The Space of Rough Paths

Graduate Seminar on Rough Paths - Summer Semester 2022
Lead by: Prof. Dr. Massimiliano Gubinelli

Throughout this talk, V is a Banach space. Moreover, $V \otimes V$ is understood to be the algebraic tensor product equipped with a symmetric and compatible tensor norm, i.e, $|w \otimes v| = |v \otimes w| \leq |v| \cdot |w|$.

1. Motivation

Definition 1.1. For a smooth path $X : [0,T] \to \mathbb{R}^d$, we define the second order iterated integral of X against itself as

$$\int_{0<r_1<r_2<T} dX_{r_1} \otimes dX_{r_2} := (\int_{0<r_1<r_2<T} dX^i_{r_1} \cdot dX^j_{r_2})_{i,j=1,\ldots,d} \tag{1}$$

where the integration on the right hand side is understood to be the Riemann–Stieltjes integral.

Definition 1.2. Given a smooth path $X : [0,T] \to \mathbb{R}^d$, the signature of X, denoted $S(X)_{0,T}$ is defined as

$$S(X)_{0,T} := \left(1, \int_{0<r_1<T} dX_{r_1}, \int_{0<r_1<r_2<T} dX_{r_1} \otimes dX_{r_2}, \ldots, \int_{0<r_1<r_2<..<r_n<T} dX_{r_1} \otimes dX_{r_2} \otimes .. \otimes dX_{r_n}, \ldots\right)$$

in the above sense.

Remark 1.3.

- We have $S(X)_{0,T} \in \mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d)^{\otimes 2} \oplus (\mathbb{R}^d)^{\otimes 3} \oplus \cdots$.
- If X is α-Hölder for $\alpha > \frac{1}{2}$, then one can define its signature via Young's integral. (Later in talk 6.)
- When the path X is not regular enough, the integrals in the definition of signature are not well defined. But, if one could construct an object

$$\mathbb{X}_{s,t} := (1 =: \mathbb{X}^0_{s,t},\ X_t - X_s =: \mathbb{X}^1_{s,t},\ \mathbb{X}^2_{s,t},\ \mathbb{X}^3_{s,t}, \cdots) \qquad \forall s,t \in [0,T], \tag{2}$$

  which mimics signature of X up to some algebraic and analytical properties, then one could use that object in the place of signature for practically all relevant purposes.

Questions:

- What basic properties of signature of smooth paths do we want to impose on that object?
- How do we construct such object?
- Up to which component $\mathbb{X}^k$ is it sufficient to construct the whole object $\mathbb{X}$?
- What consequences do we have if we can construct some object satisfying those basic properties?

2. Basic definitions

Definition 2.1. For $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, we define the space of $\alpha-$Hölder rough paths (over V), in symbol $\mathscr{C}^\alpha([0,T], V)$, as those pairs $(\mathbb{X}^1, \mathbb{X}^2) =: X$ with $\mathbb{X}^1 : [0,T] \to V$ and $\mathbb{X}^2 : [0,T]^2 \to V \otimes V$ being continuous and satisfying the following conditions:

$$\mathbb{X}^2_{s,t} - \mathbb{X}^2_{s,u} - \mathbb{X}^2_{u,t} = \mathbb{X}^1_{s,u} \otimes \mathbb{X}^1_{u,t}, \quad \forall s,u,t \in [0,T] \tag{3}$$

$$||\mathbb{X}^1||_\alpha := \sup_{s \neq t \in [0,T]} \frac{|\mathbb{X}^1_{s,t}|}{|t-s|^\alpha} < \infty, \qquad ||\mathbb{X}^2||_{2\alpha} := \sup_{s \neq t \in [0,T]} \frac{|\mathbb{X}^2_{s,t}|}{|t-s|^{2\alpha}} < \infty. \tag{4}$$

Example 2.2. (Constructions of $\mathbb{X}^2$)

- If X is a smooth V-valued path, setting $\mathbb{X}^2_{s,t} := \int_s^t \mathrm{X}_{s,r} \otimes \dot{\mathrm{X}}_r dr$ gives the canonical rough path lift $(X, \mathbb{X}^2)$ of X, where $X_{s,r} := X_r - X_s$.
- Consider the 2-dimensional Brownian motion $B = (B^1, B^2)$. Define $\mathbb{B}^2$ as the Ito integral or Stratonovich integral of B against itself which takes value in $\mathbb{R}^4$, then $(B, \mathbb{B}^2)$ will be rough paths $\mathbb{P}$-almost surely.

Remark 2.3. (a). One can also define $\mathbb{X}$ as $[0,T] \to V \oplus V^{\otimes 2}$, $t \mapsto (\mathbb{X}^1_t, \mathbb{X}^2_{0,t})$ since all "second order increments" are already determined by such information.

(b). Given $\mathbb{X}^1$ an underlying path, then $\mathbb{X}^2$ is unique up to some function $F \in C^{2\alpha}([0,T], V \otimes V)$.

(c). The *Lyons–Victoir extension theorem* states that for any $X \in C^\alpha$, with values in any Banach space V for some $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, one can find some suitable "second order increments", with which X lifts to an α-Hölder rough path. (cf. [FH] Exercise 2.14 and [LV]). However, in the setting of stochastic analysis, one does not rely on the extension theorem. Some typical constructions are via probability, e.g. via Stratonovich integral.

Definition 2.4. Given $\mathbb{X}$, $\mathbb{Y} \in \mathscr{C}^\alpha([0,T], V)$,we define the $\alpha-$Hölder rough path metric via

$$\varrho_\alpha(\mathbb{X}, \mathbb{Y}) := \sup_{s \neq t \in [0,T]} \frac{|\mathbb{X}^1_{s,t} - \mathbb{Y}^1_{s,t}|}{|t-s|^\alpha} + \sup_{s \neq t \in [0,T]} \frac{|\mathbb{X}^2_{s,t} - \mathbb{Y}^2_{s,t}|}{|t-s|^{2\alpha}}. \tag{5}$$

Remark 2.5. The space of α-Hölder rough paths equipped with the metric above is a nonlinear but closed subspace of the Banach space $C^\alpha([0,T], V) \oplus C_2^{2\alpha}([0,T], V \otimes V)$, and thus complete.

3. (Weakly) geometric rough paths

Definition 3.1. We define the space of weakly geometric (α-Hölder) rough paths,

$$\mathscr{C}^\alpha_g([0,T], V) \subset \mathscr{C}^\alpha([0,T], V)$$

as the space of α-Hölder rough paths such that for all times s, t we have the "first order calculus" condition:

$$\mathrm{Sym}(\mathbb{X}^2_{s,t}) = \frac{1}{2} \mathbb{X}^1_{s,t} \otimes \mathbb{X}^1_{s,t}. \tag{6}$$

Definition 3.2. We define the space of geometric (α-Hölder) rough paths,

$$\mathscr{C}^{0,\alpha}_g([0,T], V) \subset \mathscr{C}^\alpha([0,T], V)$$

as the closure of the space of canonical rough path lifts of smooth paths.

Remark 3.3. One has the strict inclusion $\mathscr{C}^{0,\alpha}_g([0,T], V) \subset \mathscr{C}^\alpha_g([0,T], V)$. However, the distinction between them rarely matters since $\mathscr{C}^\beta_g([0,T], V) \subset \mathscr{C}^{0,\alpha}_g([0,T], V)$ whenever $\beta > \alpha$.

4. A pure area rough path example

Proposition 4.1 (No Proof). *Assume that* ${}^n\mathbb{X} \in \mathscr{C}^\beta$*, for* $1/3 < \alpha < \beta$*, with uniform bounds*

$$\sup_n \left\| {}^n\mathbb{X}^1 \right\|_\beta < \infty \quad \textit{and} \quad \sup_n \left\| {}^n\mathbb{X}^2 \right\|_{2\beta} < \infty$$

and uniform convergence ${}^n\mathbb{X}^1_{s,t} \to \mathbb{X}^1_{s,t}$ *and* ${}^n\mathbb{X}^2_{s,t} \to \mathbb{X}^2_{s,t}$*, i.e. uniformly over* $s, t \in [0,T]$*. Then we have* $\mathbb{X} := (\mathbb{X}^1, \mathbb{X}^2) \in \mathscr{C}^\beta$ *and* ${}^n\mathbb{X} \to \mathbb{X}$ *in* $\mathscr{C}^\alpha$*. Furthermore, the assumption of uniform convergence can be weakened to pointwise convergence:*

$$\forall t \in [0,T] : \quad {}^n\mathbb{X}^1_{0,t}{}^n \to \mathbb{X}^1_{0,t} \quad \textit{and} \quad {}^n\mathbb{X}^2_{0,t} \to \mathbb{X}^2_{0,t}.$$

Example 4.2. Identify $\mathbb{R}^2$ with the complex numbers and consider

$$[0,1] \ni t \mapsto n^{-1} \exp\left(2\pi i n^2 t\right) \equiv X^n.$$

Set $\mathbb{X}^n_{s,t} := \int_s^t X^n_{s,r} \otimes dX^n_r$. Then for fixed $s < t$,

$$X^n_{s,t} \to 0, \quad \mathbb{X}^n_{s,t} \to \pi(t-s)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $(X^n, \mathbb{X}^n)$ converges in $\mathscr{C}^\alpha$, any $\alpha < 1/2$.

References

[FH] Fritz, Hairer: *A Course on Rough Paths: With an Introduction to Regularity Structures.* Springer, 2014.

[LV] Lyons, Victoir: An extension theorem to rough paths. *Ann. Inst. H. Poincare Anal. Non Linéaire* **24**, no. 5, (2007), 835–847