Stochastic Analysis – Problem Sheet 5.

Exercise 1. Let $M$ be a positive continuous supermartingale such that $\mathbb{E}[M_0] < \infty$. Let $M_\infty = \lim_{t \to \infty} M_t$ (assumed to exist $\mathbb{P}$-a.s.). Show that if $\mathbb{E}[M_\infty] = \mathbb{E}[M_0]$ then $M$ is a martingale and $\mathbb{E}[M_\infty | \mathcal{F}_t] = M_t$. [Hint: prove that $\mathbb{E}[M_\infty | \mathcal{F}_t] \leq M_t$ and that $\mathbb{E}[M_t] = \mathbb{E}[M_0]$ and conclude.]

Exercise 2. Assume that $\Omega = C(R_\geq 0; \mathbb{R}^d)$, $\mathbb{P}$ is the $d$-dimensional Wiener measure and that $X$ is the canonical process on $\Omega$ and that the filtration $\mathcal{F}_\bullet$ is generated by $X$. Consider a predictable $\mathbb{R}^d$-valued drift $b$ given by a function $b: \mathbb{R}_\geq 0 \times \Omega \to \mathbb{R}^d$. By tilting $\mathbb{P}$ via $Z = \mathcal{E}(\int_0^\cdot b(X) \, dX)$ we obtain that, under the tilted measure $\mathbb{P}^b$, the process $X$ is a solution of the SDE

$$dX_t = b_t(X) \, dW_t, \quad t \geq 0$$

where $W$ is a $\mathbb{P}^b$-Brownian motion.

a) Prove that if

$$|b_t(x)| \leq C(1 + |x_t|), \quad t \geq 0, x \in \Omega,$$

then Novikov’s condition holds conditionally on $\mathcal{F}_s$ for intervals $[s, t]$ such that $|t - s|$ is small enough, i.e.

$$\mathbb{E}\left[ \exp\left( \frac{1}{2} \int_s^t |b_u(X)|^2 \, du \right) | \mathcal{F}_s \right] < +\infty.$$

b) Deduce that $Z$ is a martingale. [Hint: prove that $\mathbb{E}[Z_t | \mathcal{F}_s] = Z_s$ for small time intervals $[s, t]$ and the conclude].

c) Prove that

$$\mathbb{P}(\|X\|_{[0, t]} > r) \leq 2 e^{-r^2/2t} \quad t \geq 0, r \geq 0,$$

where $\|X\|_{[0, t]}$ denotes the supremum wrt. the Euclidean norm of $(X_s)_{s \in [0, t]}$.

[Hint: use Doob’s inequality for the submartingale $e^{\lambda X_t}$ and optimize over $\lambda > 0$]

d) Prove the same result as in (a) under the more general assumption that $b$ is a previsible drift such that

$$|b_t(x)| \leq C(1 + \|x\|_{\infty, [0, t]}), \quad t \geq 0, x \in \Omega$$

where $C < +\infty$. 