Stochastic Analysis – Problem Sheet 7.

Tutorial classes: Mon June 13th in SemR 0.008. Philipp Boos <s6phboos@uni-bonn.de>. Solutions will be collected at the beginning of the tutorial session. At most in groups of 3.

Let \( (\Omega := C(\mathbb{R}_{\geq 0}; \mathbb{R}), \mathcal{F}, \mathcal{F}_*, \mathbb{P}) \) the one dimensional Wiener space and \( X \) the canonical process.

**Exercise 1.** Find a predictable process \( F \) such that
\[
\Phi = \mathbb{E}[\Phi] + \int_0^\infty F_s dX_s
\]
when \( \Phi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}) \) is each of the following r.v. (with \( T > 0 \) fixed)
\[
X_T^2, \quad e^{X_T}, \quad \int_0^T X_t dt, \quad X_T^3, \quad \sin(X_T).
\]

**Exercise 2.** We want to prove that the linear span of r.v. of the form
\[
E(h) = \cos \left( \int h_s dX_s \right) \exp \left( \frac{1}{2} \int h_s^2 ds \right), \quad F(h) = \sin \left( \int h_s dX_s \right) \exp \left( \frac{1}{2} \int h_s^2 ds \right), \quad h \in L^2(\mathbb{R}_{\geq 0}),
\]
is dense in \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \) (\( h \) a deterministic function and the integrals are over \( \mathbb{R}_{\geq 0} \)).

a) Show that if \( G \in L^2(\Omega, \mathcal{F}, \mathbb{P}) \) is orthogonal to all \( \{ E(h), F(h) : h \in L^2(\mathbb{R}) \} \), then in particular
\[
\mathbb{E}[G \exp(i \lambda_1 B_{t_1} + \cdots + i \lambda_n B_{t_n})] = 0
\]
for all \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \) and \( t_1, \ldots, t_n > 0 \).

b) Deduce from this that \( G \) is orthogonal to all functions of the form \( \phi(B_{t_1}, \ldots, B_{t_n}) \) with \( \phi \in C_0^\infty \).
[Hint: use Fourier transform]

c) Conclude.

**Exercise 3.** Use the class of functions introduced in Exercise 2 to reprove the Brownian martingale representation theorem.

a) Determine the martingale representation for functions \( \Phi \) of the form
\[
\Phi = \sum_i (a_i E(h_i) + b_i F(h_i))
\]
where \( a_i, b_i \in \mathbb{R}, h_i \in L^2(\mathbb{R}_{\geq 0}) \) and the sum is finite.

b) Use the density of such functions to approximate an arbitrary element \( \Phi \in L^2 \) and conclude.