
Tutorial classes: Daria Frolova (Wed 16-18, SemR 1.007), Min Liu (Monday 16-18, SemR 1.007) Online until further notice.

Handling of sheets via eCampus. Upload the sheet when you completed it, and download there the corrections. Sheet must be handled in LATEX.

Subscribe also to the Exercise in Stochastic Analysis in eCampus

Prerequisites

“Foundations/Introduction on Stochastic Analysis”. Probability measures, continuous time stochastic processes, Kolmogorov's construction of stoch. proc., continuous time martingales, stochastic integration, Itô formula, SDE. Give a look at

https://www.iam.uni-bonn.de/abteilung-gubinelli/teaching/found-stoch-analy-sis-ws1920/

Introduction

Stochastic Analysis: set of tools to study stochastic (continuous) processes (i.e. Brownian motion, semimartingales, solutions to SDE, random fields).

Wiener ‘40 (Brown. mot., Lebesgue's theory)
Doob's/Levy/Ito ('40-'50) / Kunita/Watanabe/McKean/Malliavin/…
Malliavin derivative / White-noise calculus
Generalisation of analysis adapted to the study of stoch. proc.

Content of the course

- **Stoch. Diff. equations**: weak, strong, martingale problems. Links between the various notions. Including questions of uniqueness of solutions (pathwise uniq, weak uniq, uniq. of mart. problem).

  \[
  dX_t = b(X_t)dt + \sigma (X_t)dB_t \\
  X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma (X_s)dB_s
  \]
• **Techniques for SDEs.** time-change \((X_t = Y_t(t))\), Girsanov's theorem \((Q \ll P, (\mathcal{F}_t)_t)\), Tanaka's formula, conditioning (Doob's \(h\)-transform), singular conditioning (cond. on events of prob. zero). Doss–Sussmann technique (exact solutions to SDEs, link with control theory and ODE theory). Relation with PDE theory.

• **Martingale representation theorem.** (every mart. on a Brownian filtration is a stoch. integral). The formula of Boué–Dupuis (90) - gives a variational formula for expectation values over a Brownian filtration. Large deviations for SDE:

\[
dX^\varepsilon_t = b(X^\varepsilon_t)dt + \varepsilon \sigma(X^\varepsilon_t)dB_t
\]

\(\varepsilon > 0\) small. \((X^\varepsilon)_{\varepsilon > 0}\) What happens for \(\mu^\varepsilon(A) := \mathbb{P}(X^\varepsilon \in A)\) as \(\varepsilon \to 0\). \(\mu^\varepsilon \to \delta_{\text{ODE}}\). How fast is a question for large deviations theory.

\[
\mu^\varepsilon(A) = \exp \left( \frac{I(A)}{\varepsilon^2} \right), \quad I(A) = \inf_{f \in A} I(f).
\]

• **Diffusions on manifolds.** \((X_t \in \mathcal{M})_{t \geq 0}\) SDE?? Brownian motion on \(\mathcal{M}\), relation with differential geometry. \(\Delta\) Laplace–Beltrami.

• **Numerical methods for SDE.** \((X^n_t)_{t \geq 0}\) Euler-Maruyama method. Strong, weak approximations. As \(n \to \infty\),

\[
\mathbb{E}(f(X_t)) \approx \mathbb{E}(f(X^n_t)), \quad \mathbb{E}\|X_t - X^n_t\| \approx 0.
\]

Stochastic Taylor expansion (iterated stochastic integrals)

\[
f(B_t) = f(B_s) + f'(B_s)(B_t - B_s) + f''(B_s) \int_s^t \left( \int_u^s dB_v \right) dB_u + \cdots
\]

• **Rough path theory** (?). (robust and path-wise integration theory for irregular processes) (T. Lyons '98)

• **Malliavin calculus** (?). (P. Malliavin '80) Analysis on infinite dimensional measure spaces. Wiener measure \(\mathcal{W}(A) = \mathbb{P}(B \in A) \ A \in \mathcal{B}(C([0, 1]; \mathbb{R}))\). \(\mathcal{W}\) is a probability measure on \(C([0, 1]; \mathbb{R})\). Wiener measure is a replacement for Lebesgue measure in \(C([0, 1]; \mathbb{R})\). Quasi-invariant under shift. Lebesgue/Sobolev type spaces on \(C([0, 1]; \mathbb{R})\). Notion of derivative: Malliavin derivative. Link to the martingale rep. theorem and to iterated stochastic integrals.

### 1 Stochastic differential equations

Setting. Probability space \((\Omega, \mathcal{F}, \mathbb{P})\), filtration \((\mathcal{F}_t)_{t \geq 0}\) right-continuous, completed.
Definition 1. A weak solution of the SDE in $\mathbb{R}^n$

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad t \in [0, T]$$

$$X_0 = x \in \mathbb{R}^n$$

is a pair of adapted processes $(X, B)$ where $(B_t)_{t \geq 0}$ is a $m$-dimensional Brownian motion and $b, \sigma$ are coefficients $b: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\sigma: \mathbb{R}^n \rightarrow \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$ such that almost surely

$$\int_0^t |b(X_s)| ds < \infty, \quad \int_0^t \text{Tr}(\sigma(X_s)\sigma(X_s)^T) ds < \infty, \quad t \in [0, T]$$

and that

$$X_t = x + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dB_s, \quad t \in [0, T].$$

$\sigma = (\sigma_a)_{a=1, \ldots, m}$ family of vector-fields $\sigma_a: \mathbb{R}^n \rightarrow \mathbb{R}^n$ (this is the right point of view on manifolds)

Control-theory point of view:

$$dX_t = b(X_t)dt + \sum_{a=1}^m \sigma_a(X_t)dB_t^a.$$  

$$\sum_{a=1}^m \int_0^t |\sigma_a(X_s)|^2 ds < \infty.$$

Definition 2. A strong solution to the SDE above is a weak solution such that $X$ is adapted to the filtration $(\mathcal{F}_t^B)_{t \geq 0}$ generated by $B$, $\mathcal{F}_t^B = \sigma(B_s; s \in [0, t])$.

$$X_t \in \mathcal{F}_t \Rightarrow X_t(\omega) = \Phi_t((B_s(\omega))_{s \in [0,t]}$$

$\Phi_t: C([0, t]; \mathbb{R}^m) \rightarrow \mathbb{R}^n$. While in general we could have

$$X_t(\omega) = \Phi_t((B_s(\omega))_{s \in [0,t]}, N(\omega)).$$

Facts.

- There are weak solutions which are not strong. (Tanaka’s example)
- There are SDEs which do not have strong solutions.
- A weak solution is really the data $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0}, X, B)$.

Definition 3. An SDE has uniqueness in law iff two solutions $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0}, X, B)$ $(\Omega', \mathcal{F}', \mathbb{P}', (\mathcal{F}'_t)_{t \geq 0}, X', B')$ are such that

$$\text{Law}_{\mathbb{P}}(X) = \text{Law}_{\mathbb{P}'}(X').$$
Definition 4. An SDE has pathwise uniqueness if for any two solutions $X, X'$ defined on the same filt. prob. space and with the same BM $B$ we have that they are indistinguishable, i.e.

$$\mathbb{P} (\exists t \in [0, T]: X_t \neq X'_t) = 0.$$