Stochastic Analysis – Problem Sheet 9.

Tutorial classes: Mon July 11th in SemR 0.008. Philipp Boos <s6phboos@uni-bonn.de>. Solutions will be collected at the beginning of the tutorial session. At most in groups of 3.

**Exercise 1.** Let \((B_t)_{t \in [0,1]}\) be a standard Brownian motion. Note that if \(\lambda < 1/2\) we have

\[
\mathbb{E} \int_{[0,1]^2} dt \, ds \, e^{\lambda |B_t - B_s|^2 / |t-s|} < +\infty.
\]

Use this observation and the Garsia–Rodemich–Rumsey inequality with \(\Psi(x) = e^{\lambda x^2} - 1\) to derive an almost sure modulus of continuity for Brownian motion.

**Exercise 2.** Prove (the upper bound of) Burkholder–Davis–Gundy inequality. Let \(M\) be a continuous local martingale (with \(M_0 = 0\)). For any \(p \geq 2\) we have

\[
\mathbb{E} \left[ \sup_{t \leq T} |M_t|^p \right] \leq C_p \mathbb{E} \left[ \left( [M]_T^{p/2} \right)^p \right]
\]

where \(C_p\) is a universal constant depending only on \(p\).

a) Assume that the martingale \(M\) is bounded. Use Itô formula on \(t \mapsto (\varepsilon + |M_t|^2)^{p/2} \) to prove that

\[
\mathbb{E} \left[ \sup_{t \leq T} |M_t|^p \right] \leq \int_0^T \mathbb{E} [ |M_t|^{p-2} d[M]_t ].
\]

(why we need \(\varepsilon > 0\)?)

b) Use Hölder’s and Doob’s inequality to conclude.

c) Remove the assumption of boundedness.

**Exercise 3.** Let us continue with the setting of Exercise 1 and prove now a complementary lower bound when \(p \geq 4\), that is

\[
\mathbb{E} \left[ \left( [M]_T^{p/2} \right)^p \right] \leq C_p \mathbb{E} \left[ \sup_{t \leq T} |M_t|^p \right].
\]

where again \(C_p\) is a universal constant depending only on \(p\) (not the same as that of the upper bound).

a) Use the relation

\[
[M]_T = M_T^2 - 2 \int_0^T M_s dM_s
\]

to estimate \(\mathbb{E} \left[ \left( [M]_T^{p/2} \right)^p \right] \) and then use the BDG upper bound for the stochastic integral.

b) Prove that if we let \(N_T = \int_0^T M_s dM_s\) then for any \(\varepsilon > 0\) there exists \(\lambda_\varepsilon > 0\) such that

\[
[N]_T^{1/2} \leq \lambda_\varepsilon \sup_{t \leq T} |M_t| + \varepsilon [M]_T
\]
Exercise 4. Prove a substitution lemma for stochastic integrals. Let us be given a standard basis $(\Omega, \mathcal{F}, \mathcal{F}_*, \mathbb{P})$ and let $Y$ be an $\mathcal{F}_0$ measurable r.v. with values in $\mathbb{R}^d$ and $f: \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$ a stochastic process such that for any $x \in \mathbb{R}^d$ the process $s \mapsto f_s(x)$ is predictable, $\int_0^T |f_s(x)|^2 d[M]_s < \infty$ a.s. and $(s, x) \mapsto f_s(x)$ is continuous. Let $M$ be a continuous local martingale. Then

$$\left[ \int_0^T f_s(x) dM_s \right]_{x=Y} = \int_0^T f_s(Y) dM_s.$$ 

a) Consider a partition $\Delta$ of [0, T] and let $f^\Delta(x)$ be piecewise linear approximations to $f(x)$. Prove that for any compact $K \subset \mathbb{R}^d$, $\int_0^T \sup_{x \in K} |f_s(x) - f^\Delta_s(x)|^2 d[M]_s \rightarrow 0$ a.s. as the size of the partition $|\Delta| \rightarrow 0$.

b) Deduce that there exists a sequence of partitions $(\Delta_n)_{n \geq 1}$ such that $|\Delta_n| \rightarrow 0$ and that, letting $J^n(x) := \int_0^T f^\Delta_s(x) dM_s$ and $J(x) := \int_0^T f_s(x) dM_s$ we have for any compact $K$

$$\sup_{x \in K} |J^n(x) - J(x)| \rightarrow 0 \quad \text{a.s.}$$

c) Prove that the substitution formula is true for $f^\Delta$ and conclude.

Exercise 5. (Bonus) Prove a Fubini theorem for stochastic integrals. Let $(\Lambda, \mathcal{A})$ be a measure space and $(\Omega, \mathcal{F}, \mathcal{F}_*, \mathbb{P})$ a filtered probability space.

a) Let $(X_n)_{n}$ a sequence of functions $X_n: \Omega \times \Lambda \rightarrow \mathbb{R}$ which are $\mathcal{F} \otimes \mathcal{A}$ measurable (product $\sigma$-field) and such that $(X_n(\cdot, \lambda))_n$ converges in probability for any fixed $\lambda \in \Lambda$. Prove that there exists an $\mathcal{F} \otimes \mathcal{A}$ measurable function $X: \Omega \times \Lambda \rightarrow \mathbb{R}$ for which $X_n(\cdot, \lambda) \xrightarrow{\mathbb{P}} X(\cdot, \lambda)$ for any $\lambda \in \Lambda$. [Hint: here the difficulty is the measurability of the limit $X$, consider the sequence $n_k(\lambda)$ defined by $n_0(\lambda) = 1$ and

$$n_{k+1}(\lambda) = \inf \left\{ m > n_k(\lambda): \sup_{p, q \geq m} \mathbb{P}(|X_p(\cdot, \lambda) - X_q(\cdot, \lambda)| > 2^{-k}) \leq 2^{-k} \right\}$$

and prove that $\lim_k X_{n_k(\cdot, \lambda)}$ exists a.s. and conclude]

b) Let $H: \Lambda \times \mathbb{R}_{\geq 0} \times \Omega \rightarrow \mathbb{R}$ be a bounded function which is measurable w.r.t. $\mathcal{A} \otimes \mathcal{P}$ where $\mathcal{P}$ is the predictable $\sigma$-field on $\mathbb{R}_{\geq 0} \times \Omega$. Let $M$ be a continuous martingale on $(\Omega, \mathcal{F}, \mathcal{F}_*, \mathbb{P})$. Prove that there exists a function $J: \Lambda \times \Omega \rightarrow \mathbb{R}$ measurable for $\mathcal{A} \otimes \mathcal{F}_T$ which is a version of the stochastic process $\lambda \mapsto J(\lambda) := \int_0^T H(\lambda, s) dM_s$ and for which it holds

$$\int_{\Lambda} J(\lambda) m(d\lambda) = \int_0^T \int_{\Lambda} H(\lambda, s, \cdot) m(d\lambda) dM_s, \quad \text{a.s.}$$

for any bounded measure $m$ on $(\Lambda, \mathcal{A})$. Hint: prove that

$$\mathbb{E} \left[ \left( \int_0^T \int_{\Lambda} H(\lambda, s, \cdot) m(d\lambda) dM_s - \int_{\Lambda} J(\lambda) m(d\lambda) \right)^2 \right] = 0.$$