Institute for Applied Mathematics – WS19/20
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V3F1 Elements of Stochastic Analysis – Problem Sheet 1

Distributed October 11th, 2019. In groups of 2. Solutions have to be handed in before 4pm on Thursday October 17th into the marked post boxes opposite to the maths library. Please clearly specify your names and your tutorial group on top of your homework.

Exercise 1. [2+2 Pts] Consider a random variable X with law $\mathcal{N}(0,\ 1)$ and let Z an independent random variable taking values $\{-1,1\}$ with equal probabilities. Let $Y=ZX$. Clearly $Z\sim\mathcal{N}(0,1)$.

a) Show that X,Y are uncorrelated, but not independent.

b) Show that (X,Y) is not a Gaussian vector.

Exercise 2. [2+3+4+2+2+2 pts] Let $(B_t)_{t\geq 0}$ be a (one-dimensional) Brownian motion. Prove the following properties

a) $(B_t)_{t\geq 0}$is a continuous Gaussian process on $\mathbb{R}_+$ with covariance $\mathrm{Cov}(B_s,B_t)=\min\{s,t\}$.

b) For any $t>s$ and any bounded measurable $f:\mathbb{R}\to\mathbb{R}$ we have

$$\mathbb{E}[f(B_t)|B_s]=(P_{t-s}f)(B_s),$$

where the transition kernel P_t is given by

$$P_t(x,\mathrm{d}y)=\frac{1}{\sqrt{2\pi t}}e^{-(x-y)^2/2t}\mathrm{d}y.$$

c) The process $(B_t)_t$ is a Markov process wrt. the filtration $(\mathscr{F}_t)_t$ given by $\mathscr{F}_t=\sigma(B_s: s\in[0,t])$. Here the relevant Markov property is

$$\mathbb{E}[f(B_t)|\mathscr{F}_s]=\mathbb{E}[f(B_t)|B_s]=(P_{t-s}f)(X_s).$$

d) The process $(-B_t)_{t\geq 0}$ is also a Brownian motion (*symmetry property*).

e) For any $c>0$, the process defined by

$$X_t:=\frac{1}{\sqrt{c}}B_{c\cdot t},\qquad t\geqslant 0,$$

is also a Brownian motion (*scaling invariance property*).

f) The process $Z_t:=B_{t+r}-B_r$ is also a Brownian motion.

Exercise 3. [2+2+1 pts] Let $(B_t)_{t\geqslant 0}$ a standard Brownian motion. Let $Z:=\sup_{t\geqslant 0}B_t$.

a) Show that Z and cZ have the same law for all $c>0$. Conclude that the law of Z is supported on $\{0,\infty\}$.

b) Show that $\mathbb{P}(Z=0)\leqslant\mathbb{P}(B_1\leqslant 0)\mathbb{P}(\sup_{t\geqslant 0}(B_{t+1}-B_1)=0)$ and conclude that $\mathbb{P}(Z=0)=0$.

c) Conclude that $\mathbb{P}(\sup_{t\geqslant 0}B_t=+\infty,\ \inf_{t\geqslant 0}B_t=-\infty)=1$. That is the Brownian motion oscillates a.s. infinitely often between $+\infty$ and $-\infty$. (see also the law of iterated logarithm).