Exercise 1 (Pts 4) Prove that if $B$ is a Brownian motion, then we have the relation $L^{|B|,0}_t = 2L^B_t$ where $L^X_{t,a}$ denotes the local time in $a \in \mathbb{R}$ of the semimartingale $X$.

Exercise 2 (Pts 2+4) Let $y : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function and let
$$a(t) = \sup_{s \in [0,t]} (y(s))_-, \quad z(t) = y(t) + a(t).$$

a) Prove that $a, z$ are continuous functions and $a$ is non-decreasing.

b) Prove that $a$ is of bounded variation and that $\int_0^\infty 1_{s>0} \, da_s = 0$. (Hint: use the fact that $da_s$ is a Borel measure).

c) Remove the assumption of boundedness.

Exercise 3 [Pts 2+2+2] Prove (the upper bound of) Burkholder–Davis–Gundy inequality. Let $M$ be a continuous local martingale (with $M_0 = 0$). For any $p \geq 2$ we have
$$\mathbb{E}[\sup_{t \leq T}|M_t|^p] \leq C_p \mathbb{E}[(|M|^p_T)^{p/2}]$$
where $C_p$ is a universal constant depending only on $p$.

a) Assume that the martingale $M$ is bounded. Use Itô formula on $t \mapsto (\varepsilon + |M_t|^2)^{p/2}$ to prove that
$$\mathbb{E}[\sup_{t \leq T}|M_t|^p] \leq \int_0^T \mathbb{E}[(|M_t|^p-2d[M]_t)].$$

(why we need $\varepsilon > 0$?)

b) Use Hölder’s and Doob’s inequality to conclude.

c) Remove the assumption of boundedness.

Exercise 4 (Pts 2+2+2) Let us continue with the setting of Exercise 3 and prove now a complementary lower bound when $p \geq 4$, that is
$$\mathbb{E}[(|M|^p_T)^{p/2}] \leq C_p \mathbb{E}[\sup_{t \leq T}|M_t|^p].$$
where again $C_p$ is a universal constant depending only on $p$ (not the same as that of the upper bound).

a) Use the relation
$$[M]_T = M^2_T - 2 \int_0^T M_s \, dM_s$$
to estimate $\mathbb{E}[(|M|^p_T)^{p/2}]$ and then use the BDG upper bound for the stochastic integral.

b) Prove that if we let $N_T = \int_0^T M_s \, dM_s$ then for any $\varepsilon > 0$ there exists $\lambda_\varepsilon > 0$ such that
$$[N]^{1/2}_T \leq \lambda_\varepsilon \sup_{t \leq T}|M_t| + \varepsilon |M|_T$$
c) Conclude by choosing $\varepsilon$ small enough.