V4F1 Stochastic Analysis – Problem Sheet 5

Exercise 1 (Pts 2+2+2) (Passage time to a sloping line) Let $X$ be a one–dimensional Brownian motion with $X_0 = 0$ and let $a > 0$, $b \in \mathbb{R}$.

a) Let $T_L = \inf\{t \geq 0 : X_t = a + bt\}$ denote the first passage time to the line $y = a + bt$. Show that
\[ \mathbb{P}(T_L \leq t) = \mathbb{E}[e^{-bX_t - b^2t/2}\mathbb{1}_{T_a \leq t}], \quad (1) \]
where $T_a = \inf\{t \geq 0 : X_t = a\}$ is the first passage time to level $a$.

b) Recall that, by the reflection principle, the law of $T_a$ is absolutely continuous with density
\[ f_{T_a}(t) = at^{-3/2}\phi\left(a/\sqrt{t}\right)1_{(0,\infty)}(t), \]
where $\phi$ is the standard normal density. Deduce that the law of $T_L$ is absolutely continuous with density
\[ f_{T_L}(t) = at^{-3/2}\phi\left((a+bt)/\sqrt{t}\right)1_{(0,\infty)}(t). \]
[Hint: in (1) take the conditional expectation w.r.t. $F_{T_a}$].

c) Show that, for $b > 0$,
\[ \mathbb{E}[e^{-bX_t\max_{s\leq t}(X_s)}] \simeq \frac{e^{b^2t/2}}{2b}, \quad \text{and} \quad \mathbb{E}[e^{bX_t\max_{s\leq t}(X_s)}] \simeq be^{b^2t/2}, \quad \text{as} \ t \to \infty. \]

Exercise 2 (Pts 2+2+3) (Brownian Bridge) Let $X$ be a $d$–dimensional Brownian motion with $X_0 = 0$.

a) Show that, for any $y \in \mathbb{R}^d$, the process
\[ X^y_t = X_t - t(X_1 - y) \quad t \in [0, 1] \]
is independent of $X_1$.

b) Let $\mu_y$ denote the law of $X^y$ on $C([0, 1]; \mathbb{R}^d)$. Show that $y \mapsto \mu_y$ is a regular version of the conditional distribution of $X$ given $X_1 = y$.

c) Compute the SDE satisfied by the canonical process $Y$ under the probability measure $\mu_y$ on the space $C([0, 1]; \mathbb{R}^d)$. (Hint: use Doob’s $h$-transform argument from the lectures)

Exercise 3 (Pts 3) Let $M$ be a positive continuous supermartingale such that $\mathbb{E}[M_0] < \infty$. Let $M_\infty = \lim_{t \to \infty} M_t$. Show that if $\mathbb{E}[M_\infty] = \mathbb{E}[M_0]$ then $M$ is a martingale and $\mathbb{E}[M_\infty | F_t] = M_t$. [Hint: prove that $\mathbb{E}[M_\infty | F_t] \leq M_t$ and that $\mathbb{E}[M] = \mathbb{E}[M_0]$ and conclude.]

Exercise 4 (Pts 4) Prove directly that the $h$-transform gives a transformation of martingale problems from the one with drift $b$ and diffusion $\sigma$ to another with same diffusion coefficient $\sigma$ but different drift $\tilde{b}$. 

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