**Exercise 1 (Pts 3) (Constant quadratic variation)** Let $M$ be a continuous local martingale and $S \leq T$ two stopping times. Prove that $[M]_T = [M]|_S < \infty$ a.s implies $M_t = M_s$ for all $t \in [S,T]$ a.s.

*Hint: consider the continuous local martingale $N_t = \int_0^t |S,T| (s) dM_s].$

**Exercise 2 (Pts 3+3) (Feynman–Kac formula for Ito diffusions)**

a) Consider the solution $X$ of the SDE in $\mathbb{R}^n$

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad X_0 = x,$$

where $B$ is a $d$-dimensional Brownian motion and $b : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}^{n \times d}$ locally bounded continuous coefficients. Let $\mathcal{L}$ be the associated infinitesimal generator. Fix $t > 0$ and assume that $\varphi : \mathbb{R}^n \to \mathbb{R}$ and $V : [0,t] \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ are continuous functions. Show that any bounded $C^{1,2}$ solution $u : [0,t] \times \mathbb{R}^n \to \mathbb{R}$ of the equation

$$\frac{\partial}{\partial t} u(s, x) = \mathcal{L}u(s, x) - V(s, x)u(s, x), \quad (s, x) \in (0, t] \times \mathbb{R}^n,$$

$$u(0, x) = \varphi(x),$$

has the stochastic representation

$$u(t, x) = \mathbb{E} \left[ \varphi(X_t) \exp \left( - \int_0^t V(t - s, X_s)ds \right) \right].$$

*Hint: show that $M_t = \exp \left( - \int_0^t V(t - s, X_s)ds \right) u(t - r, X_r)$ is a local martingale*.

b) The price of a security is modeled by a geometric Brownian motion $X$ with parameters $\alpha, \sigma > 0$:

$$dX_t = \alpha X_t dt + \sigma X_t dB_t, \quad X_0 = x > 0.$$  

At price $y$ we have a running cost of $V(y)$ per unit time. The total cost up to time $t$ is then

$$A_t = \int_0^t V(X_s)ds.$$  

Suppose that $u$ is a bounded solution to the PDE

$$\frac{\partial}{\partial t} u(s, x) = \mathcal{L}u(s, x) - \beta V(x)u(s, x), \quad (s, x) \in (0, t] \times \mathbb{R}_{\geq 0},$$

$$u(0, x) = 1,$$

where $\mathcal{L}$ is the generator of $X$. Show that the Laplace transform of $A_t$ is given by

$$\mathbb{E}[e^{-\beta A_t}] = u(t, x).$$

**Exercise 3 (Pts 3+3+3+2) (Continuous Branching Process)** Consider a family of diffusions $(X_t(x))_{t>0, x>0}$ satisfying the SDE

$$dX_t(x) = \alpha X_t(x)dt + \sqrt{\beta X_t(x)}dB_t, \quad X_0(x) = x,$$

where $\alpha \in \mathbb{R}, \beta \in \mathbb{R}_{>0}$. Existence of strong solutions to this equation follows from the Yamada–Watanabe theorem. Let $(\tilde{X}, \tilde{B})$ be an independent copy of $(X, B)$ and let $Y_t(x, y) = X_t(x) + \tilde{X}_t(y)$ for $t > 0, x > 0, y > 0.$
a) (Branching) Compute the SDE satisfied by $Y$ and prove that $(Y(x,y))_{t \geq 0}$ has the same law of $(X_t(x+y))_{t \geq 0}$. [Hint: use martingale characterization of weak solutions and pathwise uniqueness]

b) (Duality) Show that this implies that there exists a function $u : \mathbb{R}_{\geq 0} \times \mathbb{R}_{> 0} \to \mathbb{R}_{\geq 0}$ such that

$$E[e^{-\lambda X_t(x)}] = e^{-x u(t,\lambda)}, \quad x \in \mathbb{R}_{> 0}$$

(1)

if we assume that the map $x \mapsto E[e^{-\lambda X_t(x)}]$ is continuous.

c) Assume that $u : \mathbb{R}_{\geq 0} \times \mathbb{R}_{> 0} \to \mathbb{R}_{\geq 0}$ is differentiable with respect to its first parameter. Apply Ito formula to $s \mapsto G_s = e^{-u(t-s,\lambda)X_s(x)}$ and determine which differential equation $u$ should satisfy in order for $G$ to be a local martingale. Prove that in this case eq. (1) is satisfied (in particular, if a solution of the equation exists then it is unique).

d) (Extinction probability) Find the explicit solution $u$ for the differential equation and using eq. (1) prove that if $\alpha = 0$ then

$$P(X_t(x) = 0) = e^{-2x/(\beta t)}, \quad x,t > 0.$$