from "Uniqueness and Nonuniqueness Criteria for ODEs" by Agarwal + Lakshmikantham

nondecreasing in y for each x in $0 \le x < \infty$. However, in Example 1.2.2 we have seen that (1.2.8) has a unique solution.

1.4 OSGOOD'S UNIQUENESS THEOREM

In this section we shall study a generalization of the Lipschitz uniqueness theorem which is due to Osgood . For this, we require the following :-

Lemma 1.4.1. Let $g(z)$ be a continuous and nondecreasing function in the interval $[0,\infty)$ and $g(0)=0$, $g(z)>0$ for $z>0$. Also,

$$\lim_{\varepsilon\to 0^+}\int_\varepsilon \frac{dz}{g(z)} = \infty. \tag{1.4.1}$$

Let $\phi(x)$ be a nonnegative continuous function in $[0,a]$. Then,

$$\phi(x) \le \int_0^x g(\phi(t))dt,\ 0 < x \le a \tag{1.4.2}$$

implies that $\phi(x) = 0$ in $[0,a]$.

Proof. Define $\Phi(x) = \max_{0\le t\le x}\phi(t)$ and assume that $\Phi(x) > 0$ for $0 < x \le a$. Then, $\phi(x) \le \Phi(x)$ and for each x there is an $x_1 \le x$ such that $\phi(x_1) = \Phi(x)$. From this, we have

$$\Phi(x) = \phi(x_1) \le \int_0^{x_1} g(\phi(t))dt \le \int_0^x g(\Phi(t))dt,$$

i.e., the increasing function $\Phi(x)$ satisfies the same inequality as $\phi(x)$ does. Let us set $\bar\Phi(x) = \int_0^x g(\Phi(t))dt$, then $\bar\Phi(0) = 0, \Phi(x) \le \bar\Phi(x)$ and $\bar\Phi'(x) = g(\Phi(x)) \le g(\bar\Phi(x))$. Hence, for $0 < \delta < a$, we have

$$\int_\delta^a \frac{\bar\Phi'(x)}{g(\bar\Phi(x))}dx \le a - \delta < a.$$

However, from (1.4.1), it follows that

$$\int_\delta^a \frac{\bar\Phi'(x)}{g(\bar\Phi(x))}dx = \int_\varepsilon^\alpha \frac{dz}{g(z)},\ \ \bar\Phi(\delta) = \varepsilon, \bar\Phi(a) = \alpha$$

becomes infinite when $\varepsilon \to 0^+(\delta \to 0)$. This contradiction shows that $\Phi(x)$ cannot be positive and so $\Phi(x) \equiv 0$, and hence $\phi(x) = 0$ in $[0,a]$. ∎

Theorem 1.4.2 (Osgood's Uniqueness Theorem). Let $f(x,y)$ be continuous in $\bar S$ and for all $(x,y),(x,\bar y) \in \bar S$ it satisfies **Osgood's condition**

$$|f(x,y) - f(x,\bar y)| \le g(|y-\bar y|), \tag{1.4.3}$$

where $g(z)$ is the same as in Lemma 1.4.1. Then, the initial value problem (1.1.1) has at most one solution in $|x - x_0| \le a$.

Proof. Suppose $y(x)$ and $\bar y(x)$ are two solutions of (1.1.1) in $|x-x_0| \le a$. We shall show that $y(x) = \bar y(x)$ in $[x_0, x_0+a]$. From (1.4.3) it follows that

$$\begin{aligned}|y(x_0+x) - \bar y(x_0+x)| &\le \int_{x_0}^{x_0+x} |f(t,y(t)) - f(t,\bar y(t))|dt \\ &\le \int_{x_0}^{x_0+x} g(|y(t)-\bar y(t)|)dt \\ &= \int_0^x g(|y(z+x_0) - \bar y(z+x_0)|)dz.\end{aligned}$$

For x in $[0,a]$, we set $\phi(x) = |y(x+x_0) - \bar y(x+x_0)|$. Then, the nonnegative continuous function $\phi(x)$ satisfies the inequality (1.4.2), and therefore, Lemma 1.4.1 implies that $\phi(x) = 0$ in $[0,a]$, i.e., $y(x) = \bar y(x)$ in $[x_0, x_0+a]$. If x is in $[x_0-a, x_0]$, then the proof remains the same except that we need to define the function $\phi(x) = |y(x_0-x) - \bar y(x_0-x)|$ in $[0,a]$. ∎

Example 1.4.1. Consider the initial value problem

$$y' = Ly,\ y(0) = 0 \tag{1.4.4}$$

where $L > 0$. For this problem, we choose $g(z) = Lz$, which is clearly continuous and nondecreasing in the interval $[0,\infty)$. Further, since $g(0) = 0$, $g(z) > 0$ for $z > 0$, and $\lim_{\varepsilon\to0^+}\int_\varepsilon [g(z)]^{-1}dz = \frac{1}{L}\lim_{\varepsilon\to0^+}\ \ln\frac{1}{\varepsilon} = \infty$. This function $g(z)$ satisfies the conditions of Lemma 1.4.1. Next, for any y and $\bar y$ we have

$$|f(x,y) - f(x,\bar y)| = |Ly - L\bar y| = g(|y-\bar y|)$$

and hence Osgood's condition (1.4.3) is also satisfied. Therefore, from Theorem 1.4.2 the problem (1.4.4) has a unique solution, namely, $y(x) \equiv 0$. ∎

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