SDE techniques: Doob's transform

Let \((X_t, B_t)_{t \geq 0}\) be the solution of an SDE with Markovian drift \(b: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n\) and diffusion coefficient \(\sigma: \mathbb{R}_+ \times \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)\) where \(B\) is the Brownian motion driving the SDE. Let \(h \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_{>0})\) be a strictly positive function such that
\[
(\partial_t + \mathcal{L})h(t, x) = 0,
\]
for all \(t \in [0, t_\ast]\) and \(x \in \mathbb{R}^n\) where \(\mathcal{L}\) is the generator of the SDE, i.e. \(\mathcal{L} = b \cdot \nabla + \frac{1}{2} \text{Tr} [\sigma \sigma^T] \).

By Itô formula the process \(Z_t := h(t, X_t)\) is a positive local martingale. Let us assume that \((Z_t)_{t \in [0, t_\ast]}\) is a (true) martingale and that \(Z_0 = h(0, X_0) = 1\) (this can be always arranged by normalizing \(h\)). Then we can use the process \((Z_t)\) to define a new measure
\[
d\mathbb{Q} := Z_t d\mathbb{P}.
\]
(If needed we can extend \(Z_t = Z_\tau\) if \(t > t_\ast\).) Note that by construction the process \(Z\) is continuous and \(Z_0 = 1\).

By using Girsanov's theorem we know that the process
\[
\tilde{B} = B - [B, L]
\]
is a \(\mathbb{Q}\)-Brownian motion where \(L\) is the only local martingale such that \(Z = \mathcal{F}(L)\). Since \(dZ_t = Z_t dB_t\) we have that
\[
dZ_t = \sigma(t, X_t) \nabla h(t, X_t) \cdot dB_t, \quad dL_t = Z_t^{-1} dZ_t = \frac{\sigma^T(t, X_t) \nabla h(t, X_t)}{h(t, X_t)} dB_t, \quad dB_t = \sigma^T(t, X_t) \nabla \log h(t, X_t) \cdot dB_t
\]
for \(t \leq t_\ast\) and \(dZ_t = 0\) if \(t > t_\ast\). Therefore
\[
d\tilde{B}_t = dB_t - \sigma^T(t, X_t) \nabla \log h(t, X_t) dt, \quad t \in [0, t_\ast],
\]
and \(d\tilde{B}_t = dB_t\) if \(t > t_\ast\). As consequence the process \(X\) solves now a new SDE (under \(\mathbb{Q}\))
\[
dx_t = \left[ b(t, X_t) + \sigma(t, X_t) \sigma^T(t, X_t) \nabla \log h(t, X_t) \right] dt + \sigma(t, X_t) d\tilde{B}_t, \quad t \in [0, t_\ast]
\]
with the same diffusion coefficient \(\sigma\) but a new drift
\[
\tilde{b}(t, x) = b(t, x) + \mathbb{E}_{t \in [0, t_\ast]}(\sigma \sigma^T \nabla \log h)(t, x), \quad t \geq 0, x \in \mathbb{R}^n.
\]
This construction is called Doob's \(h\)-transform.

**Exercise 1.** Try to perform the same construction for a martingale problem, i.e. not relying on the process \(B\) but only on \(X\). I.e. starting from a measure \(\mathbb{P}\) on the canonical path space \(C(\mathbb{R}_+, \mathbb{R}^n)\) solving the martingale problem for \(\mathcal{L}\) construct a new measure \(\mathbb{Q}\) which solves a new martingale problem with a modified drift as above.

**Example 1.** Take \(h(t, x) = \exp \left\{ y \cdot x - \frac{1}{2} |y|^2 \right\}\) where \(y \in \mathbb{R}^n\) and \(t \geq 0\). Then the Doob's \(h\)-transformed process of a Brownian motion with this function gives a Brownian motion with drift.

If \((Z_t)\) is only a martingale in an open interval \(I = [0, t_\ast)\) with possibly \(t_\ast = +\infty\). Then we can still define \(\mathbb{Q}\) on \(\mathcal{F}_t\) to be given by \(d\mathbb{Q}|_{\mathcal{F}_t} := Z_t d\mathbb{P}|_{\mathcal{F}_t}\) and check that this gives a well-defined probability measure on \(\mathcal{F}_\infty = \bigvee_{\tau \geq 0} \mathcal{F}_\tau\). In this case is natural to restrict all the measures to \(\mathcal{F}_\infty\) i.e. to require \(\mathcal{F}_\infty = \mathcal{F}\).
Remark 2. We do not need to require that $h$ is positive everywhere (actually this will not be the case in the applications). What we need is that the process $Z_t^i = h(t, X_t)$ is a local martingale, i.e. $(\partial_t + \mathcal{F})h(t, X_t) = 0$ a.s. and for almost every $t \geq 0$ and that $Z_t > 0$ almost surely. If $h$ is not strictly positive we can always define the stopping time $T = \inf\{t \geq 0 : Z_t = 0\}$, then the stopped process $(Z_t^i)_{t \geq 0}$ is a positive local martingale and some condition is needed to ensure that it is a martingale. Remember that we require that $Z_0 = 1$ and by construction $(Z_t^i)_{t \geq 0}$ is continuous. In this setting one can perform the Doob’s transform up to the random stopping time $T$. Note that under the measure $\mathbb{Q}$ we always have $T = +\infty$ almost surely.

1 Diffusion bridges

We use now Doob’s transform to describe the regular conditional law of a Markovian diffusion $(X_t)_{t \geq 0}$ conditioned on the event that $X_T = y$ with $T > 0$ and deterministic, and $y \in \mathbb{R}^n$. I will assume also that $X_0 = x_0$. We need to assume that the process $(X_t)_{t \geq 0}$ is a Markov process with transition density given by

$$P(X_t \in dx^i | X_s = x) = p(s, x; t, x^i)dx^i, \quad s < t \in [0, T], x, x^i \in \mathbb{R}^n,$$

for some measurable and positive function $p$. Note that we cannot take $s = t$ here. Recall that $P(X_t \in dy|X_s = x)$ means the regular conditional probability kernel for the conditional law of $X_t$ given $X_s$.

Define now the function

$$h^i(s, x) \equiv \frac{p(s, x; T, y)}{p(0, x_0; T, y)}, \quad s \in [0, T), x \in \mathbb{R}^n.$$

Let $Z_t^i = h^i(t, X_t)$, this is non-negative process, and it is also a martingale, indeed by the Markov property of $X$

$$E[Z_t^i | \mathcal{F}_s] = E[h^i(t, X_t) | \mathcal{F}_s] = E[h^i(t, X_t) | X_s] = \int_{\mathbb{R}^n} h^i(t, x') p(s, x^i; t, x') dx^i$$

$$= \int_{\mathbb{R}^n} p(s, X_s^i; t, x') p(t, x', T, y) dx' = \int_{\mathbb{R}^n} p(s, X_s^i; T, y)$$

by Chapman–Kolmogorov equations (the consistency condition for the transition density of a Markov process).

We want to define a probability kernel $(Q^i)_{x \in \mathbb{R}^n}$ on $(\Omega, \mathcal{F})$ such that they are the regular conditional probability of $P$ given $X_T$, that is they have to satisfy

$$P(A) = E[P(A | X_T)] = E[Q^i(A)] = \int_{\mathbb{R}^n} Q^i(A) P(X_T \in dy) = \int_{\mathbb{R}^n} Q^i(A) p(0, x_0; T, y) dy$$

for all $A \in \mathcal{F}$. Take $A \in \mathcal{F}_t$ for some $s < T$, by Markov property we have for any bounded measurable $g$,

$$E[1_A g(X_T)] = E[1_A E[g(X_T) | \mathcal{F}_s]] = E[1_A E[g(X_T) | X_s]] = E[1_A \int_{\mathbb{R}^n} g(y) p(s, x^i; T, y) dy]$$

$$= \int_{\mathbb{R}^n} g(y) E[1_A p(s, x^i; T, y)] dy$$

since

$$E[g(X_T) | X_s] = \int_{\mathbb{R}^n} g(y) p(s, x^i; T, y) dy.$$

This means that we have $P(A | X_T) = q(A)$ and we can take

$$q(y) = E \left[ 1_A \frac{p(s, x^i; T, y)}{p(0, x_0; T, y)} \right]$$

since we have proven that

$$E[q(X_T) g(X_T)] = E[1_A g(X_T)] = \int_{\mathbb{R}^n} g(y) q(y) p(0, x_0; T, y) dy.$$
As a consequence we can take

\[ Q^y(A) := E \left[ \mathbf{1}_A \frac{p(s,X_t; T,y)}{p(0,x_0; T,y)} \right], \quad A \in \mathcal{F}_t \]

and have that \( y \mapsto Q^y \) identify a well-defined probability kernel on \( \mathcal{F}_T \) since for any \( A \in \mathcal{F}_T \) the function \( y \mapsto Q^y(A) \) is measurable in \( y \) and for any \( y \), \( Q^y \) is a probability in \( A \).

**Remark 3.** Is it possible with some care to extend \( Q^y \) to the full \( \mathcal{F} \), but we refrain to do so here.

We have now the formula

\[ P(A|X_T) = Q^{X_T}(A), \quad A \in \mathcal{F}_T. \]

I want now to describe better the measure \( Q^y \) (at least up to time \( T \)), we observe that \( Q^y \) is obtained as the Doob's \( h \)-transform of \( P \) in the interval \([0, T)\) with \( h = h^y \) function

\[ h^y(s,x) := \frac{p(s,x; T,y)}{p(0,x_0; T,y)}, \quad s \in [0, T), x \in \mathbb{R}^n. \]

As a consequence we can show that the process \( X \) under \( Q^y \) satisfies an SDE provided I can apply Itô formula to \( h^y \), that is I have to require that \((s,x) \mapsto p(s,x; T,y)\) is \( C^{1,2}([0, T) \times \mathbb{R}^n) \). Given that Doob's transform give that \( X \) under \( Q^y \) solves the new SDE (or an equivalent martingale problem)

\[ dX_t = b(t,X_t)dt + \sigma \sigma^T \nabla \log h^y(t,X_t)dt + \sigma(t,X_t)dB_t, \quad t \in [0, T). \]

Is easy to see from specific examples that the function \( \sigma \sigma^T \nabla \log h^y(t,x) \) is singular when \( t \uparrow T \).

**Exercise 2.** Compute the SDE satisfied by a \( n \)-dimensional Brownian motion when we condition it to reach the point \( y \) at time \( T > 0 \).

Observe that under \( Q^y \) we have that

\[ Q^y \left( \lim_{t \uparrow T} X_t = z \right) = \mathbf{1}_{z=y}, \]

for any \( y, z \in \mathbb{R}^n \). Observe also that

\[ P \left( \lim_{t \uparrow T} X_t = y \right) = P(X_T = y) = 0 \]

since \( X_T \) has density \( p(0,x_0; T, \cdot ) \). So the measures \( Q^y \) are all singular wrt. \( P \).

Next week: more complex conditionings, e.g. diffusion conditioned never to leave a given region.