]_t}_{=[M]_t}$$

we conclude that

$$\int_0^t |X_s - a| \mathrm{d}L_s^a = 0, \qquad t \geqslant 0$$

which proves that the measure $(\mathrm{d}L_s^a)_{s \geqslant 0}$ is supported in the (random) set $\{s \in \mathbb{R} : X_s = a\}$ of times. The process L^a increases only when the process X visits a (in general this will be a "fractal-like" and with zero Lebesgue measure).

For Brownian motion is it true (we will not prove it) that the set $\{s \in \mathbb{R} : X_s = a\}$ is the support of the measure $(L_t^a)_{t \geqslant 0}$.

Theorem 27. *(Ito–Tanaka formula) If f is the difference of two convex functions and X a continuous semimartingale, then*

$$f(X_t) = f(X_0) + \int_0^t f'_-(X_s) \mathrm{d}X_s + \frac{1}{2}\int_{\mathbb{R}} L_t^a f''(\mathrm{d}a)$$

and in particular $(f(X_t))_{t \geqslant 0}$ is a semimartingale.

In this formula $f''(\mathrm{d}a)$ denotes the measure associated to the second derivative of a convex function (and therefore of a difference of two convex functions, by linearity).

Recall that for any convex function f we have the formula

$$f(x) = \alpha + \beta x + \frac{1}{2}\int |x-a| f''(\mathrm{d}a), \qquad x \in \mathbb{R}$$

and

$$f_-'(x) = \beta + \frac{1}{2}\int \operatorname{sgn}(x-a) f''(\mathrm{d}a), \qquad x \in \mathbb{R}$$

for some $\alpha, \beta \in \mathbb{R}$ and $f''(\mathrm{d}a)$ is the measure associated to the increasing function $(f_-'(x))_{x\in\mathbb{R}}$ (convex functions are those functions whose second distributional derivative is a positive Radon measure).

The idea is that

$$\frac{\mathrm{d}}{\mathrm{d}x}|x-a| = 2\delta(x-a)$$

which justify intuitively the $1/2$ in the formula.

Proof. We can write

$$f(X_t) = \alpha + \beta X_t + \frac{1}{2}\int |X_t - a| f''(\mathrm{d}a)$$

by Tanaka's formula

$$f(X_t) = \alpha + \beta X_0 + \beta \int_0 \mathrm{d}X_s + \frac{1}{2}\int |X_0 - a| f''(\mathrm{d}a) + \frac{1}{2}\int \left(\int_0^t \operatorname{sgn}(X_s - a)\mathrm{d}X_s\right) f''(\mathrm{d}a)$$
$$+ \frac{1}{2}\int L_t^a f''(\mathrm{d}a)$$

Note that this computation makes sense since the stochastic integral $\int_0^t \operatorname{sgn}(X_s - a)\mathrm{d}X_s$ is a measurable function of a, more precisely (see the relevant exercise in Sheet 7) the function

$$(a, t, \omega) \mapsto \left(\int_0^t \operatorname{sgn}(X_s - a)\mathrm{d}X_s\right)(\omega)$$

is a measurable function on $\mathcal{B}(\mathbb{R}) \otimes \mathcal{P}$ ($\mathcal{P}$ is the previsible σ field on $\mathbb{R}_+ \times \Omega$) and also a (stochastic) Fubini theorem applies so that

$$\int \left(\int_0^t \operatorname{sgn}(X_s - a)\mathrm{d}X_s\right) f''(\mathrm{d}a) = \int_0^t \left(\int \operatorname{sgn}(X_s - a) f''(\mathrm{d}a)\right)\mathrm{d}X_s$$

Moreover we note that

$$\beta \int_0^t \mathrm{d}X_s + \frac{1}{2}\int_0^t \left(\int \operatorname{sgn}(X_s - a) f''(\mathrm{d}a)\right)\mathrm{d}X_s = \int_0^t f_-'(X_s)\mathrm{d}X_s$$

which completes the proof. □

Corollary 28. *(Occupation-time formula) There is a $\mathbb{P}$-negligible set $\mathcal{N}$ outside which for any $t \geqslant 0$ and any positive Borel function $g: \mathbb{R} \to \mathbb{R}_+$ we have*

$$\int_0^t g(X_s)\mathrm{d}[X]_s = \int_{\mathbb{R}} g(a) L_t^a \mathrm{d}a.$$

Remark 29. The measure $\mathrm{d}[X]_s$ can be understood as some "intrinsic" time of the semimartingale. In particular, for Brownian motion X we have $\mathrm{d}[X]_s = \mathrm{d}s$ and if we take $g(x) = \mathbb{1}_{x \in A}$ for some set $A \in \mathscr{B}(\mathbb{R})$ we have

$$\mathrm{Leb}(\{s \in [0,t]: X_s \in A\}) = \int_0^t \mathbb{1}_{X_s \in A} \mathrm{d}s = \int_A L_t^a \mathrm{d}a.$$

In this sense $L_t^a \mathrm{d}a$ represents the time spent by X in the infinitesimal neighborhood $a \pm \mathrm{d}a$.

Proof. For any $g: \mathbb{R} \to \mathbb{R}_+$ we can find a convex function f such that $f'' = g$, i.e. we can take $f''(\mathrm{d}a) = g(a)\mathrm{d}a$ in the formula above. By Tanaka's formula we then have

$$f(X_t) = f(X_0) + \int_0^t f_-'(X_s)\mathrm{d}X_s + \frac{1}{2}\int_{\mathbb{R}} g(a) L_t^a \mathrm{d}a. \qquad \mathbb{P}-a.s.$$

Take a countable family $(g_n)_{n \geqslant 0}$ of compactly supported continuous functions which is dense in $C_0(\mathbb{R})$ and consider now f_n so that $f_n'' = g_n$, note that $f_n \in C^2$ and $f_{n,-}' = f_{n,+}' = f_n'$. I have now both Ito-Tanaka's formula and Ito formula (note that f_n is the difference of two convex functions)

$$f_n(X_t) = f_n(X_0) + \int_0^t f_n'(X_s)\mathrm{d}X_s + \frac{1}{2}\int_0^t g_n(X_s)\mathrm{d}[X]_s. \qquad \mathbb{P}-a.s.$$

So by comparing these two formulas we have

$$\ell_t(g_n) := \int_{\mathbb{R}} g_n(a) L_t^a \mathrm{d}a = \int_0^t g_n(X_s)\mathrm{d}[X]_s. \qquad \mathbb{P}-a.s.$$

This equality holds a.s. for any g_n and one can choose a $\mathbb{P}$-negligible set $\mathcal{N}$ such that the equalities holds simultaneously for all n and all $t \geqslant 0$ (since the quantity $\ell_t(g_n)$ is continuous in time and therefore can be detemined by looking to a dense set of times $(t_k)_k$).

One note now that for any $t \geqslant 0$, the functional ℓ_t is a positive linear functional on $C_0(\mathbb{R})$ which is continuous in the uniform norm on $C_0(\mathbb{R})$ so can be extended by continuity to all functions in $C_0(\mathbb{R})$ and by a monotone class argument to all Borel positive functions. □

Thursday no lecture.

Next lecture on tuesday: we prove that $a \mapsto L_t^a$ is cadlag and that there is a formula of the form

$$L_t^a = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{X_s \in (a, a+\varepsilon)} \mathrm{d}[X]_s$$

and for X a martingale

$$L_t^a = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{X_s \in (a-\varepsilon, a+\varepsilon)} \mathrm{d}[X]_s.$$

We continue to discuss some properties of local time of Brownian motion and reflected Brownian motion.

11 Regularity of local times and reflected Brownian motion, Takana's SDE

We want to look at $a \mapsto L_t^a$ where L_t^a is the local time in a of a semimartingale X.

Recall the occupation time formula

$$\int_0^t \varphi(X_t) \mathrm{d}[X]_t = \int_{\mathbb{R}} \varphi(x) L_t^x \mathrm{d}x$$

for all $t \geqslant 0$ and $\varphi: \mathbb{R} \to \mathbb{R}_+$ positive bounded Borel function.

Remark 30. Note that using this formula one can prove that Ito formula extends to any f such that f'' is locally integrable with respect to the Lebesgue measure, i.e.

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) \mathrm{d}X_s + \frac{1}{2} \int_{\mathbb{R}} f''(x) L_t^x \mathrm{d}x$$

Observe that

$$f'(x) - f'(y) = \int_x^y f''(z) \mathrm{d}z$$

so f' is of bounded variation and, by dominated convergence, continuous.

So far we know only that $a \mapsto L_t^a$ is measurable in a. Denote $X = M + V$

Tanaka's formula give

$$L_t^a = 2\left[(X_t - a)_+ - (X_0 - a)_+ - \int_0^t \mathbb{1}_{X_s > a} \mathrm{d}M_s - \int_0^t \mathbb{1}_{X_s > a} \mathrm{d}V_s \right]$$

Define

$$\hat{M}_t^a := \int_0^t \mathbb{1}_{X_s > a} \mathrm{d}M_s$$

We want to apply Kolmogorov's continuity theorem to $a \in \mathbb{R} \mapsto (\hat{M}_t^a)_{t \in [0,T]} \in C([0,T];\mathbb{R})$ seen as a random variable with values on $C_T = C([0,T];\mathbb{R})$ with norm $\|f\|_{C_T} = \sup_{t \in [0,T]} \|f(t)\|$. Recall that Kolmogorov's continuity theorem states that a stochastic process $Y: \mathbb{R} \to \mathscr{B}$ has a continuous version if

$$\mathbb{E}\|Y(a) - Y(b)\|_{\mathscr{B}}^p \leqslant C_L |a-b|^{1+c}$$

for some $p, c > 0$ and $a, b \in [0, L]$ for all L with some finite C_L. Moreover a consequence of the theorem is alos that the process Y can be chosen to be locally Hölder continuous with index $\gamma \in (0, c/p)$, namely for any $L > 0$

$$\|Y(a)(\omega) - Y(b)(\omega)\|_{\mathscr{B}} \leqslant K_L(\omega)|a-b|^{\gamma}, \qquad a, b \in [0, L]$$

almost surely.

In our case we take $\mathscr{B} = C_T$ and then we need to estimate for some $p \geqslant 2$

$$\mathbb{E}[\sup_{t \in [0,T]} |\hat{M}_t^a - \hat{M}_t^b|^p]$$

By Burkholder-David-Gundy (BDG) inequality, (see next exercise sheet) take $b > a$,

$$\mathbb{E}[\sup_{t \in [0,T]} |\hat{M}_t^a - \hat{M}_t^b|^p] \leqslant C_p \mathbb{E}[[\hat{M}^a - \hat{M}^b]_T^{p/2}]$$

$$\leqslant C_p \mathbb{E}\left[\left(\int_0^T (\mathbb{1}_{X_s > a} - \mathbb{1}_{X_s > b})^2 \mathrm{d}[M]_s\right)^{p/2}\right]$$

by occupation time formula

$$\lesssim C_p \mathbb{E}\left[\left(\int_a^b L_T^x \mathrm{d}x\right)^{p/2}\right]$$

by Jensen's inequality

$$\lesssim C_p (b-a)^{p/2} \mathbb{E}\left[\int_a^b (L_T^x)^{p/2} \frac{\mathrm{d}x}{b-a}\right] \lesssim_p (b-a)^{p/2} \sup_{x \in \mathbb{R}} \mathbb{E}[(L_T^x)^{p/2}].$$

In order to show that $\sup_{x \in \mathbb{R}} \mathbb{E}[(L_T^x)^{p/2}]$ is finite we observe that since

$$|(X_T - a)_+ - (X_0 - a)_+| \leqslant |X_T - X_0|$$

$$\mathbb{E}[(L_T^x)^{p/2}] = \mathbb{E}\left[\left(2\left[(X_T - a)_+ - (X_0 - a)_+ - \int_0^T \mathbb{1}_{X_s > a} \mathrm{d}M_s - \int_0^T \mathbb{1}_{X_s > a} \mathrm{d}V_s\right]\right)^{p/2}\right]$$

$$\lesssim_p \mathbb{E}[|X_T - X_0|^{p/2}] + \mathbb{E}\left[\left|\int_0^T \mathbb{1}_{X_s > a} \mathrm{d}M_s\right|^{p/2}\right] + \mathbb{E}\left[\left|\int_0^T \mathbb{1}_{X_s > a} \mathrm{d}V_s\right|^{p/2}\right]$$

$$\lesssim_p \mathbb{E}[|X_T - X_0|^{p/2}] + \mathbb{E}[|[M]_T|^{p/4}] + \mathbb{E}\left[\left(\int_0^T |\mathrm{d}V_s|\right)^{p/2}\right] = L_T$$

this shows that $\sup_{x\in\mathbb{R}}\mathbb{E}[(L_T^x)^{p/2}]$ is finite provided L_T is finite. In this case Kolmogorov's continuity criterion tells us that $a\mapsto \hat{M}_t^a$ is continuous in a uniformly in t. If the quantity L_T is not finite, then we introduce a suitable sequence of stopping times $(T_n)_n$, $T_n\to\infty$ and look at that stopped martingale $(\hat{M}_t^a)_t^{T_n}$. For example take

$$T_n=\inf\left\{t\geqslant 0\colon \sup_{s\in[0,t]}|X_s-X_0|+[M]_t+\int_0^t|\mathrm{d}V_s|\geqslant n\right\}$$

so that we now know that $(t,a)\mapsto(\hat{M}_t^a)^{T_n}$ is continuous in both variables and then taking the limit as $n\to\infty$ we deduce that $(t,a)\mapsto\hat{M}_t^a$ is also continuous in both variables since $T_n\to\infty$ almost surely.

Actually from this proof one could also deduce that the process $a\mapsto\hat{M}_t^a$ for fixed t is locally Hölder continuous for any $\gamma<1/2$, i.e.

$$\sup_{t\in[0,T]}|\hat{M}_t^a-\hat{M}_t^b|\leqslant C_L(\omega)|b-a|^\gamma,\qquad a,b\in[0,L].$$

holds almost surely for some random constant C_L which can be taken to be

$$C_L(\omega)=C_L^{N_T}(\omega)$$

where $N_T\coloneqq\inf_{n\geqslant 0}\{n\colon T_n>T\}$ where $C_L^{N_T}(\omega)$ is the constant appearing in the bound

$$\sup_{t\in[0,T_n]}|\hat{M}_t^a-\hat{M}_t^b|\leqslant C_L^n(\omega)|b-a|^\gamma,\qquad a,b\in[0,L]$$

which holds for any $n\geqslant 0$ by considering the stopped process.

As far as $\int_0^t\mathbb{1}_{X_s>a}\mathrm{d}V_s$ is concerned we have letting

$$\hat{V}_t^a\coloneqq\int_0^t\mathbb{1}_{X_s>a}\mathrm{d}V_s$$

and using dominated convergence

$$\hat{V}_t^{a+}=\lim_{b\searrow a}\hat{V}_t^b=\int_0^t\mathbb{1}_{X_s>a}\mathrm{d}V_s=\hat{V}_t^a$$

since $\lim_{b\searrow a}\mathbb{1}_{X_s>b}=\mathbb{1}_{X_s>a}$. However we have $\lim_{b\nearrow a}\mathbb{1}_{X_s>b}=\mathbb{1}_{X_s\geqslant a}$ so

$$\hat{V}_t^{a-}=\lim_{b\nearrow a}\hat{V}_t^b=\int_0^t\mathbb{1}_{X_s\geqslant a}\mathrm{d}V_s\neq\hat{V}_t^a$$

So the process $a\mapsto\hat{V}_t^a$ is almost surely cadlag. Additionally

$$\hat{V}_t^{a-}-\hat{V}_t^a=\int_0^t\mathbb{1}_{X_s=a}\mathrm{d}V_s=\int_0^t\mathbb{1}_{X_s=a}\mathrm{d}X_s$$

since by the occupation time formula and Ito isometry, we have $\int_0^t \mathbb{1}_{X_s=a} \mathrm{d}M_s = 0$, since

$$\left[\int_0^\cdot \mathbb{1}_{X_s=a} \mathrm{d}M_s\right]_T = \int_0^T \mathbb{1}_{X_s=a} \mathrm{d}[M]_s = \int_0^T \mathbb{1}_{X_s=a} \mathrm{d}[X]_s = \int_{\mathbb{R}} \mathbb{1}_{x=a} L_T^x \mathrm{d}x = 0$$

almost surely. Putting all together we have proven the following theorem

Theorem 31. *For any continuous semimartingale X there exists a modification of the local time process $(L_t^a)_{t,a}$ which is continuous in t and cadlag in a and moreover we have*

$$L_t^a - L_t^{a-} = 2\int_0^t \mathbb{1}_{X_s=a} \mathrm{d}X_s = 2\int_0^t \mathbb{1}_{X_s=a} \mathrm{d}V_s.$$

In particular, if X is a local martingale then the local time has bicontinuous version.

Corollary 32. *If X is a continuous semimartingale then*

$$L_t^a = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{X_s \in [a, a+\varepsilon[} \mathrm{d}[X]_s$$

and if X is a martingale

$$L_t^a = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{X_s \in]a-\varepsilon, a+\varepsilon[} \mathrm{d}[X]_s$$

Proof. Just use the occupation time formula and the continuity from the right of local times. □

Remark 33. For Brownian motion this implies that L_t^0 is measurable with respect to the filtration $\mathscr{F}^{|B|}$ generated by $|B|$ since $\mathbb{1}_{B_s \in]-\varepsilon, +\varepsilon[} = \mathbb{1}_{|B_s| < \varepsilon} \hat{\in} \mathscr{F}^{|B|}$ and $[B]_s = s$ and

$$L_t^0 = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{|B_s| < \varepsilon} \mathrm{d}s.$$

12 Brownian motion and local time

Let B be a one dimensional Brownian motion. By Ito–Tanaka formula we have

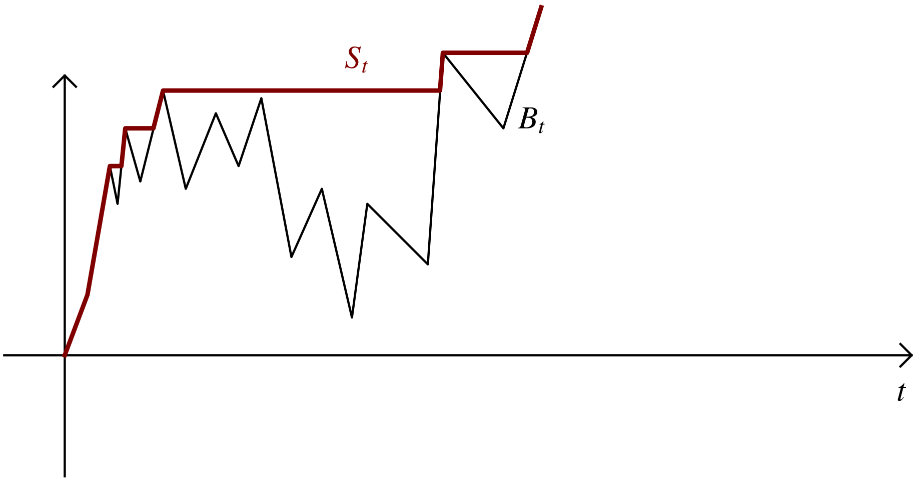
$$|B_t| = |B_0| + \int_0^t \mathrm{sgn}(B_s) \mathrm{d}B_s + L_t \tag{7}$$

where we let L_t to be the local time in zero of B. Is not important in this case to specify which version of the sign it is used since by the occupation time formula

$$\left[\int_0^\cdot \mathbb{1}_{B_s=0} \mathrm{d}B_s\right]_T = \int_0^T \mathbb{1}_{B_s=0} \mathrm{d}s = \int_{\mathbb{R}} \mathbb{1}_{x=0} L_T^x \mathrm{d}x = 0.$$

We want to show next that $R_t = |B_t|$ is an interesting process which satisfies a *reflected* SDE and is called the reflected Brownian motion, this will make link also with another process which is the maximum of the Brownian motion

$$S_t := \sup_{s \leqslant t} B_s$$



Take again

$$R_t = |B_t| = \int_0^t \operatorname{sgn}(B_s) \mathrm{d}B_s + L_t$$

and define

$$\beta_t := \int_0^t \operatorname{sgn}_{-1}(B_s) \mathrm{d}B_s$$

where we denote sgn_a the signum function which satisfy $\operatorname{sgn}_a(0) = a$. Note that sgn_{-1} is the left derivative of the absolute value.

Observe by Lévy characterisation that β is a Brownian motion, indeed $[\beta]_t = \int_0^t \operatorname{sgn}_{-1}(B_s)^2 \mathrm{d}[B]_s = [B]_t = t$, moreover

$$\int_0^t \operatorname{sgn}_0(B_s) \mathrm{d}\beta_s = \int_0^t \operatorname{sgn}_0(B_s) \operatorname{sgn}_{-1}(B_s) \mathrm{d}B_s = \int_0^t \operatorname{sgn}_0(B_s)^2 \mathrm{d}B_s = B_t - \underbrace{\int_0^t \mathbb{1}_{B_s = 0} \mathrm{d}B_s}_{=0} = B_t$$

since using the local time of B I have $[\int_0^{\cdot} \mathbb{1}_{B_s = 0} \mathrm{d}B_s]_\infty = 0$.

The first observation out of this computation is that (B, β) is a weak solution of the SDE

$$\mathrm{d}B_t = \operatorname{sgn}_0(B_t) \mathrm{d}\beta_t,$$

this is called *Tanaka's* SDE. So we have proven weak existence for this equation. This solution is unique in law (obviously) since any solution will be such that B is a Brownian motion. However this SDE do not have strong solutions. Indeed if (X, W) is a strong solution (starting in $X_0 = 0$), we have

$$\mathrm{d}X_t = \mathrm{sgn}_0(X_t)\mathrm{d}W_t,$$

and X is a Brownian motion, moreover

$$\int_0^t \mathrm{sgn}_0(X_t)\mathrm{d}X_t = \int_0^t \mathrm{sgn}_0(X_t)^2\mathrm{d}W_t = W_t - \int_0^t \mathbb{1}_{X_s=0}\mathrm{d}W_s = W_t$$

since $[\int_0^\cdot \mathbb{1}_{X_s=0}\mathrm{d}W_s]_T = \int_0^T \mathbb{1}_{X_s=0}\mathrm{d}s = 0$. By Ito-Tanaka's formula

$$|X_t| = \int_0^t \mathrm{sgn}_{-1}(X_s)\mathrm{d}X_s + L_t^{X,0}$$

where $L_t^{X,0}$ is the local time of X in 0, and this shows that

$$W_t = \int_0^t \mathrm{sgn}_0(X_t)\mathrm{d}X_t = \int_0^t \mathrm{sgn}_{-1}(X_s)\mathrm{d}X_s = |X_t| - L_t^{X,0}$$

and recalling that we have (since X is a martingale)

$$L_t^{X,0} = \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{|X_s|<\varepsilon}\mathrm{d}s,$$

which implies that W is measurable wrt. the filtration generated by $|X|$. If we had a strong solution then we would have that $\mathscr{F}_t^X \subseteq \mathscr{F}_t^W \subseteq \mathscr{F}_t^{|X|}$ which is not possible because you cannot recover the sign of a Brownian motion only knowing its absolute value.

So there are no strong solution and a consequence there is no pathwise uniqueness (by Yamada–Watanabe).

Exercise 6. Prove that if B is a Brownian motion, then we have the relation $L_t^{|B|,0} = 2L_t^{B,0}$.

We go back to the equation

$$R_t = |B_t| = \underbrace{\int_0^t \mathrm{sgn}_{-1}(B_s)\mathrm{d}B_s}_{\beta_t} + L_t$$

we want to show that in this equation both R, L are functions of the Brownian motion β_t which we think as given, according to the following definition

Definition 34. *(Reflected SDE) The family* (X, ℓ, W) *is a weak solution of the one dimensional reflected SDE*

$$\mathrm{d}X_t = \mathrm{d}W_t + \mathrm{d}\ell_t$$

if W is a Brownian motion, ℓ *a continuous positive non-decreasing process and X a continuous positive process such that*

$$\int_0^\infty \mathbb{1}_{X_s>0} \mathrm{d}\ell_s = 0.$$

The solution is strong if (X, ℓ) *is adapted to the noise W.*

Therefore (R, L, β) is a weak solution of this reflected SDE. We will need the followin analysis lemma (we use $\mathbb{R}_+ = \mathbb{R}_{\geqslant 0}$)

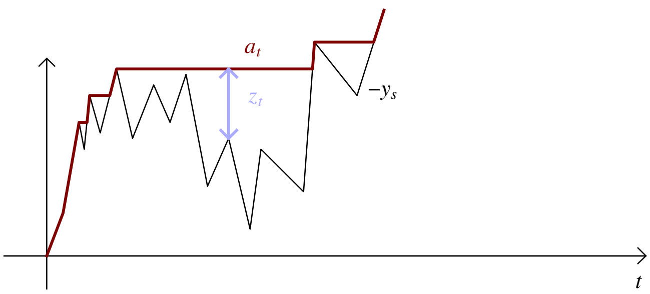
Lemma 35. *(Skorokhod lemma) Let* $y \in C(\mathbb{R}_+; \mathbb{R})$ *such that* $y(0) \geqslant 0$*. There exists a unique pair* (z, a) *with* $z \in C(\mathbb{R}_+; \mathbb{R}_+)$ *and* $a \in C(\mathbb{R}_+; \mathbb{R}_+)$ *with a non-decreasing,* $a(0) = 0$*, such that*

a) $z_t = y_t + a_t$

b) $\int_0^\infty \mathbb{1}_{z_s>0} \mathrm{d}a_s = 0.$

Moreover

$$a(t) = \sup_{s \in [0,t]} (y_s)_- = \sup_{s \in [0,t]} (-y_s \vee 0). \tag{8}$$



Proof. Exercise prove that if we let a as in eq. (8) then *a),b)* are satisfied, this settles the existence part. As for uniqueness we assume that both (z, a) and (z', a') are two solutions of this problem. Then $y_t = z_t - a_t = z_t' - a_t'$ so we have $z_t - z_t' = a_t - a_t'$ so $h_t = z_t - z_t'$ is of bounded variation (as a difference of two increasing functions) and we can write (by Ito formula)

$$\mathrm{d}(z_t - z_t')^2 = 2\int_0^t (z_s - z_s')\mathrm{d}(z_s - z_s') = 2\int_0^t (z_s - z_s')\mathrm{d}(a_s - a_s') = 2\int_0^t (z_s - z_s')\mathrm{d}a_s - 2\int_0^t (z_s - z_s')\mathrm{d}a_s'$$

$$= 2\int_0^t (-z_s')\mathrm{d}a_s - 2\int_0^t (z_s)\mathrm{d}a_s' \leqslant 0$$

where we used that $\int_0^t z_s \mathrm{d}a_s = \int_0^t z'_s \mathrm{d}a'_s = 0$ and that $z_s, z'_s \geqslant 0$. So $h_t^2 \geqslant 0$ is decreasing and since $h_0 = 0$ we have that $h_t = 0$ for any t. This establish uniqueness. □

As a consequence of this lemma we have that the reflected SDE has a unique solution in law (and pathwise) which is given therefore by

$$\ell_t = \sup_{s \in [0,t]} (-W_s)_+ = \sup_{s \in [0,t]} (-W_s) = S_t^{-W} \qquad X_t = W_t + \ell_t$$

where we note $S_t^W = \sup_{s \leqslant t} W_t$ and the solution is strong.

Definition 36. *We call the process X the reflected Brownian motion*

We deduce as a consequence that if we consider

$$R_t = |B_t| = \underbrace{\int_0^t \operatorname{sgn}_{-1}(B_s) \mathrm{d}B_s}_{\beta_t} + L_t$$

then we have

$$L_t = \sup_{s \in [0,t]} (-\beta_s)_+ = \sup_{s \in [0,t]} (-\beta_s) = S_t^{-\beta}.$$

From this we deduce

Theorem 37.

$$\mathrm{Law}(|B|, L) = \mathrm{Law}(\beta + L, L) = \mathrm{Law}(\beta + S^{-\beta}, S^{-\beta}) = \mathrm{Law}(S^W - W, S^W)$$

where W here is a generic Brownian motion. This formula allows to compute the joint law of the supremum S^W of a Brownian motion W together with the Brownian motion, in terms of the law of the reflected Brownian motion R.

Remark 38. Some of the utility of this relation come from the fact that it implies that

$$\mathrm{Law}(|B_t|, L_t) = \mathrm{Law}(S_t^W - W_t, S_t^W)$$

and that by the reflection principle one can compute explicitly the law $\mathrm{Law}(S_t^W - W_t, S_t^W)$, or moreover that

$$\mathrm{Law}(|B|) = \mathrm{Law}(S^W - W)$$

which given informations on the supremum S^W in terms of the modulus of another Brownian motion.