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ON THE UNIQUENESS IN LAW AND THE PATHWISE UNIQUENESS FOR STOCHASTIC DIFFERENTIAL EQUATIONS*

A. S. CHERNY†

(*Translated by the author*)

Abstract. We prove that the uniqueness in law for an SDE

$$(*)\qquad dX_t^i = b_t^i(X)\,dt + \sum_{j=1}^m \sigma_t^{ij}(X)\,dB_t^j, \qquad X_0^i = x^i, \quad i = 1, \dots, n,$$

implies the uniqueness of the joint distribution of a pair (X, B).

Moreover, we prove that the uniqueness in law for $(*)$, together with the strong existence, guarantees the pathwise uniqueness. This result is somehow "dual" to the theorem of Yamada and Watanabe.

Key words. stochastic differential equations, weak solutions, strong solutions, uniqueness in law, pathwise uniqueness, theorem of Yamada and Watanabe

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1. Introduction.

Let $C(\mathbf{R}_+, \mathbf{R}^n)$ be the space of continuous functions $\mathbf{R}_+ \to \mathbf{R}^n$. Recall that the *coordinate process* $Y = (Y_t)_{t \geqq 0}$ on this space is defined by

$$Y_t\colon\ C(\mathbf{R}_+, \mathbf{R}^n) \ni y \longmapsto y(t) \in \mathbf{R}^n.$$

The filtration $\mathcal{H}_t = \cap_{\varepsilon>0}\sigma(Y_s;\ s \leqq t+\varepsilon)$ is called the *canonical filtration* on $C(\mathbf{R}_+, \mathbf{R}^n)$. The *predictable σ-field* on $C(\mathbf{R}_+, \mathbf{R}^n)$ is the σ-field generated by the left-continuous $(\mathcal{H}_t)$-adapted processes on $C(\mathbf{R}_+, \mathbf{R}^n)$.

In this paper, we will deal with the multidimensional stochastic differential equations (SDEs) of the form

$$(1.1)\qquad dX_t^i = b_t^i(X)\,dt + \sum_{j=1}^m \sigma_t^{ij}(X)\,dB_t^j, \qquad X_0^i = x^i, \quad i = 1, \dots, n,$$

where $n \in \mathbf{N}$, $m \in \mathbf{N}$, $x \in \mathbf{R}^n$, and b, σ are predictable processes on $C(\mathbf{R}_+, \mathbf{R}^n)$ taking values in $\mathbf{R}^n$ and $\mathbf{R}^{n\times m}$, respectively.

Remark. We fix a starting point x together with b and σ. In our terminology, SDEs with the same b and σ and with different starting points are different SDEs.

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†Faculty of Mechanics and Mathematics, Department of Probability Theory, MSU, Vorobjevy Gory, 119899 Moscow, Russia (cherny@mech.math.msu.su).

DEFINITION 1.1. *A* solution *of* (1.1) *is a pair* (X, B) *of adapted processes on a filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqq 0}, \mathbf{P})$ *such that*

(i) B *is an* $(\mathcal{F}_t, \mathbf{P})$-$\mathrm{BM}^m(0)$; *i.e.,* B *is an* m*-dimensional Brownian motion started at zero and is an* $(\mathcal{F}_t, \mathbf{P})$*-martingale*;

(ii) *for any* $t \geqq 0$,

$$\int_0^t \left(\sum_{i=1}^n |b_s^i(X)| + \sum_{i=1}^n \sum_{j=1}^m (\sigma_s^{ij}(X))^2 \right) ds < \infty \qquad \mathbf{P}\text{-}a.s.;$$

(iii) *for any* $t \geqq 0$, $i = 1, \ldots, n$,

$$X_t^i = x^i + \int_0^t b_s^i(X)\, ds + \sum_{j=1}^m \int_0^t \sigma_s^{ij}(X)\, dB_s^j \qquad \mathbf{P}\text{-}a.s. \tag{1.2}$$

Remark. In what follows, we will use the vector form of the notation. If b is an n-dimensional process, then by $\int_0^t b_s\, ds$ we will mean the n-dimensional process whose ith component equals $\int_0^t b_s^i\, ds$. If M is an m-dimensional local martingale and σ is a predictable $\mathbf{R}^{n \times m}$-valued process, then by $\int_0^t \sigma_s\, dM_s$ we will mean the n-dimensional process whose ith component equals $\sum_{j=1}^m \int_0^t \sigma_s^{ij}\, dM_s^j$. With this form of notation, equality (1.2) can be rewritten as

$$X_t = x + \int_0^t b_s(X)\, ds + \int_0^t \sigma_s(X)\, dB_s \quad \mathbf{P}\text{-a.s.}$$

DEFINITION 1.2. *A solution* (X, B) *is called a* strong solution *if* X *is adapted to* $(\overline{\mathcal{F}}_t^B)$, *i.e., to the completed natural filtration of* B.

Remark. Solutions in the sense of Definition 1.1 are sometimes called *weak solutions*. Here we simply call them *solutions*. However, the existence of a solution will be denoted by the term *weak existence* in order to stress its difference from the *strong existence*, i.e., the existence of a strong solution.

DEFINITION 1.3. *There is* uniqueness in law *for* (1.1) *if for any solutions* (X, B) *and* $(\widetilde{X}, \widetilde{B})$ (*that may be defined on different filtered probability spaces*), *one has* $\mathrm{Law}(X_t;\, t \geqq 0) = \mathrm{Law}(\widetilde{X}_t;\, t \geqq 0)$.

DEFINITION 1.4. *There is* pathwise uniqueness *for* (1.1) *if for any solutions* (X, B) *and* $(\widetilde{X}, B)$ (*that are defined on the same filtered probability space*), *one has* $\mathbf{P}\{\forall\, t \geqq 0,\, X_t = \widetilde{X}_t\} = 1$.

Remarks. (i) If there exists no solution of (1.1), then there is both uniqueness in law and pathwise uniqueness.

(ii) An overview of sufficient conditions for various types of existence and various types of uniqueness can be found in [7, Chap. 4, section 4], [8, Chap. IX], [10], and [12].

The following two propositions clarify the advantages of the strong solutions and of the pathwise uniqueness.

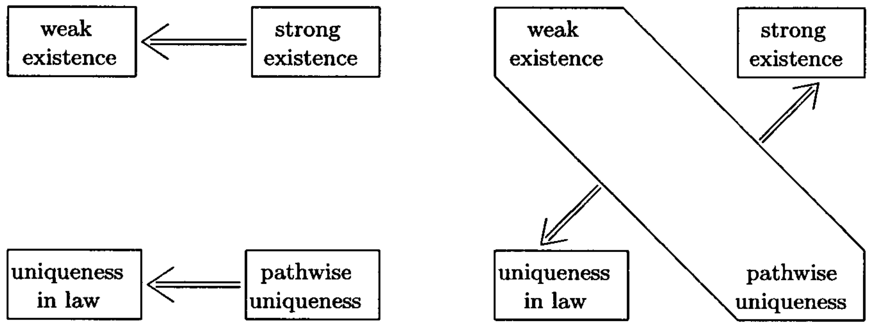

FIG. 1. *The obvious implications and the implications given by the theorem of Yamada and Watanabe.*

PROPOSITION 1.1. *Let* (X, B) *be a strong solution of* (1.1)*. Then*

(i) *there exists a measurable map*

$$\Phi\colon \bigl(C(\mathbf{R}_+, \mathbf{R}^m), \mathcal{B}\bigr) \longrightarrow \bigl(C(\mathbf{R}_+, \mathbf{R}^n), \mathcal{B}\bigr)$$

(*here,* $\mathcal{B}$ *denotes the Borel* σ*-field*) *such that* $X(\omega) = \Phi(B(\omega))$ *for* $\mathbf{P}$*-a.e.* ω;

(ii) *if* $\widetilde{B}$ *is an* $(\widetilde{\mathcal{F}}_t, \widetilde{\mathbf{P}})$-BM$^m(0)$ *and* $\widetilde{X}(\widetilde{\omega}) := \Phi(\widetilde{B}(\widetilde{\omega}))$, *then* $(\widetilde{X}, \widetilde{B})$ *is a strong solution of* (1.1) *on* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t), \widetilde{\mathbf{P}})$.

For the proof, see, for example, [1].

PROPOSITION 1.2 (Yamada and Watanabe). *Suppose that the pathwise uniqueness holds for* (1.1)*. Then*

(i) *the uniqueness in law holds for* (1.1);

(ii) *there exists a measurable map*

$$\Phi\colon \bigl(C(\mathbf{R}_+, \mathbf{R}^m), \mathcal{B}\bigr) \longrightarrow \bigl(C(\mathbf{R}_+, \mathbf{R}^n), \mathcal{B}\bigr)$$

such that, for any solution (X, B) *of* (1.1)*, we have* $X(\omega) = \Phi(B(\omega))$ *for* $\mathbf{P}$*-a.e.* ω.

Proposition 1.2 is illustrated by Figure 1.

For the proof, see [11] or [8, Chap. IX, Thm. (1.7)].

The situation with the solutions of SDEs may now be described as follows.

It may happen that there exists no solution of (1.1) on any filtered probability space (see Example 2.1). It may also happen that on some filtered probability spaces there exists a solution (or even several solutions with the same Brownian motion) while on some other filtered probability spaces there exists no solution (see Example 2.2).

If there exists a strong solution of (1.1) on some filtered probability space, then there exists a strong solution on any other filtered probability space with any Brownian motion on this space (see Proposition 1.1). However, it may happen in this case that there exist several solutions with the same Brownian motion (see Example 2.3).

If the pathwise uniqueness holds for (1.1) and there exists a solution on some filtered probability space, then on any other filtered probability space with any Brownian motion there exists exactly one solution, and this solution is strong (see Proposition 1.2). This is the best possible situation.

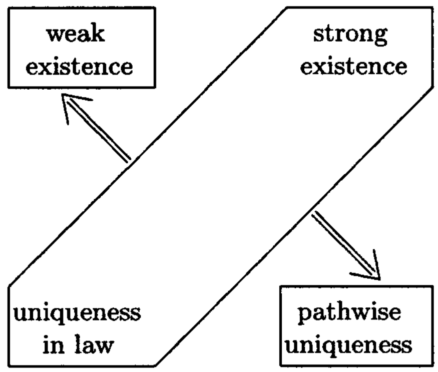

Fig. 2. *The implications give by Theorem* 3.2.

Thus, the theorem of Yamada and Watanabe shows that the pathwise uniqueness, together with the existence of a solution, guarantees that the situation is the best possible.

In this paper, we prove that the situation is the best possible provided that we have the uniqueness in law and the strong existence. Namely, we show that these two properties imply the pathwise uniqueness (Theorem 3.2). Theorem 3.2 is illustrated by Figure 2. The proof of this result is based on a statement that is of interest in itself: if there is uniqueness in law for (1.1), then the joint distribution $\mathrm{Law}(X_t, B_t;\, t \geqq 0)$ is the same for all solutions (X, B) (Theorem 3.1).

Remarks. (i) One may consider SDEs of a more general form than (1.1), i.e., the SDEs in which a Brownian motion B is replaced by a semimartingale Z. For such SDEs, the uniqueness in law is sometimes defined as the uniqueness of the joint distribution of (X, Z) (see [4], [5]). Theorem 3.1 shows that, for SDEs of the form (1.1), this strengthened version of the uniqueness in law is equivalent to Definition 1.3.

(ii) Engelbert proved in [2] that the uniqueness of the joint distribution $\mathrm{Law}(X_t, B_t;\, t \geqq 0)$, together with the strong existence, guarantees the pathwise uniqueness. Moreover, it is proved in [2], under certain additional assumptions, that the uniqueness in law for (1.1) implies the uniqueness of the joint distribution of (X, B). Theorem 3.1 in the present paper shows that this result is true with no additional assumptions.

The paper is arranged as follows. Section 2 contains several examples of SDEs. These examples illustrate various possible situations with the existence and the uniqueness of solutions. Examples 2.2 and 2.3 are well known. The main results of the paper are given in section 3. Section 4 contains an interpretation of Theorem 3.1 in terms of the martingale problems. We also present in section 4 a table that shows which combinations of existence and uniqueness are possible and which are impossible.

2. Examples.

Example 2.1 (no solution). *For the SDE*

$$dX_t = -\frac{1}{2X_t}\, I(X_t \neq 0)\, dt + dB_t, \qquad X_0 = 0, \tag{2.1}$$

there exists no solution.

Proof. Suppose that (X, B) is a solution of (2.1). Then

$$X_t = -\int_0^t \frac{1}{2X_s} I(X_s \neq 0)\, ds + B_t, \qquad t \geqq 0.$$

By Itô's formula,

$$\begin{aligned} X_t^2 &= -\int_0^t 2X_s \frac{1}{2X_s} I(X_s \neq 0)\, ds + \int_0^t 2X_s\, dB_s + \int_0^t 1\, ds \\ &= \int_0^t I(X_s = 0)\, ds + \int_0^t 2X_s\, dB_s, \qquad t \geqq 0. \end{aligned}$$

The process X is a continuous semimartingale with $\langle X \rangle_t = t$. Hence, by the occupation times formula (see [8, Chap. VI, Cor. 1.6]),

$$\int_0^t I(X_s = 0)\, ds = \int_{\mathbf{R}} I(x = 0)\, L_t^x(X)\, dx = 0, \qquad t \geqq 0,$$

where $L_t^x(X)$ denotes the local time spent by the process X at the point x by the time t. As a result, X^2 is a local martingale. Since $X^2 \geqq 0$ and $X_0^2 = 0$, we conclude that $X^2 = 0$ a.s. This means that (X, B) is not a solution of (2.1).

EXAMPLE 2.2 (no strong solution and no pathwise uniqueness; Tanaka). *For the SDE*

$$dX_t = \operatorname{sign} X_t\, dB_t, \qquad X_0 = 0, \tag{2.2}$$

where

$$\operatorname{sign} x = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x \leqq 0, \end{cases}$$

there exists a solution and there is uniqueness in law while there exists no strong solution and there is no pathwise uniqueness.

Proof. Let W be a Brownian motion on $(\Omega, \mathcal{F}, \mathbf{P})$. Set

$$X_t = W_t, \quad B_t = \int_0^t \operatorname{sign} W_s\, dW_s, \quad t \geqq 0,$$

and take $\mathcal{F}_t = \mathcal{F}_t^B$. Then (X, B) is a solution of (2.2) on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$.

If (X, B) is a solution of (2.2) on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$, then X is a continuous $(\mathcal{F}_t, \mathbf{P})$-local martingale with $\langle X \rangle_t = t$. It follows from Lévy's characterization theorem that X is a Brownian motion. This implies the uniqueness in law.

If (X, B) is a solution of (2.2), then

$$B_t = \int_0^t \operatorname{sign} X_s\, dX_s, \qquad t \geqq 0.$$

This implies that $\mathcal{F}_t^B = \mathcal{F}_t^{|X|}$ (see [8, Chap. VI, Cor. 2.2]). Hence, there exists no strong solution.

If (X, B) is a solution of (2.2), then $(-X, B)$ also is a solution. Thus, there is no pathwise uniqueness.

Remark. Let B be a Brownian motion on $(\Omega, \mathcal{F}, \mathbf{P})$. Set $\mathcal{F}_t = \mathcal{F}_t^B$. Then there exists no solution of (2.2) on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$ with the Brownian motion B. Indeed, if (X, B) is a solution, then X is $(\mathcal{F}_t)$-adapted, and hence, (X, B) is a strong solution. On the other hand, (2.2) possesses no strong solution.

Example 2.3 (no uniqueness). *For the SDE*

$$dX_t = I(X_t \neq 0)\, dB_t, \qquad X_0 = 0, \tag{2.3}$$

there exists a strong solution while there is no uniqueness in law, and there is no pathwise uniqueness.

Proof. It is sufficient to note that (B, B) and $(0, B)$ are solutions of (2.3) whenever B is a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$.

Remark. Let B be a Brownian motion and ξ be a random variable independent of B with $\mathbf{P}\{\xi = 1\} = \mathbf{P}\{\xi = -1\} = \frac{1}{2}$. Set

$$X_t(\omega) = \begin{cases} B_t(\omega) & \text{if } \xi(\omega) = 1, \\ 0 & \text{if } \xi(\omega) = -1 \end{cases}$$

and take $\mathcal{F}_t = \mathcal{F}_t^X$. Then (X, B) is a solution of (2.3) that is not strong. Indeed, for each $t > 0$, ξ is $\sigma(X_t)$-measurable while ξ is not $\mathcal{F}_t^B$-measurable.

Example 2.4 (no strong solution and no uniqueness). *For the SDE*

$$dX_t = I(X_t \neq 1)\, \operatorname{sign} X_t\, dB_t, \qquad X_0 = 0, \tag{2.4}$$

there exists a solution while there exists no strong solution, there is no uniqueness in law, and there is no pathwise uniqueness.

Proof. If W is a Brownian motion, then the pair

$$X_t = W_t, \quad B_t = \int_0^t \operatorname{sign} W_s\, dW_s, \qquad t \geqq 0, \tag{2.5}$$

is a solution of (2.4).

Let (X, B) be the solution given by (2.5). Set $\tau = \inf\{t \geqq 0\colon\ X_t = 1\}$, $\widetilde{X}_t = X_{t\wedge\tau}$. Then $(\widetilde{X}, B)$ is another solution. Thus, there is no uniqueness in law and there is no pathwise uniqueness.

Suppose that (X, B) is a strong solution of (2.4). Set $\tau = \inf\{t \geqq 0\colon\ X_t = 1\}$, $X_t^\tau = X_{t\wedge\tau}$, $B_t^\tau = B_{t\wedge\tau}$. The random variable $\tau' = \inf\{t \geqq 0\colon\ X_t = 1/2\}$ is an $(\mathcal{F}_t^X)$-stopping time. Since X is $(\overline{\mathcal{F}}_t^B)$-adapted, τ' is also an $(\overline{\mathcal{F}}_t^B)$-stopping time. Hence, there exists an $(\mathcal{F}_t^B)$-stopping time τ'' such that $\tau'' = \tau'$ a.s. (see [6, Chap. I, Lem. 1.19]). It follows from Galmarino's test (see [3, section 3.2]) that τ'' is also an $(\mathcal{F}_t^{B^\tau})$-stopping time. On the other hand,

$$B_t^\tau = \int_0^t \operatorname{sign} X_t^\tau\, dX_t^\tau, \qquad t \geqq 0.$$

In view of the theory of local times for continuous semimartingales, this equality yields that $\mathcal{F}_t^{B^\tau} \subseteq \mathcal{F}_t^{|X^\tau|}$ (see [8, Chap. VI, section 1]). But obviously, τ' and τ'' are not stopping times with respect to $(\mathcal{F}_t^{|X^\tau|})$. Hence, there exists no strong solution.

3. The main results.

THEOREM 3.1. *Suppose that the uniqueness in law holds for* (1.1). *Then, for any solutions* (X, B) *and* $(\widetilde{X}, \widetilde{B})$ (*that may be defined on different filtered probability spaces*), *one has* $\mathrm{Law}(X_t, B_t;\ t \geqq 0) = \mathrm{Law}(\widetilde{X}_t, \widetilde{B}_t;\ t \geqq 0)$.

THEOREM 3.2. *Suppose that the uniqueness in law holds for* (1.1) *and there exists a strong solution. Then the pathwise uniqueness holds for* (1.1).

Remark. Consider a one-dimensional SDE of the form (1.1) such that $\sigma_t(x) \neq 0$ for any $t \geqq 0$, $x \in C(\mathbf{R}_+, \mathbf{R})$. In this case, Theorem 3.1 is almost trivial. Indeed, B is expressed as a measurable functional of X:

$$B_t = \int_0^t \frac{1}{\sigma_s(X)}\, dM_s, \qquad t \geqq 0,$$

where

$$M_t = X_t - \int_0^t b_s(X)\, ds, \qquad t \geqq 0.$$

However, if σ vanishes at some points, this reasoning does not work.

In order to prove Theorems 3.1 and 3.2, we need several auxiliary lemmas.

LEMMA 3.1. *If* B *is an* $(\mathcal{F}_t, \mathbf{P})$-$\mathrm{BM}^m(0)$, *then, for any* $0 \leqq s \leqq t$, *the random variable* $B_t - B_s$ *is independent of* $\mathcal{F}_s$.

Proof. By Itô's formula, the process $\exp\{i(\lambda, B_t) + \|\lambda\|^2 t/2\}$ is an $(\mathcal{F}_t, \mathbf{P})$-local martingale for any $\lambda \in \mathbf{R}^m$. Being bounded, it is a martingale. Hence, for any $0 \leqq s \leqq t$, $A \in \mathcal{F}_s$, $\lambda \in \mathbf{R}^m$, we have

$$\mathbf{E}\big[\exp\{i(\lambda, B_t - B_s)\}\, I_A\big] = \exp\bigg\{-(t-s)\,\frac{\|\lambda\|^2}{2}\bigg\}\, \mathbf{P}(A).$$

This leads to the desired result.

LEMMA 3.2. *Let* $t \geqq 0$ *and* $f \in L^2([0, t])$. *For* $k \in \mathbf{N}$, *set*

$$f^{(k)}(s) = \begin{cases} 0 & \text{if } s \in \Big[0, \dfrac{t}{k}\Big], \\ \dfrac{k}{t} \displaystyle\int_{(i-1)t/k}^{it/k} f(r)\, dr & \text{if } s \in \Big(\dfrac{it}{k}, \dfrac{(i+1)t}{k}\Big], \qquad (i = 1, \ldots, k-1). \end{cases}$$

Then $f^{(k)} \to f$ *in* $L^2([0, t])$ *as* $k \to \infty$.

Proof. We have

$$\begin{aligned} \|f^{(k)}\|^2_{L^2([0,t])} &= \sum_{i=1}^{k-1} \frac{t}{k} \bigg(\frac{k}{t} \int_{(i-1)t/k}^{it/k} f(r)\, dr\bigg)^2 \leqq \sum_{i=1}^{k-1} \int_{(i-1)t/k}^{it/k} f^2(r)\, dr \\ &\leqq \int_0^t f^2(r)\, dr = \|f\|^2_{L^2([0,t])}. \end{aligned} \tag{3.1}$$

Fix $\varepsilon > 0$. Then there exists $\varphi \in C([0, t])$ such that $\|\varphi - f\|_{L^2([0,t])} < \varepsilon$. Let $\varphi^{(k)}$ be constructed by φ in the same way as $f^{(k)}$ is constructed by f. Then, in view of (3.1),

$$\|\varphi^{(k)} - f^{(k)}\|_{L^2([0,t])} \leqq \|\varphi - f\|_{L^2([0,t])} < \varepsilon$$

for any $k \in \mathbf{N}$. Furthermore, as φ is continuous, there exists $K \in \mathbf{N}$ such that, for any $k \geqq K$, $\|\varphi^{(k)} - \varphi\|_{L^2([0,t])} < \varepsilon$. This leads to the desired result.

We will recall the following fact from the measure theory. Let $\xi\colon\ \Omega \to E$ be a random element on $(\Omega, \mathcal{F}, \mathbf{P})$ taking values in a Polish space $(E, \mathcal{B}(E))$. Let $\mathcal{G} \subseteq \mathcal{F}$. Then there exists a *conditional distribution* of ξ with respect to $\mathcal{G}$, i.e., a family $(\mathbf{Q}_\omega)_{\omega\in\Omega}$ of probability measures on $(E, \mathcal{B}(E))$ such that

(i) for any $A \in \mathcal{B}(E)$, the map $\omega \mapsto \mathbf{Q}_\omega(A)$ is $\mathcal{G}$-measurable;

(ii) for any $A \in \mathcal{B}(E)$, $D \in \mathcal{G}$,

$$\mathbf{P}\big(D \cap \{\xi \in A\}\big) = \int_D \mathbf{Q}_\omega(A)\,\mathbf{P}(d\omega).$$

The conditional distribution is unique in the following sense: If $(\widetilde{\mathbf{Q}}_\omega)_{\omega\in\Omega}$ is another family with the stated properties, then $\mathbf{Q}_\omega = \widetilde{\mathbf{Q}}_\omega$ for $\mathbf{P}$-a.e. ω.

Remark. Properties (i), (ii) mean that, for any bounded $\mathcal{B}(E)$-measurable function h, the random variable $\eta(\omega) := \mathbf{E}_{\mathbf{Q}_\omega}[h]$ is a version of $\mathbf{E}_{\mathbf{P}}[h(\xi)|\mathcal{G}]$. Note also that if $A \in \mathcal{B}(E)$ is such that $\mathbf{P}\{\xi \in A\} = 1$, then $\mathbf{Q}_\omega(A) = 1$ for $\mathbf{P}$-a.e. ω.

LEMMA 3.3. *Let (X, B) be a solution of (1.1) on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$. Let $(\mathbf{Q}_\omega)_{\omega\in\Omega}$ be a conditional distribution of (X, B) with respect to $\mathcal{F}_0$ (we consider (X, B) as a $C(\mathbf{R}_+, \mathbf{R}^{n+m})$-valued random element). Let Y denote the process that consists of the first n components of the coordinate process on $C(\mathbf{R}_+, \mathbf{R}^{n+m})$, and let Z denote the process that consists of the last m components. Let $(\mathcal{H}_t)$ be the canonical filtration on $C(\mathbf{R}_+, \mathbf{R}^{n+m})$ and $\mathcal{H} = \bigvee_{t\geqq 0}\mathcal{H}_t$. Then, for $\mathbf{P}$-a.e. ω, the pair (Y, Z) is a solution of (1.1) on $(C(\mathbf{R}_+, \mathbf{R}^{n+m}), \mathcal{H}, (\mathcal{H}_t), \mathbf{Q}_\omega)$.*

Proof. Let us check conditions (i)–(iii) of Definition 1.1.

(i) For any $0 \leqq s \leqq t$, $D \in \mathcal{H}_s$, $\lambda \in \mathbf{R}^m$, $A \in \mathcal{F}_0$, we have

$$\begin{aligned}&\mathbf{E}_{\mathbf{P}}\big[\exp\{i(\lambda, B_t - B_s)\}\, I\big((X,B) \in D\big)\, I_A\big]\\ &\quad= \exp\left\{-(t-s)\frac{\|\lambda\|^2}{2}\right\}\mathbf{E}_{\mathbf{P}}\big[I\big((X,B)\in D\big)\, I_A\big].\end{aligned}$$

Hence, for any $0 \leqq s \leqq t$, $D \in \mathcal{H}_s$, $\lambda \in \mathbf{R}^m$, we have

$$\mathbf{E}_{\mathbf{Q}_\omega}[\exp\{i(\lambda, Z_t - Z_s)\}\, I_D] = \exp\left\{-(t-s)\frac{\|\lambda\|^2}{2}\right\}\mathbf{Q}_\omega(I_D)$$

for $\mathbf{P}$-a.e. ω. Taking a countable collection $\{s_k, t_k, D_{kl}, \lambda_{kl};\ k, l \in \mathbf{N}\}$ such that the sequence (s_k, t_k) runs through all pairs of positive rational numbers $(s_k \leqq t_k)$, the collection $\{D_{kl};\ l \in \mathbf{N}\}$ generates $\mathcal{H}_{s_k}$, and the set $\{\lambda_{kl};\ l \in \mathbf{N}\}$ is dense in $\mathbf{R}^m$, we deduce that, for $\mathbf{P}$-a.e. ω, the process Z is an $(\mathcal{H}_t, \mathbf{Q}_\omega)$-$\mathrm{BM}^m(0)$.

(ii) For any $t \geqq 0$,

$$\int_0^t \left(\sum_{i=1}^n |b_s^i(X)| + \sum_{i=1}^n\sum_{j=1}^m (\sigma_s^{ij}(X))^2\right) ds < \infty \quad \mathbf{P}\text{-a.s.}$$

Hence, for any $t \geqq 0$,

$$\int_0^t \left(\sum_{i=1}^n |b_s^i(Y)| + \sum_{i=1}^n\sum_{j=1}^m (\sigma_s^{ij}(Y))^2\right) ds < \infty \quad \mathbf{Q}_\omega\text{-a.s.}$$

for $\mathbf{P}$-a.e. ω.

(iii) Fix $t \geqq 0$. For $k \in \mathbf{N}$, define a process

$$\sigma^{(k)}\colon\ \mathbf{R}_+ \times C(\mathbf{R}_+, \mathbf{R}^n) \ni (s, y) \longmapsto \sigma_s^{(k)}(y) \in \mathbf{R}^{n \times m}$$

by

$$\sigma_s^{(k)}(y) = \begin{cases} 0 & \text{if } s \in \left[0, \dfrac{t}{k}\right], \\ \dfrac{k}{t} \displaystyle\int_{(i-1)t/k}^{it/k} \sigma_r(y)\, dr & \text{if } s \in \left(\dfrac{it}{k}, \dfrac{(i+1)t}{k}\right], \qquad i = 1, \dots, k-1. \end{cases}$$

Then, by Lemma 3.2,

$$\int_0^t \left\|\sigma_s^{(k)}(X) - \sigma_s(X)\right\|^2 ds \xrightarrow[k\to\infty]{\mathbf{P}\text{-a.s.}} 0. \tag{3.2}$$

Consequently,

$$\int_0^t \sigma_s^{(k)}(X)\, dB_s \xrightarrow[k\to\infty]{\mathbf{P}} \int_0^t \sigma_s(X)\, dB_s,$$

which means that

$$\sum_{i=2}^{k} \sigma_{it/k}^{(k)}(X)\big(B_{it/k} - B_{(i-1)t/k}\big) \xrightarrow[k\to\infty]{\mathbf{P}} X_t - x - \int_0^t b_s(X)\, ds \tag{3.3}$$

(we use here the vector form of notation). There exists a subsequence $(k(l))$ such that, along this subsequence, the convergence in (3.3) holds $\mathbf{P}$-a.s. Therefore,

$$\sum_{i=2}^{k(l)} \sigma_{it/k(l)}^{(k(l))}(Y)\big(Z_{it/k(l)} - Z_{(i-1)t/k(l)}\big) \xrightarrow[l\to\infty]{\mathbf{Q}_\omega\text{-a.s.}} Y_t - x - \int_0^t b_s(Y)\, ds \tag{3.4}$$

for $\mathbf{P}$-a.e. ω. On the other hand, in view of (3.2),

$$\int_0^t \left\|\sigma_s^{(k)}(Y) - \sigma_s(Y)\right\|^2 ds \xrightarrow[k\to\infty]{\mathbf{Q}_\omega\text{-a.s.}} 0$$

for $\mathbf{P}$-a.e. ω, and hence,

$$\int_0^t \sigma_s^{(k)}(Y)\, dZ_s \xrightarrow[k\to\infty]{\mathbf{Q}_\omega} \int_0^t \sigma_s(Y)\, dZ_s \tag{3.5}$$

for $\mathbf{P}$-a.e. ω. Combining (3.4) and (3.5), we get

$$Y_t - x - \int_0^t b_s(Y)\, ds = \int_0^t \sigma_s(Y)\, dZ_s \quad \mathbf{Q}_\omega\text{-a.s.}$$

for $\mathbf{P}$-a.e. ω. This completes the proof.

Proof of Theorem 3.1. Let (X,B) be a solution of (1.1) on a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t),\mathbf{P})$. Let $(W_t)_{t\geqq 0}$ and $(\overline{W}_t)_{t\geqq 0}$ be two independent $(\mathcal{F}'_t,\mathbf{P}')$-$\mathrm{BM}^m(0)$. Set

$$\big(\widetilde{\Omega},\widetilde{\mathcal{F}},(\widetilde{\mathcal{F}}_t),\widetilde{\mathbf{P}}\big)=\big(\Omega,\mathcal{F},(\mathcal{F}_t),\mathbf{P}\big)\times\big(\Omega',\mathcal{F}',(\mathcal{F}'_t),\mathbf{P}'\big).$$

The processes X, B, W, and $\overline{W}$ can be defined on $\widetilde{\Omega}$ in an obvious way. The pair (X,B) is a solution of (1.1) on $(\widetilde{\Omega},\widetilde{\mathcal{F}},(\widetilde{\mathcal{F}}_t),\widetilde{\mathbf{P}})$, and W, $\overline{W}$ are independent $(\widetilde{\mathcal{F}}_t,\widetilde{\mathbf{P}})$-$\mathrm{BM}^m(0)$.

For any $t\geqq 0$, $y\in C(\mathbf{R}_+,\mathbf{R}^n)$, the matrix $\sigma_t(y)$ corresponds to a linear operator $\mathbf{R}^m\to\mathbf{R}^n$. Let $\varphi_t(y)$ denote the $m\times m$-matrix of the operator of orthogonal projection onto $(\ker\sigma_t(y))^\perp$; let $\psi_t(y)$ denote the $m\times m$-matrix of the operator of orthogonal projection onto $\ker\sigma_t(y)$. Then the processes $\varphi=\varphi_t(y)$ and $\psi=\psi_t(y)$ are predictable $\mathbf{R}^{m\times m}$-valued processes on $C(\mathbf{R}_+,\mathbf{R}^n)$.

Set

$$U_t=\int_0^t\varphi_s(X)\,dB_s+\int_0^t\psi_s(X)\,dW_s,\qquad t\geqq 0,\tag{3.6}$$

$$V_t=\int_0^t\varphi_s(X)\,d\overline{W}_s+\int_0^t\psi_s(X)\,dB_s,\qquad t\geqq 0.\tag{3.7}$$

The $2m$-dimensional process (U,V) is a continuous $(\widetilde{\mathcal{F}}_t,\widetilde{\mathbf{P}})$-local martingale. Moreover, for any $i,j=1,\dots,m$, in view of the symmetry of matrices $\varphi_t(y)$, $\psi_t(y)$, we have

$$\begin{aligned}\langle U^i,U^j\rangle_t&=\int_0^t\left(\sum_{k=1}^m\varphi_s^{ik}(X)\,\varphi_s^{jk}(X)+\sum_{k=1}^m\psi_s^{ik}(X)\,\psi_s^{jk}(X)\right)ds\\&=\int_0^t\Big(\big(\varphi_s(X)\,e_i\,,\varphi_s(X)\,e_j\big)+\big(\psi_s(X)\,e_i\,,\psi_s(X)\,e_j\big)\Big)\,ds\\&=\int_0^t\big(\varphi_s(X)\,e_i+\psi_s(X)\,e_i\,,\varphi_s(X)\,e_j+\psi_s(X)\,e_j\big)\,ds=\int_0^t(e_i\,,e_j)\,ds=\delta_{ij}t,\end{aligned}$$

where $(e_i)_{i=1}^m$ is the standard basis in $\mathbf{R}^m$. Similarly,

$$\begin{aligned}\langle U^i,V^j\rangle_t&=\int_0^t\big(\varphi_s(X)\,e_i\,,\psi_s(X)\,e_j\big)\,ds=0,\\\langle V^i,V^j\rangle_t&=\delta_{ij}t.\end{aligned}$$

By the multidimensional version of Lévy's characterization theorem (see [8, Chap. IV, Thm. 3.6]), we deduce that the process (U,V) is an $(\widetilde{\mathcal{F}}_t,\widetilde{\mathbf{P}})$-$\mathrm{BM}^{2m}(0)$.

For any $t\geqq 0$, we have

$$\begin{aligned}\int_0^t\sigma_s(X)\,dB_s&=\int_0^t\big(\sigma_s(X)\,\varphi_s(X)\big)\,dB_s=\int_0^t\sigma_s(X)\,d\Big(\int_0^s\varphi_r(X)\,dB_r\Big)\\&=\int_0^t\sigma_s(X)\,d\Big(\int_0^s\varphi_r(X)\,dU_r\Big)=\int_0^t\sigma_s(X)\,dU_s,\end{aligned}$$

where $\sigma_s(X)\,\varphi_s(X)$ denotes the product of matrices. Consequently, (X,U) is a solution of (1.1) on $(\widetilde{\Omega},\widetilde{\mathcal{F}},(\widetilde{\mathcal{F}}_t),\widetilde{\mathbf{P}})$.

Let us now consider the filtration

$$\mathcal{G}_s = \widetilde{\mathcal{F}}_s \vee \sigma(V_t;\, t \geqq 0) = \widetilde{\mathcal{F}}_s \vee \sigma(V_t - V_s;\ t \geqq s), \qquad s \geqq 0.$$

It follows from Lemma 3.1 that, for any $s \geqq 0$, the σ-fields $\widetilde{\mathcal{F}}_s$ and $\sigma(U_t - U_s;\ t \geqq s) \vee \sigma(V_t - V_s;\ t \geqq s)$ are independent. Hence, for any $0 \leqq s \leqq t$, $i = 1, \dots, m$, $A \in \widetilde{\mathcal{F}}_s$, $D \in \sigma(V_t - V_s;\ t \geqq s)$, we have

$$\mathbf{E}_{\widetilde{\mathbf{P}}}\big[(U_t^i - U_s^i)\, I_D\, I_A\big] = \mathbf{E}_{\widetilde{\mathbf{P}}}\big[(U_t^i - U_s^i)\, I_D\big]\, \widetilde{\mathbf{P}}(A) = \mathbf{E}_{\widetilde{\mathbf{P}}}[U_t^i - U_s^i]\, \widetilde{\mathbf{P}}(D)\, \widetilde{\mathbf{P}}(A) = 0.$$

Thus, U is a $(\mathcal{G}_t, \widetilde{\mathbf{P}})$-$\mathrm{BM}^m(0)$. Since the stochastic integral $\int_0^t \sigma_s(X)\, dU_s$ is the same for both filtrations $(\widetilde{\mathcal{F}}_t)$ and $(\mathcal{G}_t)$, the pair (X, U) is a solution of (1.1) on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\mathcal{G}_t), \widetilde{\mathbf{P}})$.

Let $(\mathbf{Q}_{\widetilde{\omega}})_{\widetilde{\omega} \in \widetilde{\Omega}}$ be a conditional distribution of (X, U) with respect to $\mathcal{G}_0$. By Lemma 3.3, the pair (Y, Z) is a solution of (1.1) on $(C(\mathbf{R}_+, \mathbf{R}^{n+m}), \mathcal{H}, (\mathcal{H}_t), \mathbf{Q}_{\widetilde{\omega}})$ for $\widetilde{\mathbf{P}}$-a.e. $\widetilde{\omega}$. As the uniqueness in law holds for (1.1), the distribution $\mathrm{Law}(Y_t;\, t \geqq 0 \,|\, \mathbf{Q}_{\widetilde{\omega}})$ (which is the conditional distribution of X with respect to $\mathcal{G}_0$) is the same for $\widetilde{\mathbf{P}}$-a.e. $\widetilde{\omega}$. This means that the process X is independent of $\mathcal{G}_0$. In particular, X and V are independent.

For any $t \geqq 0$, $y \in C(\mathbf{R}_+, \mathbf{R}^n)$, the restriction of the operator $\sigma_t(y)$ to $(\ker \sigma_t(y))^{\perp}$ is a bijection from $(\ker \sigma_t(y))^{\perp} \subseteq \mathbf{R}^m$ onto $\operatorname{Im} \sigma_t(y) \subseteq \mathbf{R}^n$. Let us define the operator $\chi_t(y)\colon\ \mathbf{R}^n \to \mathbf{R}^m$ as follows: $\chi_t(y)$ maps $\operatorname{Im} \sigma_t(y)$ onto $(\ker \sigma_t(y))^{\perp}$ as the inverse of $\sigma_t(y)$; $\chi_t(y)$ vanishes on $(\operatorname{Im} \sigma_t(y))^{\perp}$. Obviously, $\chi = \chi_t(y)$ is a predictable $\mathbf{R}^{m \times n}$-valued process on $C(\mathbf{R}_+, \mathbf{R}^n)$. We have $\chi_t(y)\sigma_t(y) = \varphi_t(y)$. Therefore,

$$\int_0^t \varphi_s(X)\, dB_s = \int_0^t \big(\chi_s(X)\, \sigma_s(X)\big)\, dB_s = \int_0^t \chi_s(X)\, dM_s,$$

where

$$M_t = \int_0^t \sigma_s(X)\, dB_s = X_t - x - \int_0^t b_s(X)\, ds.$$

Keeping (3.7) in mind, we get

$$B_t = \int_0^t \varphi_s(X)\, dB_s + \int_0^t \psi_s(X)\, dB_s = \int_0^t \chi_s(X)\, dM_s + \int_0^t \psi_s(X)\, dV_s. \tag{3.8}$$

The process M is a measurable functional of X while V is independent of X. Thus, (3.8) shows that the distribution $\mathrm{Law}(X_t, B_t;\, t \geqq 0)$ is the same for all solutions (X, B).

Proof of Theorem 3.2. Let (X, B) be a strong solution of (1.1) on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$. Then there exists a measurable map $\theta\colon\ C(\mathbf{R}_+, \mathbf{R}^m) \to C(\mathbf{R}_+, \mathbf{R}^n)$ such that $X(\omega) = \theta(B(\omega))$ for $\mathbf{P}$-a.e. ω. Let $(\mathbf{Q}_\omega)_{\omega \in \Omega}$ be a conditional distribution of X with respect to $\mathcal{F}_\infty^B$. Then $\mathbf{Q}_\omega = \delta_{\theta(B(\omega))}$ for $\mathbf{P}$-a.e. ω.

Now, let $(\widetilde{X}, \widetilde{B})$ be a solution of (1.1) on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t), \widetilde{\mathbf{P}})$. Let $(\widetilde{\mathbf{Q}}_{\widetilde{\omega}})_{\widetilde{\omega} \in \widetilde{\Omega}}$ be a conditional distribution of $\widetilde{X}$ with respect to $\mathcal{F}_\infty^{\widetilde{B}}$. Since $\mathrm{Law}(\widetilde{X}, \widetilde{B}) = \mathrm{Law}(X, B)$, we deduce that $\widetilde{\mathbf{Q}}_{\widetilde{\omega}} = \delta_{\theta(\widetilde{B}(\widetilde{\omega}))}$ for $\widetilde{\mathbf{P}}$-a.e. $\widetilde{\omega}$. Hence, $\widetilde{X} = \theta(\widetilde{B}(\widetilde{\omega}))$ for $\widetilde{\mathbf{P}}$-a.e. $\widetilde{\omega}$. This yields the desired statement.

4. Applications of the obtained results. We will first describe the interpretation of Theorem 3.1 in terms of the martingale problems. Let $x \in \mathbf{R}^n$. Let b be a predictable process on $C(\mathbf{R}_+, \mathbf{R}^n)$ taking values in $\mathbf{R}^n$. Let a be a predictable process on $C(\mathbf{R}_+, \mathbf{R}^n)$ taking values in the space of symmetric nonnegative $n \times n$-matrices.

DEFINITION 4.1. *A solution of the* n*-dimensional martingale problem* (x, b, a) *is a measure* $\mathbf{Q}$ *on* $C(\mathbf{R}_+, \mathbf{R}^n)$ *such that*

(i) $\mathbf{Q}\{Y_0 = x\} = 1$ (*here,* Y *denotes the coordinate process on* $C(\mathbf{R}_+, \mathbf{R}^n)$);

(ii) *for any* $t \geqq 0$,

$$\int_0^t \left(\sum_{i=1}^n |b_s^i(Y)| + \sum_{i=1}^n a^{ii}(Y) \right) ds < \infty \quad \mathbf{Q}\text{-}a.s.;$$

(iii) *for any* $i = 1, \dots, n$, *the process*

$$M_t^i = Y_t^i - \int_0^t b_s^i(Y)\, ds$$

is an $(\mathcal{H}_t, \mathbf{Q})$*-local martingale* ($(\mathcal{H}_t)$ *denotes the canonical filtration on* $C(\mathbf{R}_+, \mathbf{R}^n)$), *and, for any* $i, j = 1, \dots, n$,

$$\langle M^i, M^j \rangle_t = \int_0^t a_s^{ij}(Y)\, ds.$$

For more information on martingale problems, see [9].

Let us return to SDE (1.1). Set $a_t(y) = \sigma_t(y)\sigma_t^*(y)$. If (X, B) is a solution of (1.1), then $\mathbf{Q} := \mathrm{Law}(X_t;\, t \geqq 0)$ is a solution of the martingale problem (x, b, a). Conversely, if $\mathbf{Q}$ is a solution of the martingale problem (x, b, a), then there exists a solution (X, B) of (1.1) such that $\mathrm{Law}(X_t;\, t \geqq 0) = \mathbf{Q}$. The uniqueness in law for (1.1) is equivalent to the uniqueness of a solution of the martingale problem (x, b, a).

Now, Theorem 3.1 can be reformulated as follows.

THEOREM 4.1. *Let* (x, b, a) *be an* n*-dimensional martingale problem. Let* σ *be a predictable* $\mathbf{R}^{n\times m}$*-valued process on* $C(\mathbf{R}_+, \mathbf{R}^n)$ *such that* $\sigma_t(y)\sigma_t^*(y) = a_t(y)$. *Then the uniqueness of a solution of the martingale problem* (x, b, a) *implies the uniqueness of a solution of the* $(n+m)$*-dimensional martingale problem*

$$\left(\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} b \\ 0 \end{pmatrix}, \begin{pmatrix} a & \sigma \\ \sigma^* & I \end{pmatrix} \right).$$

Let us now mention one more application of the above results. For SDE (1.1), each of the following properties may or may not hold:

existence of a solution; existence of a strong solution;
uniqueness in law; pathwise uniqueness.

Thus, there are $16 (= 2^4)$ feasible combinations. Some of these combinations are impossible (for instance, if there is pathwise uniqueness, then there must be uniqueness in law). Using Examples 2.1–2.4 as well as Proposition 1.2 and Theorem 3.2, one can, for each of these combinations, either provide an example of the corresponding SDE or prove that this combination is impossible. It turns out that there are only five possible combinations (see Table 1).

TABLE 1

Combinations of various types of existence and of various types of uniqueness. For example, the combination "$+-+-$" *on line* 11 *corresponds to an SDE for which there exists a solution, there exists no strong solution, there is uniqueness in law, and there is no pathwise uniqueness. The table shows that such an SDE is provided by Example* 2.2.

Weak existence	Strong existence	Uniqueness in law	Pathwise uniqueness	Possible/impossible
$-$	$-$	$-$	$-$	impossible, obviously
$-$	$-$	$-$	$+$	impossible, obviously
$-$	$-$	$+$	$-$	impossible, obviously
$-$	$-$	$+$	$+$	possible, Example 2.1
$-$	$+$	$-$	$-$	impossible, obviously
$-$	$+$	$-$	$+$	impossible, obviously
$-$	$+$	$+$	$-$	impossible, obviously
$-$	$+$	$+$	$+$	impossible, obviously
$+$	$-$	$-$	$-$	possible, Example 2.4
$+$	$-$	$-$	$+$	impossible, Figure 1
$+$	$-$	$+$	$-$	possible, Example 2.2
$+$	$-$	$+$	$+$	impossible, Figure 1
$+$	$+$	$-$	$-$	possible, Example 2.3
$+$	$+$	$-$	$+$	impossible, Figure 1
$+$	$+$	$+$	$-$	impossible, Figure 2
$+$	$+$	$+$	$+$	possible, obviously

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