Brownian Motion as a Rough Path

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Outline of the seminar

- Introduction
- Kolmogorov criterion for rough paths
- Itô and Stratonovich Brownian motion
- Brownian motion in a magnetic field

Definition. *For* $\alpha \in (1/3, 1/2)$, *we define the space* $\mathcal{C}^\alpha([0,T];V)$ *of* α*-Hölder rough paths (over a Banach space* V*) as those pairs* $\boldsymbol{X} := (X, \mathbb{X})$ *such that*

$$\|X\|_\alpha := \sup_{s\neq t, s,t\in[0,T]} \frac{|X_{s,t}|}{|t-s|^\alpha} < \infty, \qquad \|\mathbb{X}\|_\alpha := \sup_{s\neq t, s,t\in[0,T]} \frac{|\mathbb{X}_{s,t}|}{|t-s|^{2\alpha}} < \infty,$$

with the notation $X_{s,t} := X_t - X_s$, and such that

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = X_{s,u} \otimes X_{u,t}.$$

Remark. In this seminar we shall be interested in *random rough path*

$$X(\omega) : [0,T] \to V, \qquad \mathbb{X}(\omega) : [0,T] \to V \otimes V.$$

In particular we shall consider the d-dimensional standard Brownian motion B (here $V = \mathbb{R}^d$) enhanced with

$$\mathbb{B}^{\text{Itô}}_{s,t} := \int_s^t B_{s,r} \otimes \mathrm{d}B_r, \qquad \text{or} \qquad \mathbb{B}^{\text{Strat}}_{s,t} := \int_s^t B_{s,r} \circ \mathrm{d}B_r.$$

Claim

- We want to prove that $\boldsymbol{B}^{\text{Itô}} = (B, \mathbb{B}^{\text{Itô}})$ and $\boldsymbol{B}^{\text{Strat}} = (B, \mathbb{B}^{\text{Strat}})$ are RP of regularity α for any $\alpha \in (1/3, 1/2)$.

  To this end we need to check:

  - regularity condition ($\sim$Kolmogorov criterion for RP);

  - algebraic condition (this follows directly from linearity and properties of the integral)

Theorem. (KC for RP)

Let $q \geqslant 2$ and $\beta > 1/q$. Suppose that for any $s, t \in [0, T]$, with $T > 0$, there exists a constant $C < \infty$ such that

$$\mathbb{E}[X_{s,t}^q]^{1/q} = \|X_{s,t}\|_{L^q} \leqslant C|t-s|^{\beta}, \qquad \mathbb{E}[\mathbb{X}_{s,t}^q]^{1/q} = \|\mathbb{X}_{s,t}\|_{L^q} \leqslant C|t-s|^{2\beta},$$

and that $(X, \mathbb{X})$ satisfies the algebraic condition.

Then for any $\alpha \in [0, \beta - 1/q)$ there exists a modification of $(X, \mathbb{X})$ and random variables $K_\alpha \in L^q$ and $\mathbb{K}_\alpha \in L^{q/2}$ such that for all $s, t \in [0, T]$ it holds that

$$|X_{s,t}| \leqslant K_\alpha(\omega)|t-s|^{\alpha}, \qquad |\mathbb{X}_{s,t}| \leqslant \mathbb{K}_\alpha(\omega)|t-s|^{2\alpha}.$$

In particular, if $\beta - 1/q > 1/3$, then for any $\alpha \in (1/3, \beta - 1/q)$ it holds that $(X, \mathbb{X}) \in \mathcal{C}^\alpha$ a.s.

For simplicity fix $T = 1$ and let

$$D_n := \{k2^{-n} \,|\, k \in \mathbb{N}, k2^{-n} \in (0,1)\}$$

be the set of integer multiples of 2^{-n} in $(0,1)$. Note that $\#D_n = 1/2^{-n} = 2^n$.

We shall consider $s, t \in \bigcup_n D_n$ (the remaining times are filled in by continuity).

We define

$$K_n := \sup_{t \in D_n} |X_{t,t+2^{-n}}|, \qquad \mathbb{K}_n := \sup_{t \in D_n} |\mathbb{X}_{t,t+2^{-n}}|.$$

Exploiting the hypothesis, it holds that

$$\mathbb{E}[K_n^q] \leqslant \mathbb{E}\left[\sum_{t \in D_n} |X_{t,t+2^{-n}}|^q\right] \leqslant \frac{1}{2^{-n}} C^q (2^{-n})^{\beta q} = C^q (2^{-n})^{\beta q - 1},$$

and similarly $\mathbb{E}\big[\mathbb{K}_n^{q/2}\big] \leqslant C^{q/2} (2^{-n})^{\beta q - 1}$.

Fix now $s, t \in \bigcup_n D_n$ with $s < t$ and take $m \in \mathbb{N}$ such that

$$2^{-(m+1)} < t - s \leqslant 2^{-m}.$$

Consider a partition of $[s, t)$ of the form $s = \tau_0 < \tau_1 < \cdots < \tau_N = t$, where $(\tau_i, \tau_{i+1}) \in D_n$ for some $n \geqslant m+1$ and where at most two sub-intervals share the same n. It follows that

$$|X_{s,t}| \leqslant \max_{0 \leqslant i < N} |X_{s,\tau_{i+1}}| \leqslant \sum_{i=0}^{N-1} |X_{\tau_i,\tau_{i+1}}| \leqslant 2 \sum_{n \geqslant m+1} K_n,$$

and thus

$$\frac{|X_{s,t}|}{|t-s|^\alpha} \leqslant \sum_{n \geqslant m+1} \frac{2K_n}{(2^{-(m+1)})^\alpha} \leqslant \sum_{n \geqslant m+1} \frac{2K_n}{(2^{-n})^\alpha} \leqslant K_\alpha,$$

with $K_\alpha := 2\sum_{n \geqslant 0} \frac{K_n}{(2^{-n})^\alpha}$. Finally, $K_\alpha \in L^q$ since, recalling that $\alpha \in [0, \beta - 1/q)$,

$$\|K_\alpha\|_{L^q}^{1/q} \leqslant \sum_{n \geqslant 0} \frac{2}{(2^{-n})^\alpha} \mathbb{E}[K_n^q]^{1/q} \leqslant \sum_{n \geqslant 0} \frac{2C}{(2^{-n})^\alpha} (2^{-n})^{\beta - 1/q} < \infty.$$

Analogously, we have

$$|\mathbb{X}_{s,t}| = \left| \sum_{i=0}^{N-1} \mathbb{X}_{\tau_i,\tau_{i+1}} + X_{s,\tau_i} \otimes X_{\tau_i,\tau_{i+1}} \right| \leqslant \sum_{i=0}^{N-1} |\mathbb{X}_{\tau_i,\tau_{i+1}}| + |X_{s,\tau_i}||X_{\tau_i,\tau_{i+1}}| \leqslant$$
$$\leqslant \sum_{i=0}^{N-1} |\mathbb{X}_{\tau_i,\tau_{i+1}}| + \max_{0\leqslant i<N} |X_{s,\tau_i}| \sum_{j=0}^{N-1} |X_{\tau_j,\tau_{j+1}}| \leqslant$$
$$\leqslant 2 \sum_{n\geqslant m+1} \mathbb{K}_n + \left(2 \sum_{n\geqslant m+1} K_n \right)^2,$$

and thus

$$\frac{|\mathbb{X}_{s,t}|}{|t-s|^{2\alpha}} \leqslant \sum_{n\geqslant m+1} \frac{2\mathbb{K}_n}{(2^{-(m+1)})^{2\alpha}} + \left(\sum_{n\geqslant m+1} \frac{2K_n}{(2^{-(m+1)})^{\alpha}} \right)^2 \leqslant \mathbb{K}_\alpha + K_\alpha^2,$$

with $\mathbb{K}_\alpha := 2\sum_{n\geqslant 0} \frac{\mathbb{K}_n}{(2^{-n})^{2\alpha}} \in L^{q/2}$ and $K_\alpha \in L^q$. This concludes the proof.

Itô Brownian motion

- Let B be a d-dimensional standard Brownian motion enhanced with its iterated integrals

$$\mathbb{B}^{\mathrm{It\hat{o}}}_{s,t} := \int_s^t B_{s,r} \otimes \mathrm{d}B_r \in \mathbb{R}^d \otimes \mathbb{R}^d,$$

where the stochastic integration is understood in the sense of Itô;

Proposition. *For any $\alpha \in (1/3, 1/2)$ and $T > 0$, with probability one*

$$\boldsymbol{B}^{\mathrm{It\hat{o}}} := (B, \mathbb{B}^{\mathrm{It\hat{o}}}) \in \mathcal{C}^\alpha([0,T]; \mathbb{R}^d).$$

***Proof.* (sketch)** *This follows from the KC for RP together with the Gaussian nature (we need to control only the case $q = 2$), the finiteness of moments and the scaling behaviour of B ($B_t = \lambda^{-1/2} B_{\lambda t}$ and $\mathbb{B}_{0,t} = \lambda^{-1} \mathbb{B}_{0,\lambda t}$).* □

Itô BM is not a geometric RP

$\boldsymbol{B}^{\text{Itô}}$ is actually a RP but not a *geometric* RP

- This comes from Itô formula

$$\mathrm{d}(B^iB^j) = B^i\mathrm{d}B^j + B^j\mathrm{d}B^i + \langle B^i, B^j\rangle \mathrm{d}t, \qquad i,j = 1,\dots,d,$$

yielding, for $s < t$,

$$\mathrm{Sym}(\mathbb{B}^{\text{Itô}}_{s,t}) = \frac{1}{2}B_{s,t}\otimes B_{s,t} - \frac{1}{2}\mathbb{I}(t-s) \neq \frac{1}{2}B_{s,t}\otimes B_{s,t}.$$

Stratonovich Brownian motion

- Stratonovich BM is defined analogously but enhanced with its iterated integrals

$$\mathbb{B}^{\mathrm{Strat}}_{s,t} := \int_s^t B_{s,r} \circ \mathrm{d}B_r \in \mathbb{R}^d \otimes \mathbb{R}^d,$$

understood in the sense of Stratonovich. This gives

$$\mathbb{B}^{\mathrm{Strat}}_{s,t} = \mathbb{B}^{\mathrm{It\hat{o}}}_{s,t} + \frac{1}{2}\mathbb{I}(t-s)$$

- Similarly, $\boldsymbol{B}^{\mathrm{Strat}} := (B, \mathbb{B}^{\mathrm{Strat}}) \in \mathcal{C}^{\alpha}([0,T];\mathbb{R}^d)$ for any $\alpha \in (1/3, 1/2)$;

- $\boldsymbol{B}^{\mathrm{Strat}}$ is a geometric RP

$$\mathrm{Sym}(\mathbb{B}^{\mathrm{Strat}}_{s,t}) = \frac{1}{2} B_{s,t} \otimes B_{s,t}.$$

Physical Brownian motion

- MODEL: particle of mass m and position $x(t)$ in $\mathbb{R}^3$, subject to a white noise $\dot{B}$ (distributional derivative of a Brownian motion B) in time and some frictions $\alpha_1, \alpha_2, \alpha_3 > 0$ in orthonormal directions.

- Described by the Newton's second law of dynamics which reads

$$m\ddot{x} = -M\dot{x} + \dot{B},$$

where M is a symmetric 3×3 matrix having spectrum $\alpha_1, \alpha_2, \alpha_3$.

- The process $x(t)$ is called *physical Brownian motion*.

- In the limit of small mass, $m \to 0$, a good approximation of $x(t)$ is the (*mathematical*) Brownian motion with a non-standard covariance (if $m = 0 \Rightarrow M\dot{x} = \dot{B} \Rightarrow \ x = M^{-1}B$).

Brownian motion in a magnetic field

- What if our particle carries a non-zero electric charge q and it moves in a (constant) *magnetic field* H?

- Newton's second law is again of the form

$$m\ddot{x} = -M\dot{x} + \dot{B},$$

but now M is no longer a symmetric matrix (due to Lorentz force $\vec{F} = q\vec{\dot{x}} \times \vec{H}$).

- Instead we shall simply assume M to be a 3×3 matrix such that

$$\mathrm{Real}\{\sigma(M)\} \subset (0, +\infty).$$

Claim

- We are studying

$$m\ddot{x} = -M\dot{x} + \dot{B};$$

- We introduce the *momentum* variable $p(t) = m\dot{x}(t)$ and get

$$\dot{p} = -\frac{1}{m}Mp + \dot{B}.$$

- Claim: we shall prove that $X = X^m$, indexed by the mass m, converges in a *non-trivial* way to BM at the level of RP as $m \to 0$.

  In particular, it converges to $\tilde{\mathbf{B}} := (B, \tilde{\mathbb{B}})$, with $\tilde{\mathbb{B}}_{s,t} := \mathbb{B}_{s,t}^{\mathrm{Strat}} + (t-s)A$ where

$$A = \frac{1}{2}(M\Sigma - \Sigma M^*), \qquad \Sigma := \int_0^\infty e^{-Ms} e^{-M^*s} \mathrm{d}s$$

  (A is anti-symmetric).

Precise formulation

Theorem 1. *Let M be a $d\times d$ square matrix whose eingenvalues have strictly positive real part. Let B be a $d-$ dimensional standard Brownian motion, $m>0$ and consider the following SDEs*

$$\mathrm{d}X=\frac{1}{m}P\mathrm{d}t,\qquad \mathrm{d}P=-\frac{1}{m}P\mathrm{d}t+\mathrm{d}B,$$

with vanishing initial conditions. For any $q\geqslant 1$ and $\alpha\in(\frac{1}{3},\frac{1}{2})$, it holds that, as $m\to 0$,

$$\left(MX,\int MX\otimes\mathrm{d}(MX)\right)\to\tilde{\mathbf{B}},\qquad \text{in } \mathcal{C}^{\alpha} \text{ and } L^{q},$$

where $\tilde{\mathbf{B}}:=(B,\tilde{\mathbb{B}})$, with $\tilde{\mathbb{B}}_{s,t}:=\mathbb{B}^{\mathrm{Strat}}_{s,t}+(t-s)A$ where

$$A=\frac{1}{2}(M\Sigma-\Sigma M^{*}),\qquad \Sigma:=\int_{0}^{\infty}e^{-Ms}e^{-M^{*}s}\mathrm{d}s.$$

Outline of the proof

In general, given $(\mathbf{X}^n)_n \subset \mathcal{C}^\beta$ for $1/3 < \alpha < \beta$ with uniform RP bounds

$$\sup_n \|X^n\|_\beta < \infty, \qquad \sup_n \|\mathbb{X}^n\|_{2\beta} < \infty,$$

and pointwise convergence

$$\forall t \in [0,T], \qquad X^n_{0,t} \to X_{0,t}, \qquad \mathbb{X}^n_{0,t} \to \mathbb{X}_{0,t},$$

this implies $\mathbf{X} \in \mathcal{C}^\beta$ and $\rho_\alpha(\mathbf{X}^n, \mathbf{X}) \to 0$.

The proof is thus divided in two steps:

1. *Pointwise convergence* in L^q

$$\left(M X^\varepsilon_t, \int_0^t M X^\varepsilon_s \otimes \mathrm{d}(M X^\varepsilon)_s \right) \to (B_{0,t}, \tilde{\mathbb{B}}_{0,t});$$

2. *Uniform RP bounds* in L^q.

Pointwise convergence in L^q

- In order to exploit Brownian scaling, we set $m = \varepsilon^2$ and we introduce the rescaled momentum

$$Y^\varepsilon := \frac{P}{\varepsilon}.$$

- We have

$$\mathrm{d}Y^\varepsilon = -\varepsilon^{-2} M Y^\varepsilon \mathrm{d}t + \varepsilon^{-1} \mathrm{d}B, \qquad \mathrm{d}X^\varepsilon = \varepsilon^{-1} Y^\varepsilon \mathrm{d}t.$$

- For a fixed ε, we define the Brownian motion $\bar{B}_t := \varepsilon B_{\varepsilon^{-2}t}$ and consider the SDEs

$$\mathrm{d}\bar{Y} = -M\bar{Y}\mathrm{d}t + \mathrm{d}\bar{B}, \qquad \mathrm{d}\bar{X} = \bar{Y}\mathrm{d}t.$$

- When solved with identical initial condition, we have the pathwise equality

$$(Y_t^\varepsilon, \varepsilon^{-1} X_t^\varepsilon) = (\bar{Y}_{\varepsilon^{-2}t}, \bar{X}_{\varepsilon^{-2}t}).$$

Pointwise convergence in L^q (II)

- We observe that since M is positive, $\bar{Y}$ is ergodic and the stationary solution has law

$$\nu \sim \mathcal{N}(0, \Sigma);$$

- To compute the covariance matrix Σ we write the stationary solution

$$\bar{Y}_t^{\mathrm{stat}} = \int_{-\infty}^{t} e^{-M(t-s)} \mathrm{d}\bar{B}_s$$

and observe that thus, *e.g.*,

$$\Sigma = \mathbb{E}[\bar{Y}_0^{\mathrm{stat}} \otimes \bar{Y}_0^{\mathrm{stat}}] = \int_0^{\infty} e^{-Ms} e^{-M^* s} \mathrm{d}s.$$

where we exploited the properties of the BM.

Pointwise convergence in L^q (III)

- Since $\sup_{t\in[0,\infty)}\mathbb{E}[|\bar{Y}|^2]<\infty$, it follows that

$$\varepsilon Y_t^\varepsilon = \varepsilon \bar{Y}_{\varepsilon^{-2}t} \to 0$$

in L^2 (and thus in any L^q, $q<\infty$) as $\varepsilon\to 0$ uniformly in t;

- From

$$\mathrm{d}Y^\varepsilon = -\varepsilon^{-2}MY^\varepsilon \mathrm{d}t + \varepsilon^{-1}\mathrm{d}B, \qquad \mathrm{d}X^\varepsilon = \varepsilon^{-1}Y^\varepsilon \mathrm{d}t.$$

it follows that $MX_t^\varepsilon = B_t - \varepsilon Y_{0,t}^\varepsilon$.

- The first part of the convergence directly follows, *i.e.*,

$$MX_t^\varepsilon \to B_t$$

as $\varepsilon\to 0$.

Pointwise convergence in L^q (IV)

- Moreover, thanks to ergodicity,

$$\int_0^t f(Y_t^\varepsilon)\mathrm{dt} \to t\int f(y)\nu(\mathrm{d}y), \qquad \text{in } L^q, \text{for } q<\infty.$$

- In particular, it holds that

$$\begin{aligned}
\int_0^t MX_s^\varepsilon \otimes \mathrm{d}(MX^\varepsilon)_s &= \int_0^t MX_s^\varepsilon \otimes \mathrm{d}B_s - \varepsilon\int_0^t MX_s^\varepsilon \otimes \mathrm{d}Y_s^\varepsilon \\
&= \int_0^t MX_s^\varepsilon \otimes \mathrm{d}B_s - MX_t^\varepsilon \otimes (\varepsilon Y_t^\varepsilon) + \varepsilon\int_0^t \mathrm{d}(MX^\varepsilon)_s \otimes Y_s^\varepsilon \\
&= \int_0^t MX_s^\varepsilon \otimes \mathrm{d}B_s - MX_t^\varepsilon \otimes (\varepsilon Y_t^\varepsilon) + \varepsilon\int_0^t MY_s^\varepsilon \otimes Y_s^\varepsilon \mathrm{d}s \\
&\to \int_0^t B_s \otimes \mathrm{d}B_s - 0 + t\int (My\otimes y)\nu(\mathrm{d}y) \\
&= \int_0^t B_s \otimes \mathrm{d}B_s + \mathrm{tM\Sigma} = \mathbb{B}_{0,t}^{\mathrm{Strat}} + t\Big(M\Sigma - \frac{1}{2}\mathbb{I}\Big),
\end{aligned}$$

where the convergence is in L^q for $q \geqslant 2$.

Pointwise convergence in L^q (V)

- But taking the symmetric part of the above equation yields

$$\frac{1}{2}MX_t^\varepsilon \otimes MX_t^\varepsilon \to \frac{1}{2}B_t \otimes B_t + \mathrm{Sym}\Big(M\Sigma - \frac{1}{2}\mathbb{I}\Big),$$

- But we already know that

$$\frac{1}{2}MX_t^\varepsilon \otimes MX_t^\varepsilon \to \frac{1}{2}B_t \otimes B_t,$$

and thus we conclude that $\mathrm{Sym}(M\Sigma - \frac{1}{2}\mathbb{I}) = 0$.

- Hence $M\Sigma - \frac{1}{2}\mathbb{I}$ is *anti-symmetric*, yielding

$$M\Sigma - \frac{1}{2}\mathbb{I} = \frac{1}{2}(M\Sigma - \Sigma M^*).$$

- Summarizing, we proved that pointwise in t, we have convergence

$$\Big(MX_t^\varepsilon, \int_0^t MX_s^\varepsilon \otimes \mathrm{d}(MX^\varepsilon)_s\Big) \to (B_{0,t}, \tilde{\mathbb{B}}_{0,t}).$$

Uniform RP bounds

- To conclude, we need to prove the following uniform bounds, for $q<\infty$

$$\sup_{\varepsilon\in(0,1]}\mathbb{E}[\|MX^\varepsilon\|_\alpha^q]<\infty, \qquad \sup_{\varepsilon\in(0,1]}\mathbb{E}\left[\left\|\int MX^\varepsilon\otimes\mathrm{d}(MX^\varepsilon)\right\|_{2\alpha}^q\right]<\infty.$$

- Thanks to the KC for RP, it suffices to prove that

$$\sup_{\varepsilon\in(0,1]}\mathbb{E}[|X_{s,t}^\varepsilon|^q]\lesssim|t-s|^{q/2}, \qquad \sup_{\varepsilon\in(0,1]}\mathbb{E}\left[\left|\int_s^t X_{s,\cdot}^\varepsilon\otimes\mathrm{d}(X_\cdot^\varepsilon)\right|^q\right]\lesssim|t-s|^q.$$

- For what concerns the first bound, we see that since X is Gaussian, it suffices to consider the case $q=2$. We observe that this follows from

$$\mathbb{E}[|\bar{X}_{s,t}|^2]\lesssim|t-s|.$$

Indeed, provided the above bound holds true, we have

$$\mathbb{E}[|X_{s,t}^\varepsilon|^2]=\mathbb{E}[|\varepsilon\bar{X}_{\varepsilon^{-2}s,\varepsilon^{-2}t}|^2]\lesssim\varepsilon^2|\varepsilon^{-2}t-\varepsilon^{-2}s|=|t-s|.$$

Uniform RP bounds (II)

- Thus we have to prove that

$$\mathbb{E}[|\bar{X}_{s,t}|^2] \lesssim |t-s|.$$

This follows from $M\bar{X}_{s,t} = \bar{B}_{s,t} - \bar{Y}_{s,t}$ together with the estimate

$$\mathbb{E}[|\bar{Y}_{s,t}|^2] = \mathbb{E}[|(e^{-M(t-s)} - \mathbb{I})\bar{Y}_s|^2] + \int_s^t \mathrm{Tr}(e^{-Mu}e^{-M^*u})\mathrm{d}u \lesssim |t-s|,$$

where the bound is uniform since $\mathrm{Real}\{\sigma(M)\} \subset (0, +\infty)$.

- An analogous computation shows that at the level of iterated integrals it holds that

$$\mathbb{E}\left[\left|\int_s^t \bar{X}_{s,u} \otimes \mathrm{d}(\bar{X}_u)\right|^2\right] \lesssim |t-s|^2,$$

which in turns implies

$$\mathbb{E}\left[\left|\int_s^t X^{\varepsilon}_{s,\cdot} \otimes \mathrm{d}(X^{\varepsilon}_{\cdot})\right|^2\right] \lesssim |t-s|^2.$$

This concludes the proof.

Summarizing...

- Kolmogorov criterion for RP;
- Itô and Stratonovich Brownian motions;
- Brownian motion in a magnetic field and limit for vanishing mass m.

Thank you for attention!