SDE techniques: martingale solutions, time change

Uniqueness of martingale solutions, one dimensional diffusions.

1 Uniqueness of the martingale problem for a diffusion

Let \( \mathcal{C} = \mathcal{C}^n = C(\mathbb{R}_+, \mathbb{R}^n) \) with its Borel \( \sigma \)-algebra \( \mathcal{F} \) and canonical process \( (X_t)_{t \geq 0} \) with associated filtration \( (\mathcal{F}_t)_{t \geq 0} \). Remember that with \( \Pi(\mathcal{C}) \) we denote the probability measures on the path space \( \mathcal{C} \).

Consider the generator \( \mathcal{L} \) defined for any \( f \in C^2(\mathbb{R}^n) \) as

\[
\mathcal{L} f(x) = b(x) \cdot \nabla f(x) + \frac{1}{2} \text{Tr}(a \nabla^2 f(x)), \quad x \in \mathbb{R}^n,
\]

with measurable and bounded coefficients.

**Definition 1.** We say that \( \mathbb{P} \) on \( (\mathcal{C}, (\mathcal{F}_t)_{t \geq 0}) \) is a solution of the martingale problem for the generator \( \mathcal{L} \) iff for any \( f \in C^1,2(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}) \)

\[
M_t^f := f(t, X_t) - f(0, X_0) - \int_0^t (\partial_x + \mathcal{L}) f(s, X_s) \, ds
\]

is a \( \mathbb{P} \)-martingale wrt. \( (\mathcal{F}_t)_{t \geq 0} \).

We want to discuss the uniqueness of such solutions, meaning the following.

**Definition 2.** We say that the martingale problem (1) has unique solution if any two solutions \( \mathbb{P}, \mathbb{Q} \in \Pi(\mathcal{C}) \) of the martingale problem such that \( \text{Law}_{\mathbb{P}}(X_0) = \text{Law}_{\mathbb{Q}}(X_0) \) then \( \mathbb{P} = \mathbb{Q} \).

This notion corresponds directly with the uniqueness in law of the corresponding weak solutions. It is enough that \( \mathbb{P}, \mathbb{Q} \) coincide on finite dimensional distributions.

Let us observe that if \( \varphi \in C^1,2(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}) \) is a solution to the (parabolic) PDE (Kolmogorov backward equation)

\[
\partial_t \varphi(t, x) = \mathcal{L} \varphi(t, x), \quad t \geq 0, x \in \mathbb{R}^n,
\]

Note that \((\partial_s + \mathcal{L}) \varphi(r-s, X_s) = 0\) for any \( r > s \), therefore for any \( r > 0 \) and any \( t \in [0, r] \) the process

\[
M_t^r := \varphi(r-t, X_t) - \varphi(r, X_0) - \int_0^t (\partial_s + \mathcal{L}) \varphi(r-s, X_s) \, ds = \varphi(r-t, X_t) - \varphi(r, X_0)
\]

is a martingale under any solution \( \mathbb{P} \) of the martingale problem associated to \( \mathbb{P} \). Now \( M_t^r = \varphi(0, X_r) - \varphi(r, X_0) \) so

\[
0 = \mathbb{E}_\mathbb{P}[M_t^r - M_0^r] = \mathbb{E}_\mathbb{P}[\varphi(0, X_r) - \varphi(r, X_0)|\mathcal{F}_t]
\]

tells me that for any \( r \geq t \) we have

\[
\mathbb{E}_\mathbb{P}[\varphi(0, X_r)|\mathcal{F}_t] = \mathbb{E}_\mathbb{P}[\varphi(r-t, X_0)|\mathcal{F}_t] = \varphi(r-t, X_t), \quad \mathbb{P} - a.s.
\]

So the value of this expectation essentially do not depends on which solution of the martingale problem we get

\[
\mathbb{E}_\mathbb{P}[\varphi(0, X_t)] = \mathbb{E}_\mathbb{P}[\mathbb{E}_\mathbb{P}[\varphi(0, X_t)|\mathcal{F}_0]] = \mathbb{E}_\mathbb{P}[\varphi(r, X_0)]
\]
and if $Q$ is another solution with $\text{Law}_Q(X_0) = \text{Law}_P(X_0)$ then we conclude that

$$E_P[\varphi(0, X_r)] = E_Q[\varphi(0, X_r)]$$

for any $r \geq 0$. Let us assume know that the Kolmogorov backward equation has solution for any initial condition $\psi \in C^\infty_0(\mathbb{R}^n)$ (where the 0 means compactly supported). This implies that if we use such solutions in the argument above we get that for any $\psi \in C^\infty_0(\mathbb{R}^n)$ we have

$$E_P[\psi(X_r)] = E_Q[\psi(X_r)]$$

and this implies that

$$\text{Law}_P(X_r) = \text{Law}_Q(X_r)$$

for any $r \geq 0$. So we deduced that the one time marginals of $P$ and $Q$ coincide. Now let $\psi \in C^\infty_0(\mathbb{R}^n)$ and let $\varphi^\psi$ to be the solution of (2) such that $\varphi(0, x) = \psi(x)$ for all $x \in \mathbb{R}^n$ then as we already seen $E_P[\psi(X_r)|\mathcal{F}_t] = \varphi^\psi(r-t, X_r)$, therefore for any $r_1 > r_2 \geq 0$ we have for any bounded and measurable $g: \mathbb{R}^n \to \mathbb{R}^n$

$$E_P[\psi(X_r)g(X_r)] = E_P[E_P[\psi(X_r)|\mathcal{F}_{r_2}]g(X_r)] = E_P[\varphi^\psi(r_1-r_2, X_r)g(X_r)]$$

since $\psi$ and $g$ are arbitrary we conclude that

$$\text{Law}_P(X_{r_1}, X_{r_2}) = \text{Law}_Q(X_{r_1}, X_{r_2}).$$

We can continue by induction and deduce that $P, Q$ have the same finite dimensional marginals, and therefore are equal as probability measures on $\mathcal{C}$. (think about it). Moreover note that we also have for any $r > t$

$$E_P[\psi(X_r)|\mathcal{F}_t] = \varphi^\psi(r-t, X_t),$$

which implies that the process $(X_t)_{t \geq 0}$ under $P$ is a Markov process, indeed for any $t_1 < \cdots < t_n < r$ we have

$$E_P[\psi(X_r)g(X_{t_1}, \ldots, X_{t_n})] = E_P[E_P[\psi(X_r)|\mathcal{F}_{t_n}]g(X_{t_1}, \ldots, X_{t_n})] = E_P[\varphi^\psi(r-t_n, X_{t_n})g(X_{t_1}, \ldots, X_{t_n})]$$

but also

$$E_P[E_P[\psi(X_r)|X_{t_n}]g(X_{t_1}, \ldots, X_{t_n})] = E_P[\varphi^\psi(r-t_n, X_{t_n})g(X_{t_1}, \ldots, X_{t_n})]$$

from which we get

$$E_P[E_P[\psi(X_r)|X_{t_n}]g(X_{t_1}, \ldots, X_{t_n})] = E_P[\varphi^\psi(r-t_n, X_{t_n})g(X_{t_1}, \ldots, X_{t_n})]$$

and by a monotone class argument one deduce that

$$E[\psi(X_r)|X_{t_n}] = E[E[\psi(X_r)|X_{t_n}]|\mathcal{F}_{t_n}] = E[\psi(X_r)|\mathcal{F}_{t_n}]$$

for any $\psi \in C^\infty_0(\mathbb{R}^n)$ which approximates any continuous function and then also indicator functions of open sets from which we conclude that it is true for any $\psi$ which is bounded and measurable. This proves the Markov property of $(X_t)_{t \geq 0}$ under $P$.

**Theorem 3.** Assume that the Kolmogorov backward PDE

$$\partial_t \varphi(t, x) = \mathcal{L} \varphi(t, x), \quad \varphi(0, \cdot) = \psi$$

has a solution $\varphi \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$ for any $\psi \in C^\infty_0(\mathbb{R}^n)$ then the martingale problem associated to $\mathcal{L}$ in the sense of Definition 1 has a unique solution in the sense of Definition 2. (and as a consequence uniqueness of weak solutions to the associated SDE).
Remark 4. This reduces the uniqueness problem to a problem about existence of enough regular solutions to a PDE. Note that the set of initial conditions \( C_\Omega^0(\mathbb{R}^n) \) could be replaced by any set \( \mathcal{D} \) with the property that if two probability measures \( \mu, \nu \in \Pi(\mathbb{R}^n) \) satisfy

\[
\int_{\mathbb{R}^n} f(x) \mu(dx) = \int_{\mathbb{R}^n} f(x) \nu(dx), \quad f \in \mathcal{D}
\]

then \( \mu = \nu \), i.e. \( \mathcal{D} \) is a determining (or separating) class for \( \Pi(\mathbb{R}^n) \).

Remark 5. What about existence of solutions to the martingale problem.

a) (Construction of the weak solution SDE) maybe strong solutions via fixpoint arguments, or time-change, or Girsanov transformation (to be seen), Doob’s transform.

b) (Compactness arguments) Assume that we have a sequence of probabilities \( (P^n)_n \) on \( \mathcal{E} \) such that \( P^n \) solve the martingale problem wrt. \( \mathcal{L}^n \) (some generator). Assume also that we can show pointwise convergence of \( \mathcal{L}^n \) to a limiting generator \( \mathcal{L} \), in the sense that for any \( f \) “in a large class of functions” we have that \( \mathcal{L}^n f(x) = \mathcal{L} f(x) \) uniformly in \( x \in \mathbb{R}^n \). Assume also that the family \( (P^n)_n \) is tight on \( \mathcal{E} \), then one can show that any accumulation point of \( (P^n)_n \) wrt. to the weak topology of probability measures is a solution of the martingale problem for \( \mathcal{L} \).

c) (Markov process theory) If one can construct the semigroup \( (P_t)_{t \geq 0} \) in the space of continuous functions \( C(\mathbb{R}^n) \), associated to the operator \( \mathcal{L} \) in the sense that \( \partial_t P_t = \mathcal{L} P_t \) in the sense of Hille–Yoshida theory. Then one can construct a measure \( P \) using \( P_t \) to specify the finite dimensional distributions and then prove that it is a solution of the martingale problem. (this is stated here very vaguely).

Theorem 6. (Stroock–Varadhan) Assume \( b, \sigma \) is bounded measurable functions and \( a \) is bounded from below away from zero (in the sense of symmetric matrices) then there exists a solution to the martingale problem for \( \mathcal{L} \) and the martingale problem for \( \mathcal{L} \) has a unique solution.

The condition on \( a \) means that there exists \( \lambda > 0 \) such that \( (v, a(x)v)_{\mathbb{R}^n} \geq \lambda \|v\|^2_{\mathbb{R}^n} \) for any \( v \in \mathbb{R}^n \) and \( x \in \mathbb{R}^n \) (ellipticity condition).

There is no further regularity requirement on the coefficients, i.e. they can be discontinuous.