Link with Stochastic Analysis

BY

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Plan:

(1) Use Rough Path to study Ito or Stratorovich SDE

(2) Wong-Zakai

(3) a quick introduction of Large Deviation Theory

(4) Freidlin - Wentzell Large Deviation Principle

1. Use Rough Path to study Ito or Stratorovich SDE

The central goal of stochastic analysis is to give a precise measing of the equation

$$dY = f_0(Y)dt + f(Y)dB$$

with dB term coming from some noisy signal and estabish the existence, uniqueness and statility results. This is accomplished by Ito and Stratonovich (without statbility).

Now we can abso give this equation a meaning in a pathwise manner by rough path theory, the question is whether this new coustruction agree the same solution as the old theories. The answer is yes!

We have shown that for d-dimensinal Brownian motion B

$$\mathbb{B}^{\text{Ito}}_{s,t} = \int_s^t B_{s,r} \otimes dB_r$$

$$\begin{aligned}\mathbb{B}^{\text{Strat}}_{s,t} &= \int_s^t B_{s,r} \otimes \circ dB_r\\ &= \mathbb{B}^{\text{Ito}}_{s,t} + \frac{1}{2}(t-s)\, I\end{aligned}$$

are two enhancement of B such that $\boldsymbol{B} = (B, \mathbb{B}) \in \mathcal{C}^{\alpha}([0,T], \mathbb{R}^d)$ almost surely, for any $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$. Then for all ω in this probability one set, we have a precise meaning of RDE

$$dY = f_0(Y)dt + f(Y)d\boldsymbol{B}$$

Then we have following theorem.

Theorem 1. *Suppose $f \in C_b^3(\mathbb{R}^e, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e))$, $f_0 \in C_b^3(\mathbb{R}^e, \mathbb{R}^e)$ and $\xi \in \mathbb{R}^e$, Then*

1) With probability one, $\boldsymbol{B}^{Ito}(\omega) \in \mathcal{C}^{\alpha}$, any $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$ and there is a unique RDE solution $(Y(\omega), f(Y(\omega)) \in \mathcal{D}^{2\alpha}_{B(\omega)}$ to

$$\begin{aligned}dY &= f_0(Y)dt + f(Y)d\boldsymbol{B}^{\text{Ito}}\\ Y_0 &= \xi\end{aligned}$$

Moreover, $Y = (Y_t(\omega))$ is a strong solution to the Ito SDE

$$\begin{aligned}dY &= f_0(Y)\,dt + f(Y)dB\\ Y_0 &= \xi.\end{aligned}$$

2) With probability one, $\boldsymbol{B}^{Strat}(\omega) \in \mathcal{C}^{\alpha}$*, any* $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$ *and there is a unique RDE solution* $(Y(\omega), f(Y(\omega)) \in \mathcal{D}^{2\alpha}_{B(\omega)}$ *to*

$$\begin{aligned} dY &= f_0(Y)dt + f(Y)d\boldsymbol{B}^{\mathrm{Strat}} \\ Y_0 &= \xi \end{aligned}$$

Moreover, $Y = (Y_t(\omega))$ *is a strong solution to the Stratnovich SDE*

$$\begin{aligned} dY &= f_0(Y)\,dt + f(Y)dB \\ Y_0 &= \xi. \end{aligned}$$

Proof. 1) We can assume $f_0 = 0$, otherterise we conside

$$\tilde{Y} = \begin{pmatrix} t \\ Y \end{pmatrix}. \tilde{f} = \begin{pmatrix} 1 & 0 \\ f_0 & f \end{pmatrix}, \tilde{\boldsymbol{B}} = \begin{pmatrix} t \\ \boldsymbol{B} \end{pmatrix}$$

$$\Rightarrow d\tilde{Y} = \tilde{f}(\tilde{Y})d\tilde{\boldsymbol{B}}$$

So we only need to show the result for

$$dY = f(Y)\,d\boldsymbol{B}^{\mathrm{Ito}} \quad Y_0 = \xi$$

I. $(Y, f(Y))$ is adapted to the natural filtration generated by B. Since

$$\mathbb{B}^{\mathrm{Ito}}_{s,r} \ \hat{\in} \sigma(B_u, 0 \leqslant u \leqslant r)$$

$$\Rightarrow \sigma(B_s, \mathbb{B}^{\mathrm{Ito}}_{s,r} : 0 \leqslant s \leqslant r \leqslant t) = \sigma(B_u : 0 \leq u \leq t)$$

$\Rightarrow [0, T] \times \Omega \rightarrow (B, \mathbb{B}^{\mathrm{Ito}})$ is adapted.

Since $(B, \mathbb{B}^{\mathrm{Ito}}) \rightarrow (Y, f(Y)$ is contimuous by Ito-Lyons map $\Rightarrow (Y, f(Y))$ is $\sigma(B_u, 0 \leq u \leq t)$ adapted.

II. Since Ito integral = Rough integral almost surely.

$$\begin{aligned} \Rightarrow Y_t &= \xi + \int_0^t f(Y_s)\,d\boldsymbol{B}^{\mathrm{Ito}}_s \\ &= \xi + \int_0^t f(Y_s) dB_s \end{aligned}$$

The case for Stratonovich SDE can be show similarly. □

2. Wong-Zakai

We have shown that the dyadic piecewise linear approximations $(B^{(n)})$ defined by

$$\begin{cases} B^{(n)}_t = B_t \quad \text{when} \quad t = \dfrac{iT}{2^n} \\ \text{linearly interppolated on } \left[\dfrac{iT}{2^n}, \dfrac{(i+1)T}{2^n}\right] \end{cases}$$

with enhancment $\boldsymbol{B}^{(n)} = (B^{(n)}, \int_0^{\cdot} B^{(n)} \otimes d B^{(n)})$ appoximates $(B, \mathbb{B}^{\text{Stat}})$ in $\mathcal{C}^{\alpha}$ with probability one.

Then a natural question is whether the solution of the RDE driven by these approximations, approximates the solution of corresponding Stratorovich SDE? This is given by Wong-Zakai.

Theorem 2. *Let f, f_0, ξ be as above. Let $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$ Consider dyadic piecewise linear approximations (B^n) to B on $[0, T]$ as above. Write Y^n for the random ODE solutions to*

$$\begin{aligned} d Y^n &= f_0(Y^n) dt + f(Y^n) dB^n \\ Y_0^n &= \xi \end{aligned}$$

and Y for the Stratonovich SDE solution t_0.

$$\begin{aligned} d Y &= f_0(Y) dt + f(Y) \circ dB \\ Y_0 &= \xi \end{aligned}$$

Then with probability one

$$\|Y - Y^n\|_{\alpha;[0,T]} \rightarrow 0$$

Proof. By stability of Ito-Lyons map

$$\|Y - Y^n\|_{\alpha;[0,T]} \leqslant C(\underbrace{|\xi - \xi|}_{=0}| + \rho_{\alpha}(\boldsymbol{B}, \boldsymbol{B}^{(i)}) \rightarrow 0 \,\text{as}\, n \rightarrow \infty$$

note that C depends on $\omega \in \Omega$. Since $C = C(M, \alpha, f, f_0)$

$$M = \sup_{n \in \mathbb{N}} \left(\|\boldsymbol{B}\|_{\alpha}, \left\|\boldsymbol{B}_{\alpha}^{(n)}\right\| \right)$$

□

3. a quick introduction of Large Deviation Theory

Large deviation originated from the study of insurance mathematics, when Cramer studied ruin theory. Suppose you have an insurance product so that you get money from buyers at constant rate x per month after months, you get money nx. Model the random pay out of this product by a sequence of random variables: $\{X_n\}$. Then the total payout up to n months, is $S_n = X_1 + \cdots + X_n$ For simplicity X_i i.i.d. X Clearly you have to set up the price so that

$$\mathbb{E}[X_i] < x$$

The company makes money from it when

$$S_n \leqslant n\, x \Leftrightarrow \frac{S_n}{n} \leqslant x$$

loses money when

$$S_n > n\, x \Leftrightarrow \frac{S_n}{n} > x$$

Strong Law of Large number

$$\frac{1}{n} S_n \underset{n \rightarrow +\infty}{\longrightarrow} \mathbb{E}[X] \quad a.s.$$

But it is also very important to know how fast $\frac{S_n}{n} \to \mathbb{E}[X]$, since there are certain probabilities $\mathbb{P}\left(\frac{S_n}{n} > x\right)$ to lose some money. The correct estimation of $\mathbb{P}\left(\frac{f_n}{n} > x\right)$ is given by Cramer's theorem.

Theorem 3. *Assume $X_1, X_2, \ldots$ are i.i.d. and the $\mathbb{E}[e^{\lambda X_1}] < \infty$ for all $\lambda \in \mathbb{R}$ Then for all $x > \mu := \mathbb{E}[X_1]$,*

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left(\frac{S_n}{n} \geqslant x\right) = -I(x)$$

where $I(x) = \sup_{\lambda \in \mathbb{R}} [\lambda x - \log \mathbb{E}[e^{\lambda X_1}] > 0$.

This means

$$\mathbb{P}\left(\frac{s_n}{n} \geqslant x\right) = e^{-nI(x) + o(n)}$$

Here $I(x)$ is called the rate function, we define

Definition 4. *A function $I: S \to [0, \infty]$ on some polish space S (competely metrizable topological space) if for any $\alpha \geqslant 0$, the set $\{x: I(x) \leqslant \alpha\}$ are compact.*

Note that the essence of estimating $\mathbb{P}\left(\frac{S_n}{n} \geqslant x\right)$ is to bound $\frac{1}{n} \log \mathbb{P}\left(\frac{S_n}{n} \geqslant x\right)$, so we have the following definition. Note that it can also be stated for a sequence of measure.

Definition 5. *A family of $S-$valued random variables $\{X_n\}$ satisfies the LDP (Large deviation principle) with rate function I, if for any open set $A \subset S$ and closed set $B \subset S$, we have*

1. $\liminf_{n \to +\infty} \frac{1}{n} \log \mathbb{P}(X_n \in A) \geqslant -\inf_{x \in A} I(x)$
2. $\limsup_{n \to +\infty} \frac{1}{n} \log \mathbb{P}(X_n \in B) \leqslant -\inf_{X \in B} I(x)$

The reason for considering Polish space S here is that we are going to be interested in LDP for stochastic process, as a random variable in path space, as we can see in following.

Theorem 6. (Schilder) *B is a d-dimessional Brownian motion, consider $B_t^{\varepsilon} = \sqrt{\varepsilon} B_t$ as a sequence of random variables with value in $C_0[0,1]$ (space of continuous functions, $\phi: [0,1] \to \mathbb{R}$, s,t, $\phi(0) = 0$, with supremum norm) satisfred LDP with good rate function*

$$I(\phi) = \begin{cases} \frac{1}{2} \int_0^1 |\dot{\phi}(t)|^2 dt, & \phi \in H \\ \infty, & \textit{otherwise} \end{cases}$$

where $H := \{\int_0^t f(s) ds: f \in L^2[0,1]\}$.

Another important theorem in Large deviation is the contraction principle.

Theorem 7. *$S, \tilde{S}$ two Polish spaces, $\Phi: S \to \tilde{S}$ continuous map, and (X_i) a sequence of S-valued random variables that satisfied LDP with good rate funetron I. Then $\Phi(X_i)$ also satisfies LDP with rate function $\tilde{I}(\tilde{x}) = \inf_{x: \Phi(x) = \tilde{x}} I(x)$.*

The Freidlin-Wentzell large deviation principle asks the question that, we now know the LDP for Brownian motion. What about the solution of SDE driven by BM ? This can be answered easily by using Rough Path theory and contracton prinaple with some generalisation.

Theorem 8. (Freidlin-Wentzell) *Let f_1 f_0, ξ be as above Let $\frac{1}{3} < \alpha < \frac{1}{2}$, B is a d-dimensional Brownian motion, and consider the unique Stratonovich SDE solution Y^ε on $[0, T]$ to*

$$
\begin{aligned}
dY &= f_0(Y)\,dt + f(Y) \circ \varepsilon dB \\
Y_0 &= \xi
\end{aligned}
$$

Write Y^h for the ODE solution obtained by replacing $\circ\varepsilon dB$ with dh where $h \in H$. Then (Y^ε) satisfres LDP with good rate function on path space

$$J(y) = \inf_h \{I(h): Y^h = y\}$$

Where I is Schilder's rate function.

The proof requires some Schilder LDP for Stratonovich Lift of εB, please see the book. The most important thing is that RPT gives an clearn and easy proof of FW LDP, where Ito's theory can not done easily, since lack of stability.