Stochastic Analysis – Problem Sheet 1.

Exercise 1. (Martingale problem) Consider the solution \( X \) of the SDE in \( \mathbb{R}^d \)

\[
dx_t = b(X_t)dt + \sigma(X_t)dB_t,
\]

where \( B \) is a \( d \)-dimensional Brownian motion and \( b: \mathbb{R}^d \to \mathbb{R}^d, \sigma: \mathbb{R}^d \to \mathbb{R}^{d \times d} \) locally bounded coefficients. Let \( \mathcal{L} \) be the associated infinitesimal generator. By Theorem 2.3 in [Eberle, Stochastic Analysis notes SS2015] we know that the following two conditions are equivalent

i. For any \( f \in C^2(\mathbb{R}^d) \), the process \( M^f_t = f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)ds \) is a local martingale.

ii. For any \( v \in \mathbb{R}^d \), the process \( M^v_t = v \cdot X_t - v \cdot X_0 - \int_0^t v \cdot b(X_s)ds \) is a local martingale with quadratic variation

\[
[M^v]_t = \int_0^t v \cdot a(X_s)v ds.
\]

a) Show that these conditions are also equivalent to the fact that for any \( v \in \mathbb{R}^d \) the process

\[
Z^v_t = \exp \left( M^v_t - \frac{1}{2} \int_0^t v \cdot a(X_s)v ds \right)
\]

is a local martingale. [Hint: use the fact that linear combinations of exponentials are dense in \( C^2 \) w.r.t. uniform convergence on compacts for the functions and its first two derivatives (assumed without proof)]

b) Show that these conditions imply that

\[
(f(X_t)/f(X_0))\exp \left( -\int_0^t \frac{\mathcal{L}f(X_s)}{f(X_s)}ds \right)
\]

is a local martingale for every strictly positive \( C^2 \) function \( f \).

Exercise 2. (Variation of constants) Consider the nonlinear SDE

\[
dx_t = f(t, X_t)dt + c(t)X_t dB_t, \quad X_0 = x,
\]

where \( f: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \) and \( c: \mathbb{R}_+ \to \mathbb{R} \) are continuous deterministic functions.

a) Find an explicit solution \( Z_t \) in the case \( f = 0 \) and \( Z_0 = 1 \).

b) Use the Ansatz \( X_t = C_t Z_t \) to show that \( X \) solves the SDE provided \( C \) solves an ODE with random coefficients.

c) Apply this method to solve the SDE

\[
dx_t = X_t^{-1}dt + \alpha X_t dB_t, \quad X_0 = x
\]
where $\alpha$ is a constant.

d) Apply the method to study the solution of the SDE

$$dX_t = X_t^\gamma dt + \alpha X_t dB_t, \quad X_0 = x > 0$$

where $\alpha$ and $\gamma$ are constants. For which values of $\gamma$ do we get explosion?

**Exercise 3. (Exit distribution of Bessel process)** Let $X$ be the solution of the SDE

$$dX_t = \frac{d-1}{2} \frac{1}{X_t} dt + dB_t, \quad X_0 = x_0 > 0$$

where $B$ is a standard Brownian motion and $d > 1$ is a constant.

a) Find a non–constant function $u$ such that $u(X_t)$ is a local martingale.

b) Compute the ruin probability $\mathbb{P}(T_a < T_b)$ for $0 < a < b$ with $x_0 \in [a, b]$ where $T_a = \inf\{t \geq 0 : X_t \leq a\}$ and $T_b = \inf\{t \geq 0 : X_t \geq b\}$.

c) Proceed similarly to determine the mean exit time $\mathbb{E}[T]$ where $T = \min(T_a, T_b)$.