Markov Processes – Problem Sheet 8.

Exercise 1. (ASYMPTOTIC VARIANCES OF ERGODIC AVERAGES) [5pts] We consider a stationary Markov chain \((X_n, \mathbb{P}_\mu)\) with state space \((E, \mathcal{E})\), transition kernel \(\pi\), and initial distribution \(\mu\).

a) For \(f \in L^2(\mu)\) let \(f_0 = f - \mu(f)\) and let \(A_n f = t^{-1} \sum_{k=0}^{n-1} f(X_k)\). Prove (without assuming the CLT) that if \(Gf_0 = \sum_{k=0}^{\infty} \pi^k f_0\) converges in \(L^2(\mu)\) then

\[
\lim_{t \to \infty} t \text{Var}(A_t f) = 2(f_0, Gf_0)_{L^2(\mu)} - (f_0, f_0)_{L^2(\mu)} = \text{Var}_\mu(f) + \sum_{k=1}^{\infty} \text{Cov}_\mu(f, \pi^k f).
\]

b) Let \(E = \{1, 2\}\) and \(\pi(1, 1) = \pi(2, 2) = p\) and \(\pi(2, 1) = \pi(1, 2) = 1 - p\) with \(p \in (0, 1)\). Show that the unique stationary distribution \(\mu\) is given by \(\mu(1) = \mu(2) = 1/2\) for all values of \(p\). Now consider \(S_n = A_n - B_n\) where \(A_n = \#\{k \leq n; X_k = 1\}\) and \(B_n = \#\{k \leq n; X_k = 2\}\). Show that \(S_n/\sqrt{n}\) satisfies a CLT and compute the limiting variance \(\sigma^2(p)\) as a function of \(p\). How does this variance behaves as \(p \to 0\)? Can you explain it? What is the value of \(\sigma^2(1/2)\)? Could you have guessed it?

Exercise 2. (RANDOM WALKS ON \(\mathbb{Z}_+\)) [5pts] Let \(\delta \in (0, 1)\). We consider a random walk on the nonnegative integers with transition probabilities

\[
\pi(x, y) = \frac{1}{2}(\mathbb{I}_{x+y} + \mathbb{I}_{x=0, y=1}) + \frac{(1-\delta)}{4} \mathbb{I}_{y=x+1, x\geq 1} + \frac{(1+\delta)}{4} \mathbb{I}_{y=x-1, x\geq 1}.
\]

a) Find the invariant probability measure \(\mu\) explicitly.

b) Let \(f: \mathbb{Z}_+ \to \mathbb{R}\) be a function with compact support. Solve the Poisson equation \(-Lg = f\) explicitly (e.g. using the Ansatz \(g = uh\) where \(h\) is a solution to \(Lh = 0\)). Show that for large \(x\) a solution \(g\) either grows exponentially, or it is a constant.

c) Show that there is a solution \(g\) that is a constant for large \(x\) if and only if \(\mu(f) = 0\). What can you say about the asymptotic variance and the central limit theorem for \(\sum_{k=0}^{n-1} f(X_k)\) for such functions \(f\)?

Exercise 3. (COUPLINGS ON \(\mathbb{R}^d\)) [5pts] Let \(W: \Omega \to \mathbb{R}^d\) be a random variable on \((\Omega, \mathcal{F}, \mathbb{P})\), and let \(\mu_a = \text{Law}(a + W)\).

a) (Synchronous coupling) Let \(X = a + W\) and \(Y = b + W\) for \(a, b \in \mathbb{R}^d\). Show that

\[
\mathcal{W}^2(\mu_a, \mu_b) = |a - b| = \mathbb{E}(|X - Y|^2)^{1/2},
\]

i.e. that \((X, Y)\) is an optimal coupling wrt. \(\mathcal{W}^2\).
b) (Reflection coupling) Assume that \(\text{Law}(W) = \text{Law}(-W)\). Let \(\tilde{Y} = \tilde{W} + b\) where \(\tilde{W} = W - 2(e \cdot W)e\) with \(e = (a - b) / |a - b|\). Prove that \((X, \tilde{Y})\) is a coupling of \(\mu_a\) and \(\mu_b\) and if \(|W| \leq |a - b| / 2\) a.s. then

\[
\mathbb{E}(f(|X - \tilde{Y}|)) \leq f(|a - b|) = \mathbb{E}[f(|X - Y|)]
\]

for any concave, increasing function \(f: \mathbb{R}_+ \rightarrow \mathbb{R}_+\) such that \(f(0) = 0\).

**Exercise 4. (Structure of invariant measures) [5pts]** Let \(\pi\) be a probability kernel on \((E, \mathcal{E})\) and let \(S(\pi) = \{\mu \in \mathcal{P}(E): \mu \pi = \mu\}\).

a) Show that \(S(\pi)\) is convex.

b) Prove that \(\mu \in S(\pi)\) is extremal if and only if every set \(B \in \mathcal{E}\) such that \(\pi 1_B = 1_B\) \(\mu\)-a.e. satisfies \(\mu(B) \in \{0, 1\}\).

c) Show that every \(\mu \in S(\pi)\) is a convex combination of extremals (*Hint: you can use exercise 4c of Sheet 6*).