Stochastic Analysis – Problem Sheet 4.

Exercise 1. (Brownian motion on the unit sphere) Let \( Y_t = B_t / |B_t| \) where \( B \) is a Brownian motion in \( \mathbb{R}^n \) and \( n > 2 \). Prove that the time–changed process

\[
Z_a = Y_{T_a}, \quad T = A^{-1}, \quad A_t = \int_0^t |B_s|^{-2} ds,
\]

is a diffusion taking values in the unit sphere \( S^{n-1} = \{ x \in \mathbb{R}^n : |x| = 1 \} \) with generator

\[
L f = \frac{1}{2} \left( \Delta f(x) - \sum_{i,j} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right) - \frac{n-1}{2} \sum_i x_i \frac{\partial f}{\partial x_i}(x), \quad x \in S^{n-1}.
\]

Exercise 2. (Polar points of Brownian motion for \( d \geq 2 \)) Let \((X,Y)\) be a Brownian motion on \( \mathbb{R}^2 \) starting at \((0,0)\). Let

\[
(M_t, N_t) := e^{X_t}(\cos(Y_t), \sin(Y_t)).
\]

We will assume without proof that

\[
\int_0^\infty e^{2X_s} ds = +\infty, \quad \text{a.s.}
\]

a) Prove that \((M, N)\) is a Brownian motion on \( \mathbb{R}^2 \) changed of time (starting from where?) ;

b) Compute the Euclidean norm \(|(M_t, N_t)|\) of the vector \((M_t, N_t)\) and deduce that a Brownian motion \(B\) in \( \mathbb{R}^2\) never visit the point \((-1,0)\), that is

\[
\mathbb{P}(\exists t > 0: B(t) = (-1,0)) = 0.
\]

c) Conclude that \(B\) never visit any given point \(x \neq (0,0)\).

d) Use the Markov property to deduce from (c) that \(\mathbb{P}(\exists t > 0: B(t) = (0,0)) = 0. [\text{Hint: consider } \mathbb{P}(\exists t \geq 1/n: B(t) = (0,0)) \text{ as } n \to 0.]\)

e) Prove that a Brownian motion in \( \mathbb{R}^d \) with \( d > 2 \) does not visit any given point \(x \in \mathbb{R}^d\).

Exercise 3. (Transience of Brownian motion in \( d \geq 3 \)) Let \(X\) be a Brownian motion in \( \mathbb{R}^3 \) starting from \( a \in \mathbb{R}^3 \neq 0 \). We admit that every continuous positive surmartingale has an almost sure limit. Moreover we say that a process \(Y\) is transient if \(|Y_t| \to \infty\) as \(t \to \infty\) almost surely.

a) Prove that the process \(M_t = 1/|X_t|\) is a positive local martingale.
b) Prove that $M_\infty = \lim_{t \to \infty} M_t$ exists almost surely.

c) Compute $\mathbb{E}[M_t]$ and deduce that $M_\infty = 0$. This implies that $X$ is transient.

d) Show that whatever the starting point is, $X$ is always transient.

e) Prove that a Brownian motion in $\mathbb{R}^d$ with $d \geq 3$ is transient.

Exercise 4. (Conformal invariance of Brownian motion) Let $f: \mathbb{C} \to \mathbb{C}$ be an holomorphic function and $Z = X + iY$ be a planar Brownian motion (with the identification of $\mathbb{C}$ with $\mathbb{R}^2$). Prove that the process $M_t = f(Z_t)$ is a continuous local martingale with values in $\mathbb{C}$. Deduce that it is a complex Brownian motion changed of time. This property is called conformal invariance of Brownian motion.