Exercise 1 (Pts 4+2) (Martingale problem) Let \( b : \mathbb{R}^n \to \mathbb{R}^n \), \( \sigma : \mathbb{R}^n \to \mathbb{R}^{n \times n} \) locally bounded coefficients. Let \( a(x) = \sigma(x)\sigma(x)^T \in \mathbb{R}^{n \times n} \) and for all \( f \in C^2(\mathbb{R}^n) \) let
\[
L f(x) = b(x) \cdot \nabla f(x) + \frac{1}{2} \mathrm{Tr} [a(x) \nabla^2 f(x)], \quad x \in \mathbb{R}^n
\]
where \( \nabla^2 f(x) \) is the \( \mathbb{R}^{n \times n} \) matrix of second derivatives of \( f \).

a) Prove that the following conditions are equivalent
i. For any \( f \in C^2(\mathbb{R}^d) \), the process \( M^f_t = f(X_t) - f(X_0) - \int_0^t L f(X_s) \, ds \) is a local martingale.

ii. For any \( v \in \mathbb{R}^d \), the process \( M^v_t = v \cdot X_t - v \cdot X_0 - \int_0^t v \cdot b(X_s) \, ds \) is a local martingale with quadratic variation
\[
[M^v]_t = \int_0^t v \cdot a(X_s) v \, ds.
\]

iii. For any \( v \in \mathbb{R}^d \) the process
\[
Z^v_t = \exp \left( M^v_t - \frac{1}{2} \int_0^t v \cdot a(X_s) v \, ds \right)
\]
is a local martingale.

[Hint: use the fact that linear combinations of exponentials are dense in \( C^2 \) w.r.t. uniform convergence on compacts for the functions and its first two derivatives (assumed without proof)]

b) Show that any of conditions a,b,c imply that
\[
(f(X_t)/f(X_0)) \exp \left( - \int_0^t \frac{L f}{f} (X_s) \, ds \right)
\]
is a local martingale for every strictly positive \( C^2 \) function \( f \).

Exercise 2 (Pts 2+2+2) Let \( (B_t)_{t \geq 0} \) be a one dimensional Brownian motion. Find the SDEs satisfied by the following processes: (for all \( t \geq 0 \))
a) \( X_t = B_t/(1 + t) \),
b) \( X_t = \sin(B_t) \)
c) \( (X_t, Y_t) = (a \cos(B_t), b \sin(B_t)) \) where \( a, b \in \mathbb{R} \) with \( ab \neq 0 \)

Exercise 3 (Pts 2+2+2+2) (Variation of constants) Consider the nonlinear SDE
\[
dX_t = f(t, X_t) \, dt + c(t) X_t \, dB_t, \quad X_0 = x,
\]
where \( f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \) and \( c : \mathbb{R}_+ \to \mathbb{R} \) are continuous deterministic functions.
a) Find an explicit solution \( Z_t \) in the case \( f = 0 \) and \( Z_0 = 1 \).
b) Use the Ansatz \( X_t = C_t Z_t \) to show that \( X \) solves the SDE provided \( C \) solves an ODE with random coefficients.
c) Apply this method to solve the SDE

\[ dX_t = X_t^{-1} dt + \alpha X_t dB_t, \quad X_0 = x \]

where \( \alpha \) is a constant.

d) Apply the method to study the solution of the SDE

\[ dX_t = X_t^\gamma dt + \alpha X_t dB_t, \quad X_0 = x > 0 \]

where \( \alpha \) and \( \gamma \) are constants. For which values of \( \gamma \) do we get explosion, i.e. the solution tends to \(+\infty\) for finite time?