
Boué–Dupuis formula (continued)

We assume that \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) is the canonical \(d\)-dimensional Wiener space, i.e. \(\Omega = \mathbb{C}^d = C(\mathbb{R}_+, \mathbb{R}^d), X_t(\omega) = \omega(t)\), \(\mathbb{P}\) is the law of the Brownian motion and \((\mathcal{F}_t)_{t \geq 0}\) is the filtration generated by the canonical process \((X_t)_{t \geq 0}\) in particular we have \(\mathcal{F}_\infty = \mathcal{F} = \mathcal{B}(\Omega)\). We will also use the notation \(\mu\) for the Wiener measure \(\mathbb{P}\). Recall this lemma proven in the last lecture

**Lemma 1.** Let \(\nu\) be a probability measure which is absolutely continuous wrt. \(\mu\) with density \(Z\) such that \(Z \in \mathbb{C}^d\) (defined last week) and \(Z \geq \delta\) for some \(\delta > 0\). Let us call \(\mathcal{P}_\mu \subseteq \Pi(\Omega)\) the set of all such measures. Then under \(\nu \in \mathcal{P}_\mu\) the canonical process \(X\) is a strong solution of the SDE

\[
\frac{dX_t}{dt} = u_t(X)dt + dW_t, \quad t \geq 0
\]

where \(W\) is a \(\nu\)-Brownian motion and \(u\) a drift such that

\[
\|u_t(x) - u_t(y)\| \leq L\|x - y\|_{C([0,t];\mathbb{R}^d)} \quad x, y \in \Omega
\]

for some finite constant \(L\). Moreover

\[
H(\nu|\mu) = \frac{1}{2}\mathbb{E}_\nu \|u(X)\|_H^2.
\]

Recall that \(H = L^2(\mathbb{R}_+; \mathbb{R}^d)\).

We go on now to reconsider a last lemma before the actual proof.

Recall that

\[
\log \mu[e^f] = \sup_\nu \{\nu(f) - H(\nu|\mu)\}
\]

**Lemma 2.** Let \(f: \Omega \to \mathbb{R}\) which is measurable and bounded from below. Assume \(\mu(e^f) < \infty\). For every \(\epsilon > 0\) there exists \(\nu \in \mathcal{P}_\mu\) such that

\[
\log \mu[e^f] \leq \nu(f) - H(\nu|\mu) + \epsilon.
\]

If \(\mu(e^f) = +\infty\) then there exist a sequence \((\nu_n) \subseteq \mathcal{P}_\mu\) such that

\[
+\infty = \log \mu[e^f] = \sup_n \{(\nu_n(f) - H(\nu_n|\mu))\}.
\]

**Proof.** We start by assuming that \(\log \mu[e^f] < \infty\). By monotone convergence it is enough to consider only bounded functions \(f\) and moreover such that \(\mu(e^f) = 1\). Indeed if \(f\) is bounded below I can introduce \(f_n = (f \wedge n)\) which is now a bounded function for any \(n\) and if we prove the claim for bounded functions then we have that for any \(n\) and \(\epsilon > 0\) we have

\[
\log \mu[e^f] \leq (\nu_n(f_n) - H(\nu_n|\mu)) + \epsilon / 2
\]
for some $\nu_n$. But then we observe that $f_n \leq f$ so

$$\log \mu[e^f] \leq \nu_n(f) - H(\nu_n|\mu) + \varepsilon / 2.$$ 

Moreover by monotone convergence we have $\log \mu[e^f] \to \log \mu[e^f]$. Then there exist $n$ finite such that

$$\log \mu[e^f] \leq \log \mu[e^{f_n}] + \varepsilon / 2 \leq \nu_n(f) - H(\nu_n|\mu) + \varepsilon.$$

Note also that

$$\log \mu[e^{f-c}] - \nu(f-c) = \log \mu[e^f] - \nu(f)$$

so this shows that we can take $c$ such that $\log \mu[e^{f-c}] = 0$, namely we can assume that $f$ is such that $\mu[e^f] = 1$. Let $F = e^f$ and let $\nu$ be a probability measures on $\Omega$. Note that

$$x \log(x) \leq |x-1| + \frac{1}{2} |x-1|^2, \quad x \geq 0,$$

and using this we get

$$H(\nu|\mu) - \nu(f) = \int_{\Omega} \left( \log \left( \frac{d\nu}{d\mu}(\omega) \right) - f(\omega) \right) \nu(d\omega)$$

$$= \int_{\Omega} \left( \log \left( \frac{d\nu}{d\mu}(\omega) \right) - \log F(\omega) \right) \nu(d\omega) = \int_{\Omega} \left( \log \left( \frac{1}{F(\omega)} \frac{d\nu}{d\mu}(\omega) \right) \right) \nu(d\omega)$$

$$= \int_{\Omega} \left( \log \left( \frac{G(\omega)}{F(\omega)} \right) \right) \left( \frac{G(\omega)}{F(\omega)} \right) F(\omega) \mu(d\omega)$$

where $G = \frac{d\nu}{d\mu} \in \mathcal{C}$ since $\nu \in \mathcal{P}_\mu$. Using the inequality above we get

$$H(\nu|\mu) - \nu(f) \leq \int_{\Omega} \left[ \frac{G}{F} - 1 + \frac{1}{2} \left( \frac{G}{F} - 1 \right)^2 \right] F(\omega) \mu(d\omega) \leq \|F - G\|_{L^1(\mu)} + C_f \|F - G\|_{L^2(\mu)}^2$$

where the constant $C_f$ depends only on the lower bound on $f$. Moreover $\|F - G\|_{L^1(\mu)} \leq \|F - G\|_{L^2(\mu)}$. This proves that $H(\nu|\mu) - \nu(f)$ can be made as small as we want since $\mathcal{C}$ is dense in $L^2(\mu)$ and we can always find $G \in \mathcal{C}$ such that $G \geq \delta$ and $\|e^f - G\|_{L^1(\mu)} \leq \varepsilon$.

If $\log \mu[e^f] = +\infty$ the above argument allows to conclude the existence of the claimed sequence by using $f_n$ as lower bound of $f$. \hfill \Box
Now we are going to complete the proof of

**Theorem 3. (Boué–Dupuis formula)** For any function $f: \Omega \to \mathbb{R}$ measurable and bounded from below. We have

$$
\log \mathbb{E}_\mu[\mathbf{e}^f] = \sup_{u \in \mathbb{H}} \mathbb{E}_\mu\left[f(X + I(u(X))) - \frac{1}{2} \|u(X)\|^2_{\mathbb{H}}\right]
$$

where the supremum on the r.h.s. is taken wrt. all the predictable functions $u: \mathbb{R}_+ \times \Omega \to \mathbb{R}$ such that

$$
\|u\|^2_{\mathbb{H}} = \int_0^\infty |u_s|^2 \, ds < \infty, \quad \mu - a.s. \tag{2}
$$

and we write $u(\omega) = u(X(\omega))$ to stress the measurability wrt. the filtration $\mathcal{F}$ generated by $X$ and where

$$
I(u)(t) = \int_0^t u_s(X) \, ds, \quad t \geq 0.
$$

We call a function $u$ as above, a drift (wrt. $\mu$).

**Proof.** We are going to prove that we have $\leq$ with an arbitrarily small loss $\epsilon$ and then we have also the reverse inequality. Recall that we proved that if $u$ is a drift and $\nu$ is the law of $X + I(u)$ then we have

$$
H(\nu|\mu) \leq \frac{1}{2} \mathbb{E}_\mu[\|u(X)\|^2_{\mathbb{H}}]
$$

then using this measure $\nu$ in the variational characterisation of $\log \mathbb{E}_\mu[\mathbf{e}^f]$ we have

$$
\log \mathbb{E}_\mu[\mathbf{e}^f] = \sup_{\rho} (\rho(f) - H(\rho|\mu)) \geq \nu(f) - H(\nu|\mu)
$$

$$
\geq \nu(f) - \frac{1}{2} \mathbb{E}_\nu[\|u(X)\|^2_{\mathbb{H}}] = \mathbb{E}_\nu\left(f(X + I(u(X))) - \frac{1}{2} \|u(X)\|^2_{\mathbb{H}}\right)
$$

so we have one of the bounds because we can now optimize over all drifts $u$. In order to prove the reverse inequality we use the Lemma 2. Assume that $\log \mathbb{E}_\mu[\mathbf{e}^f] < \infty$. For any $\epsilon > 0$ there exists $\nu \in \mathcal{F}_\mu$ satisfying

$$
\log \mathbb{E}_\mu[\mathbf{e}^f] - \epsilon \leq \nu(f) - H(\nu|\mu)
$$

Now recall by Lemma 1 under $\nu$ the canonical process satisfies the SDE $dX = z(X) \, dt + dW$ for a “nice” drift $z$ (which is Lipshitz) and a process $W$ which is a Brownian motion under $\nu$. This SDE has a unique strong solution, so we can write $X = \Phi(W)$ with some adapted functional $\Phi$. Therefore we conclude that

$$
X = W + I(z(X)) = W + I(u(W))
$$

where we let $u(x) = z(\Phi(x))$ for all $x \in \Omega$. With this new expression we have that

$$
\nu(f) = \mathbb{E}_\nu(f(X)) = \mathbb{E}_\nu(f(W + I(z(X)))) = \mathbb{E}_\nu(f(W + I(u(W)))) = \mathbb{E}_\mu(f(X + I(u(X))))
$$

since Law$_\nu(W) = $ Law$_\mu(X)$. Moreover we have also (for similar reasons)

$$
H(\nu|\mu) = \frac{1}{2} \mathbb{E}_\nu\|z(X)\|^2_{\mathbb{H}} = \frac{1}{2} \mathbb{E}_\nu\|\Phi(W)\|^2_{\mathbb{H}} = \frac{1}{2} \mathbb{E}_\nu\|u(W)\|^2_{\mathbb{H}} = \frac{1}{2} \mathbb{E}_\mu\|u(X)\|^2_{\mathbb{H}}.
$$

Therefore putting pieces together we have

$$
\log \mathbb{E}_\mu[\mathbf{e}^f] - \epsilon \leq \nu(f) - H(\nu|\mu) = \mathbb{E}_\mu(f(X + I(u(X)))) - \frac{1}{2} \mathbb{E}_\mu\|u(X)\|^2_{\mathbb{H}}.
$$

So, for any $\epsilon > 0$ we have found a particular drift $u$ such that

$$
\log \mathbb{E}_\mu[\mathbf{e}^f] \leq \mathbb{E}_\mu(f(X + I(u(X)))) - \frac{1}{2} \mathbb{E}_\mu\|u(X)\|^2_{\mathbb{H}} + \epsilon.
$$
While if \( \log \mathbb{E}_\mu[e^f] = +\infty \) then by the same lemma one has that there exists a sequence of drifts \((u_n)_{n \geq 1}\) such that

\[
+\infty = \log \mathbb{E}_\mu[e^f] = \sup_n \left[ \mathbb{E}_\mu(f(X + I(u_n(X)))) - \frac{1}{2} \mathbb{E}_\mu\|u_n(X)\|_H^2 \right].
\]

In both cases putting together the two inequalities we conclude that

\[
\log \mathbb{E}_\mu[e^f] = \sup_u \mathbb{E}_\mu(f(X + I(u(X)))) - \frac{1}{2} \mathbb{E}_\mu\|u(X)\|_H^2
\]

which is our claim. \(\square\)

**Applications to functional analysis**

This formula and similar formulas can be used (amazingly) to prove functional inequalities for finite dimensional measures, see for example


We will not look into these, but they are very interesting.

**Applications to probabilistic problems**

Gaussian bounds on functional of Brownian motion.

**Theorem 4.** Let \((E,d)\) a metric space and \(f: \Omega \to E\) such that there an \(e \in E\) for which

\[
d(f(x + I(h)), e) \leq c(x)(g(x) + \|h\|_H), \quad h \in H,
\]

for \(\mu\)-almost every \(x \in \Omega\) where \(\mu(g) < \infty\) and \(\mu(\xi^2) < \infty\). Then for all \(\lambda > 0\) we have

\[
\mathbb{E}_\mu[e^{\lambda d(f(X),e)}] \leq e^{\lambda \mu(\xi^2) + \lambda \mu(g)}.
\]

In particular the r.v. \(d(f(X),e)\) has Gaussian tails, i.e.

\[
\mathbb{P}_\mu(d(f(X),e) > k) \leq C_1 e^{-C_2 k^2}
\]

for some \(C_1, C_2 > 0\).

**Remark 5.** Note that if we let \(y = x + I(h)\) then \(y(t) = x(t) + \int_0^t h(s)ds\). Note that the natural norm on \(y\) is given by the sup norm, i.e.

\[
\|y\|_{C^1([0,1],\mathbb{R}^d)} = \sup_{t \in [0,1]} \|x(t) + \int_0^t h(s)ds\|
\]
but on the r.h.s. of the inequality you have to control the $L^2$ norm of $h$ which corresponds to the $H^1$ norm of $I(h)$, i.e.

$$\|h\|_{H^1} = \|I(h)\|_{H^1[\mathbb{R}, \mathbb{R}^d]} = \left\| \frac{d}{dt} I(h) \right\|_{L^2(\mathbb{R}^d)}.$$

This is coherent with the fact that increments of Brownian motion are independent so formally the Wiener measure can be understood as given by

$$\mu(d\omega) \propto \exp \left( -\frac{1}{2} \int_0^\infty |\dot{\omega}(s)|^2 ds \right) d\omega.$$

**Proof.** By Boué–Dupuis formula and the hypothesis on $f$

$$\log \mathbb{E}_\mu [e^{\lambda d f(X),\omega}] = \sup_u \mathbb{E}_\mu \left[ \lambda d (f(X+I(u)), e) - \frac{1}{2} |u|_2^2 \right]$$

$$= \sup_u \mathbb{E}_\mu \left[ \lambda c(X)(g(X) + |u|_2) - \frac{1}{2} |u|_2^2 \right]$$

We observe now that the polynomial $\lambda c(X)(g(X)+t) - \frac{1}{2} t^2$ is upperbounded by

$$\lambda c(X)g(X) + \lambda c(X)t - \frac{1}{2} t^2 \leq \lambda c(X)g(X) + \frac{1}{2} \lambda^2 c(X)^2 - \frac{1}{2} \left( t - \lambda c(X) \right)^2 \leq \lambda c(X)g(X) + \frac{1}{2} \lambda^2 c(X)^2$$

therefore

$$\log \mathbb{E}_\mu [e^{\lambda d f(X),\omega}] \leq \sup_u \mathbb{E}_\mu \left[ \lambda c(X)g(X) + \frac{1}{2} \lambda^2 c(X)^2 \right] = \mathbb{E}_\mu \left[ \lambda c(X)g(X) + \frac{1}{2} \lambda^2 c(X)^2 \right]$$

$$= \lambda \mu (c g) + \frac{1}{2} \lambda^2 \mu (c^2).$$

Exercise 1. Take

$$f(t) = \sup_{t \in [0,1]} \frac{|x(t) - x(s)|}{|t-s|^a}$$

and prove that is satisfies the hypothesis of the previous theorem. Conclude that

$$\mathbb{E}_\mu \left[ \exp \left( \lambda \sup_{t \in [0,1]} \frac{|X(t) - X(s)|}{|t-s|^a} \right) \right] \leq e^{C\lambda^2 + C\lambda^a}$$

for any $a \in (0, 1/2)$ any $\lambda > 0$. From this you can also conclude that

$$\mathbb{E}_\mu \left[ \exp \left( \rho \sup_{t \in [0,1]} \frac{|X(t) - X(s)|^2}{|t-s|^a} \right) \right] < \infty$$

for some $\rho > 0$ small.

Thursday: we continue with applications and with large deviations.