Integration with respect to rough paths

Seminar on Rough Paths - 27/05/2022

Aim: Define $\int_0^\cdot Y_s d\mathbb{X}_s$ for an α-rough path $\mathbb{X} = (X, \mathbb{X}^{(2)})$. More precisely, we adress the following points:

- For which paths $\mathbb{X}$ and integrands Y_s can we define an integral?
- Can we define this integral to be continuous in a suitable topology?

Mainly following [2].

Notation.

- $C_n(K) = C([0,T]^n_\leqslant, K)$ where $[0,T]^n_\leqslant$ is the set of ordered times $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n \leqslant T$. We will leave K implicit when the space is clear from context.
- The first and second order increments: for $f \in C_1$: $\delta f_{st} := f_t - f_s$ and $F \in C_2$: $\delta F_{sut} := F_{st} - F_{su} - F_{ut}$.

A first look at Riemann-Stieltjes Integrals. If the paths X, Y are smooth, we can define

$$I_T := \int_0^T Y_r \mathrm{dX}_r := \lim_{|\mathcal{P}| \to 0} \sum_{[s,t] \in \mathcal{P}} \underbrace{Y_s \delta X_{st}}_{A_{st}}. \tag{1}$$

Denoting by

$$R_{st} = \delta I_{st} - A_{st} \quad I_0 = 0, \tag{2}$$

this means that $R_{st} = \int_s^t \delta Y_{rs} \dot{X}_r \mathrm{dr} = o(|t-s|)$. In other words, for this definition we rely on the fact that A_{st} is a good local approximation for δI_{st}. We call

- A the *germ*,
- R the *remainder*,
- I the *integral*.

For a fixed germ A, let us collect some simple observations:

- If we want to define I via (1), the remainder at the very least should not affect the limit. That is, we need $\sum_{[s,t] \in \mathcal{P}} R_{st} \to 0$ as $|\mathcal{P}| \to 0$ and thus at least $R_{st} = o(|t-s|)$.
- If $R_{st} = o(|t-s|)$, then the integral I_T is uniquely determined by (2).
- $R_{st} - R_{su} - R_{ut} = -(A_{st} - A_{su} - A_{ut})$. More precisely: If (I, R) satisfies (2), then $\delta R_{sut} = -\delta A_{sut}$. And conversely, if R satisfies $\delta R_{sut} = -\delta A_{sut}$, then $I_t = R_{0t} + A_{0t}$ satisfies (2).
- If (2) holds with $R_{st} = o(|t-s|)$, then $|\delta R_{sut}| = |\delta A_{sut}| = o(t-s)$.

Motivated by the last point, we define

Definition 1. *A germ $A \in C_2$ is called* coherent *if there is a $\gamma > 1$ such that $|\delta A_{sut}| \lesssim |t-s|^\gamma$, i.e. $\delta A \in C_3^\gamma$.*

For a coherent $A \in C_2$ the *sewing lemma* guarantees the existence of a pair (I, R) satisfiying (2) with $R_{st} = o(|t-s|)$, which we informally state below.

Lemma 2. *For a coherent germ $A \in C_2^\gamma$, there is a unique pair $(I, R) \in C_1 \times C_2$ such that (2) holds with $R_{st} = o(|t-s|)$. Moreover, the integral I can be defined via the Riemann sums (along arbitrary partitions) of A and $R \in C_2^\gamma$.*

Theorem 3. *For $\gamma > 1$, there is a unique map $\Lambda: C_3^\gamma \cap \delta C_2 \to C_2^\gamma$ such that $\delta \circ \Lambda = \mathrm{id}_{C_3^\gamma \cap \delta C_2}$. The map is bounded and linear and for a coherent germ A, the unique solution to (2) is given by $R = -\Lambda(\delta A)$ and $I = A_{0t} + R_{0t}$ and*

$$|\delta I_{st} - A_{st}| \leqslant C \|\delta A\|_\gamma |t-s|^\gamma.$$

Remark 4. In case X is a semimartingale (and the conditional expectation of the increments is much smaller than the increments), the conditions can be relaxed at the cost of trading pointwise convergence for L^2- convergence. Taking e.g. the nonanticipating left-point approximation, this corresponds to the Itô-integral (Proposition 4.19, [1]).

Return to the rough setting: For $A_{st} = Y_s \delta X_{st}$, we can compute $\delta A_{sut} = -\delta Y_{su} \delta X_{ut}$ and thus

$$\|\delta A\|_{\alpha+\beta} \lesssim \|\delta Y\|_{\beta} \|\delta X\|_{\alpha}.$$

This estimate is sharp, which means that for $\alpha + \beta \leqslant 1$, we cannot apply the sewing lemma with this choice for A.
→ We need to improve the approximation A.

Example. $$Y_s = F(X_s), \quad F \in C_b^2(V, \mathcal{L}(V, W))^{1}.$$

Adding another term to the Taylor expansion, we have

$$\underbrace{F(X_r) - F(X_s)}_{\delta Y_{rs}} = \underbrace{\mathrm{D}F(X_s)\, \delta X_{sr}}_{Y_s' \delta X_{sr}} + R_{sr}^Y, \tag{3}$$

where

$$\|Y\|_{\alpha} \leqslant \|\mathrm{D}F\|_{\infty} \|X\|_{\alpha},$$
$$\|Y'\|_{\alpha} \leqslant \|\mathrm{D}^2 F\|_{\infty} \|X\|_{\alpha},$$
$$\|R^Y\|_{2\alpha} \leqslant \frac{1}{2} \|\mathrm{D}^2 F\|_{\infty} \|X\|_{\alpha}^2.$$

With these estimates we readily verify that

$$A_{st} := Y_s \delta X_{st} + Y' \mathbb{X}_{st}^{(2)}, \text{ and thus } \delta A_{sut} = -R_{su}^Y \delta X_{ut} - \delta Y_{su}' \mathbb{X}_{ut}^{(2)} \tag{4}$$

is coherent provided $\alpha > 1/3$ and we can define the integral via the compensated Riemann sum over A and Lemma 2.

The example suggests that the integral can be defined for *controlled rough paths* satisfying a relation like (3). We fix a path $X \in C^{\alpha}([0,T], V)$ and a rough path lift $\mathbb{X} = (X, \mathbb{X}^{(2)})$ for $\alpha > 1/3$.

Definition 5. *We say $Y \in C^{\alpha}([0,T], \mathcal{L}(V,W))$ is* controlled by *X if there is a $Y' \in C^{\alpha}([0,T], \mathcal{L}(V, \mathcal{L}(V,W)))$ such that*

$$\delta Y_{st} = Y' \delta X_{st} + R_{st}^Y, \quad |R_{st}^Y| \lesssim |t-s|^{2\alpha}. \tag{5}$$

For a fixed X, we define the space of rough paths $\mathcal{D}_X^{2\alpha}([0,T], \mathcal{L}(V,W))$ as the space of all pairs (Y, Y') satisfying (5) equipped with the norm $\|Y, Y'\|_{X,2\alpha} := \|Y'\|_{\alpha} + \|R^Y\|_{2\alpha}$.

Theorem 6. *For a controlled path $(Y, Y') \in \mathcal{D}_X^{2\alpha}([0,T], \mathcal{L}(V,W))$, we can define the integral via the Riemann sums of (4) with the bounds given by Theorem 3. Moreover, $(Y, Y') \mapsto (\int Y d\mathbb{X}, Y)$ is a continuous linear map $\mathcal{D}_X^{2\alpha}([0,T], \mathcal{L}(V, W)) \to \mathcal{D}_X^{2\alpha}([0,T], W)$.*

Remark.

- Y' is not unique if $\mathbb{X}$ is "too smooth".
- Contrary to the space of rough paths (not even linear!), $\mathcal{D}_X^{2\alpha}$ is a Banachspace (at the cost of fixing the signal X) and we can e.g. use fixed point arguments.
- There is a continuous canonical injection $\mathcal{D}^{2\alpha} \hookrightarrow \mathcal{C}^{\alpha}$, that is a controlled path can itself be interpreted as a rough path.
- For two controlled paths $Y, Z \in \mathcal{D}_X^{\alpha}$ we can also define $\int Y dZ$ via $A_{st} = Y_s \delta Z_{st} + Y_s' Z_s' \mathbb{X}_{st}^{(2)}$.
- Including additional information $(\delta X, \mathbb{X}^{(2)}, \mathbb{X}^{(3)}, \dots)$ in the approximation, we can define the integral also for paths of lower regularity.
- The integral defined in Theorem 6 is locally Lipschitz in (Y, Y') (and $\mathbb{X}$) with the rough path topology (c.f. Theorem 4.17 [1]).

References

[1] Peter K Friz and Martin Hairer. *A course on rough paths*. Springer, 2020.
[2] Massimiliano Gubinelli. Controlling rough paths. *Journal of Functional Analysis*, 216(1):86–140, 2004.

1. using the identification $\mathrm{D}F \in \mathcal{L}(V, \mathcal{L}(V,W)) \simeq \mathcal{L}(V \otimes V, W)$